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THE THEORY AND PRACTICE

OF

SURVEYING.

DESIGNED FOR THE USE OF

SURVEYORS AND ENGINEERS GENERALLY,

BUT ESPECIALLY FOR THE USE OF

STUDENTS IN ENGINEERING.

BY

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ELEVENTH EDITION, REVISED AND ENLARGED.

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PREFACE

TO

THE ELEVENTH EDITION.

SINCE the issue of the seventh edition of this work, in 1890, there was added in the tenth edition (1892) Appendix F, being the instructions for field work issued by the Mississippi River Commission, and there is now added in this edition Appendix G, upon the Essential Requirements of Surveys and Maps, and the Ownership of Surveys, by Prof. William G. Raymond, of the Rensselaer Polytechnic Institute. This last appendix will be found to contain some very valuable principles and rules which should control the making and mapping of surveys, and will be found especially helpful to young surveyors. All students, also, who use this work are recommended to read this appendix with care. There has also been added to this edition a table of Azimuths of Polaris from 1895 to 1910. A number of other minor changes have also been made in this edition.

J. B. J.

MAY, 1894.

PREFACE

TO

THE SEVENTH EDITION.

IN each new edition of this volume some additions, corrections, and minor changes have been made. In the present edition there are so many changes and additions that they deserve to be specially mentioned.

To Part I., on Surveying Instruments, have been added descriptions and cuts of the architect's level, new level-rod targets and bubbles, Wood's double sextant, and the cross-section polar protractor used in the New York aqueduct tunnel.

The table of Magnetic Declination Formulæ, on pages 25 to 28 inclusive, has been replaced entire with the new table issued by the United States Coast and Geodetic Survey, 1890, and Plate I. has been redrawn and brought down to 1890.

The chapter on Land Surveying has been entirely recast. A considerable amount of new matter on the subject of monuments, and the principles and laws governing the re-survey of lands, have been added. The author wishes here to acknowledge his debt to Bellows and Hodgeman's little work on Land Surveying for much of the original matter from which he has deduced his general rules. In that work decisions are abstracted, and references given to the cases themselves, and the land surveyor would do well to obtain a copy of this valuable work. In preparing these general rules the author has had in mind the student and young surveyor rather than the experienced practitioner; and the reader must remember that judicial decisions are judgments on particular cases, and generalizations from such decisions must be made and received with caution. The rules here laid down apply rather to the inexact methods of the compass and chain than to those of the transit and steel tape.

The description of the United States Land Surveys has been entirely re-written and expanded, and an appendix added giving the location of all the principal meridians and accompanying base lines which have been used in laying out the public lands.

A method of running out parallels of latitude, with suitable tables, has been added, and also tables and descriptions by which an observation for azimuth may be made on Polaris at any hour. This latter table has but recently been computed, and is published in the last edition (1890) of the *Manual of Instructions* issued by the Commissioner of the General Land Office, Washington, D. C. By means of this table the great objections to stellar observations for azimuth are removed, as they may be made at any hour, and all tedious computations are avoided.

A description of Porro's Telescope has also been added to the chapter on Topographical Surveying. This telescope reads distances by stadia correctly from the center of the instrument instead of from a point in front of the objective. It is not now manufactured in this country, but it may again come into use.

J. B. J.

DECEMBER, 1890.

PREFACE TO THE FIRST EDITION

No apology is necessary for the appearance of a new book on Surveying. The needs of surveyors have long been far beyond the accessible literature on this subject, to say nothing of that which has heretofore been formulated in text-books. The author's object has been to supply this want so far as he was able to do it.

The subject of surveying, both in the books and in the schools, has been too largely confined to Land Surveying. The engineering graduates of our technical schools are probably called upon to do more in any one of the departments of Railroad, City, Topographical, Hydrographical, Mining, or Geodetic Surveying than in that of Land Surveying. Some of these subjects, as for example City, Geodetic, and Hydrographical Surveying, have not been formulated hitherto, in any adequate sense, in either English or any other language, to the author's knowledge. In the case of Geodetic Surveying there has been a wide hiatus between the matter given in text-books and the treatment of the subject in works on Geodesy and in special reports of geodetic operations. The latter was too technical, prolix, and difficult to give to students, while the former was entirely inadequate to any reasonable preparation for this kind of work on even a small scale. The subjects of City and Hydrographical Surveying as here presented are absolutely new.

Part I. treats of the adjustment, use, and care of all kinds of instruments used in surveying, either in field or office.* In describing the adjustments of instruments the object has been to present to the mind of the reader the geometrical relations from which a rule or method of adjustment would naturally follow. The author has no sympathy with descriptions of adjustments as mechanical processes simply to be committed to memory, any more than he has with that method of teaching geometry wherein the student is allowed to memorize the demonstration.

Many surveying instruments not usually described in books on surveying are fully treated, such as planimeters, pantographs, barometers, protractors, etc. The several sets of problems given to be worked out by the aid of the corresponding instruments are designed to teach the capacity and limitations of such instruments, as well as the more important sources of error in their use. This work is such as can be performed about the college campus, or in the near vicinity, and is supposed to be assigned for afternoon or Saturday practice while the subject is under consideration by the class. More extended surveys require a special field-season for their successful prosecution.†

The methods of the differential and integral calculus have been sparingly used, as in the derivation of the barometric formula for elevations, and of the LMZ formulæ in Appendix D. Such demonstrations may have to be postponed to a later period of the course.

* Certain special appliances, as for example heliotropes, filar micrometers, current-meters, etc., are treated in the subsequent chapters.

† At Washington University all the engineering Sophomores go into the field for four weeks at the end of the college year, and make a general land and topographical survey, such as shown in Plate II. At the end of the Junior year the civil-engineering students go again for four weeks, making then a geodetic and railroad survey. Some distant region is selected where the ground, boarding facilities, etc., are suitable.

Part II. includes descriptions of the theory and practice of Surveying Methods in the several departments of Land, Topographical, Railroad, Hydrographical, Mining, City, and Geodetic Surveying; Surveys for the Measurement of Volumes; and the Projection of Maps, Map Lettering, and Topographical Signs. The author has tried to treat these subjects in a concise, scientific, and practical way, giving only the latest and most approved methods, and omitting all problems whose only claim for attention is that of geometrical interest.

In treating the trite subject of Land Surveying many problems which are more curious than useful have been omitted, and several new features introduced. The subjects of computing areas from the rectangular co-ordinates, and the supplying of missing data, are made problems in analytical geometry, as they should be. A logarithmic Traverse Table for every minute of arc from zero to 90°, arranged for all azimuths from zero to 360°, to be used in connection with a four-place logarithmic table, serves to compute the co-ordinates of lines when the transit is the instrument used. A traverse table computed for every 15 minutes of arc is no longer of much value. The isogonic declination curves shown on Plate I. will be found to embody all the accessible data up to 1885, and are reduced from the U. S. Coast Survey chart. Appendix A will be found of great value as outlining the Judicial Functions of the Surveyor by the best possible authority.

The chapter on Mining Surveying was written by Mr. C. A. Russell, C.E., U. S. Deputy Mineral Surveyor of Boulder, Colorado. He has had an extended experience in Hydrographic Surveying, in addition to many years' practice in surveying mines and mining claims.

The chapter on City Surveying was written by Mr. Wm. Bouton, C.E., City Surveyor of St. Louis, Mo. Mr. Bouton has done a large proportion of the city surveying in St. Louis

[redacted]

for the last twenty years, and has gained an enviable reputation as a reliable, scientific, and expert surveyor.

It is believed that the ripe experience of these gentlemen which has been embodied in their respective chapters will materially enhance the value of the book.

The author also desires to acknowledge his indebtedness to his friend H. S. Pritchett, Professor of Astronomy in Washington University, for valuable assistance in the preparation of the matter on Time in Chapter XIV.

Although the theorems and the notation of the method of least squares are not used in this work, yet two problems are solved by what is called the *method of the arithmetic mean* (which, when properly defined, is the same as the method of least squares), which will serve as a good introduction to the study of the method of least squares. These problems are the Rating of a Current-meter, in Chapter X., and the Adjustment of a Quadrilateral, in Chapter XIV. The author has found that such solutions as these serve to make clear to the mind of the student exactly what is accomplished by the least-square methods of adjusting observations.

The chapter on Measurement of Volumes is not intended to be an exhaustive treatment of the subject of earthwork, but certain fundamental theorems and relations are established which will enable the student to treat rationally all ordinary problems. The particular relation between the Henck prismoid and the warped-surface prismoid is an important one, but one which the author had nowhere found.

An earthwork table (Table XI.) has also been prepared which gives volumes directly, without correction, for the warped-surface prismoid. The author has no knowledge that such a table has ever been prepared before.

A former work by the author on Topographical Surveying by the Transit and Stadia is substantially included in this book.

The methods recommended for measuring base-lines with steel-tapes are new; but they have been thoroughly tested, and are likely to work a material change in geodetic methods.

The author wishes to acknowledge his obligations to many instrument-manufacturers for the privilege they have very kindly accorded to him of having electrotype copies made from the original plates, for many of the cuts of instruments given throughout the book; persons familiar with the valuable catalogues published by these firms will recognize the makers among the following: W. & L. E. Gurley, Troy, N. Y.; Buff & Berger, Boston, Mass.; Fauth & Co., Washington, D. C.; Queen & Co. and Young & Sons, Philadelphia, Pa.; Keuffel & Esser, New York; and F. E. Brandis Sons & Co., Brooklyn, N. Y. Also to Mr. W. H. Searles for the privilege of using copies of plates from his Field-book for Tables I., VI., and VII.

Hoping this work will assist in lifting the business of surveying to a higher professional plane, as well as to enlarge its boundaries, the author submits it to surveyors and engineers generally, but especially to the instructors and students in our polytechnic schools, for such crucial tests as the field and the class-room only can give.

J. B. J.

ST. LOUIS, Sept. 23, 1886.

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SURVEYING.

INTRODUCTION.

SURVEYING is the art of making such field observations and measurements as are necessary to determine positions, areas, volumes, or movements on the earth's surface. The field operations employed to accomplish any of these ends constitute a survey. Accompanying such survey there is usually the field record, the computation, and the final maps, plats, profiles, areas, or volumes. The art of making all these belongs, therefore, to the subject of surveying.

Inasmuch as all fixed engineering structures or works involve a knowledge of that portion of the earth's surface on which they are placed, together with the necessary or resulting changes in the same, so the execution of such works is usually accompanied by the surveys necessary to obtain the required information. Thus surveying is seen to be intimately related to engineering, but it should not be confounded with it. All engineers should have a thorough knowledge of surveying, but a surveyor may or may not have much knowledge of engineering.

The subject of Surveying naturally divides itself into—

I. The Adjustment, Use, and Care of Instruments.

II. Methods of Field Work.

III. The Records, Computations, and Final Products.

All the ordinary instruments that a surveyor may be called upon to use in any of the departments of the work will be discussed in the following pages. The most approved methods

[redacted]

only will be given for obtaining the desired information, and many problems that are more curious than useful will not be mentioned. The student is assumed to possess a knowledge of geometry, and of plane and spherical trigonometry. He is also supposed to be guided by an instructor, and have access to most of the instruments here mentioned, with the privilege of using them in the field.

The field work of surveying consists wholly of *measuring* distances, angles, and time, and it is well to remember that no measurement can ever be made exactly. The first thing the young surveyor needs to learn, therefore, is the *proportionate error* allowable in the special work assigned him to perform. It is of the utmost importance to his success that he shall thoroughly study this subject. He should know what all the sources of error are, and their relative importance; also the relative cost of diminishing the size of such errors. Then, with a given standard of accuracy, he will know how to make the survey of the required standard with the least expenditure of time and labor. He must not do all parts of the work as accurately as possible, or even with the same care. For, if the expense is proportioned to the accuracy of results, then he is the most successful surveyor who does his work just good enough for the purpose. The relative size of the various sources of error is of the utmost importance. One should not expend considerable time and labor to reduce the error of *measurement* of a line to 1 in 10,000 when the unknown error in the length of the measuring unit may be as high as 1 in 1000.

The surveyor must carefully discriminate, also, between compensating errors and cumulative errors. A compensating error is one which is as likely to be plus as minus, and it is therefore largely compensated in, or eliminated from, the result. A cumulative error is one which always enters with the same sign, and therefore it accumulates in the result. Thus, in chaining, the error in setting the pin is a compensating error, while the error from erroneous length of chain is a cumulative error. If a mile is chained with a 66-foot chain, there are 80 measurements taken. Suppose the error of setting the pin be 0.5 inch, and the error in the length of the chain be 0.1 inch. Now the theory of probabilities shows us that in the case of compensating errors the square root of the number of errors *probably** remains uncompensated. The probable error from setting the pins is therefore 9×0.5 inch $= 4.5$ inches. The error from erroneous length of chain is 80×0.1 inch $= 8$ inches. Thus we see that although the error from setting the pins was five times as great as that from erroneous length of chain, yet in running one mile, the resulting error from the latter cause was nearly twice that from the former. A careful study of the various sources of error affecting a given kind of work will usually enable the surveyor either to add greatly to its accuracy without increasing its cost, or to greatly diminish its cost without diminishing its accuracy.

The surveyor should have no desire except to arrive at the truth. This is the true scientific spirit. He should be most severely honest with himself. He should not allow himself to change or "fudge" his notes without sufficient warrant, and then a full explanation should be made in his note-book. Neither should he make his results appear more accurate than they really are. He should always know what was about the relative accuracy with which his field work was done, and carry his results only so far as the accuracy of the work would warrant. He is either foolish or dishonest who, having made a survey of an area, for instance, with an error of closure of 1 in 300, should carry his results to six significant figures, thus giving the area to 1 in 500,000. It is usual to carry the computations one place farther than the results are known, in order that no additional error may come in from the computation. It is not unusual, however, to see results given in published documents to two, three, or even four places farther than the observations would warrant.

* The meaning of this statement is that *on the average* this will occur oftener than any other combination, and that any *single* result will, *on the average*, be nearer to this result than to any other.

The student should make himself familiar with the structure and use of every part of every instrument put into his hands. The best way of doing this is to take the instrument all apart and put it together again. This, of course, is not practicable for each student in college, but when he is given an instrument in real practice, he should then make himself thoroughly familiar with it before attempting to use it.

The adjustments of instruments should be studied as problems in descriptive geometry and not as mechanical manipulations, learned in a mechanical way; and when adjusting an instrument the geometry of the problem should be in the mind rather than the rule in the memory.

Students of engineering in polytechnic schools are urged to make themselves familiar with every kind of instrument in the outfit of the institution, and to do in the field every kind of work herein described if possible. Otherwise he may be called upon to do, or to direct others to do, what he has never done himself, and he will then find that his studies prove of little avail without the real knowledge that comes only from experience.

BOOK I.

ADJUSTMENT, USE, AND CARE OF INSTRUMENTS.

CHAPTER I.

INSTRUMENTS FOR MEASURING DISTANCES.

THE CHAIN.

1. **The Engineer's Chain** is 50 or 100 feet long, and should be made of No. 12 steel wire. The links are one foot long, including the connecting rings. All joints in rings and links should be brazed to prevent giving. The connections are designed so as to admit of as little stretch as possible. Every tenth foot is marked by a special form of brass tag. If the chain is adjustable in length, it should be made of standard length by measuring from the inside of the handle at one end to the outside of the handle at the other. If it is not adjustable, measure from the outside of the handle at the rear end to the standard mark at the forward end.

Fig. 1.

2. **Gunter's Chain** is 66 feet long, and is divided into 100 links, each link being 7.92 inches in length. This chain is mostly used in land-surveying, where the acre is the unit of measure. It was invented by Edmund Gunter, an English

astronomer, about 1620, and is very convenient for obtaining areas in acres or distances in miles. Thus,

$$\begin{aligned}\text{One mile} &= 80 \text{ chains; also,}\\ \text{One acre} &= 160 \text{ square rods,}\\ &= 10 \text{ square chains,}\\ &= 100{,}000 \text{ square links.}\end{aligned}$$

If, therefore, the unit of measure be chains and hundredths (links), the area is obtained in square chains and decimals, and by pointing off one more place the result is obtained in acres. This is the length of chain used on all the U. S. land surveys. In all deeds of conveyance and other documents, when the word *chain* is used it is Gunter's chain that is meant.

3. **Testing the Chain.**—No chain, of whatever material or manufacture, will remain of constant length. The length changes from temperature, wear, and various kinds of distortion. A change of temperature of 70° F. in a 100-foot chain will change its length by 0.05 foot, or a change of 1 in 2000.

If the links of a chain are joined by three rings, then there are eight wearing surfaces for each link, or eight hundred wearing surfaces for a 66- or 100-foot chain. If each surface should wear 0.01 inch, the chain is lengthened by eight inches. It is not uncommon for a railroad survey of, say, 300 miles to be run with a single chain. If such a chain were of exactly the right length at the beginning of the survey, it might be six inches too long at the end of it.

The change of length from distortion may come from a flattening out of the connecting rings, from bending the links, or from stretching the chain beyond its elastic limit, thus giving it a permanent set. Both the wear and the distortion are likely to be less for a steel chain than for an iron one. When a bent link is straightened it is permanently lengthened.

When we remember that all unknown changes in the length of the chain produce cumulative errors in the measured lines, we see how important it is that the true length of the chain should be always known, or better, that the standard length (50, 66, or 100 feet) should be properly measured from one end of the chain and marked at the other. This chain test is most readily accomplished by the aid of a standard steel tape, which is at least as long as the chain. By the aid of such a tape a standard length may be laid off on the floor of a large room, or two stones may be firmly set in the ground at the proper distance apart and marks cut upon their upper surfaces. If stones are used they should reach below the frost-line. Or a short tape, or other standard measuring unit, may be used for laying off such a base-line. By whatever means it is accomplished, some ready means should at all times be available for testing the chain. Since a chain always grows longer with use, the forward end of the chain will move farther and farther from the standard mark. A small file-mark may be made on the handle or elsewhere, and then removed when a new test gives a new position. Care must be exercised to see that there are no kinks in the chain either in testing or in use.

In laying out the standard base the temperature at which the unit of measure is standard should be known (this temperature is stamped on the better class of steel tapes), and if the base is not laid out at this temperature, a correction should be made before the marks are set. The coefficient of expansion of iron and steel is very nearly 0.0000065 for 1° F. If T_s be the temperature at which the tape is standard, T the temperature at which the base is measured, and L the length of the base, then $0.0000065\ (T_s - T)L$ is the correction to be applied to the measured length to give the true length.

When the chain is tested by this standard base the temperature should be again noted, and if this is about the mean temperature for the field measurements no correction need be made to the field work. If it is known, at the time the chain is tested, that the temperature is very different from the probable mean of the field work, then the standard mark can be so placed on the chain as to make it standard when in use.

4. **The Use of the Chain.**—The chain is folded by taking it by the middle joint and folding the two ends simultaneously. It is opened by taking the two handles in one hand and throwing the chain out with the other.

Since horizontal distances are always desired in surveying, the chain should be held horizontally in measuring. Points vertically below the ends of the chain are marked by iron pins, the head chainman placing them and the rear chainman removing them after the next pin is set. The chain is lined in either by the head or rear chainman, or by the observer at the instrument, according as the range-pole is in the rear, or in front, or not visible by either chainman. When chaining on level ground, the rear chainman brings the outside of the handle against the pin, and the head chainman sets the forward side of his pin even with the standard mark on the chain. By this means the centres of the pins are the true distance apart. On uneven ground both chainmen cannot hold to the pin; one end being elevated in order to bring the chain to a horizontal position. In this case there are three difficulties to be overcome. The chain should be drawn so taut that the stretch from the pull would balance the shortening from the sag; the chain should be made horizontal; the elevated end-mark must be transferred vertically to the ground. It is practically impossible to do any of these exactly. The first could be determined by trial. Stretch the chain between two points at the same elevation, having it supported its entire length. Then remove the supports, and see how strong a pull is required to bring it to the marks again. This should be done by the chainmen themselves, thus enabling them to judge how hard to pull it when it is off the ground. To hold the chain horizontal on sloping ground is very difficult, on account of the judgment being usually very much in error as to the position of a horizontal line. In all such cases the apparently horizontal line is much too nearly parallel with the ground. Sometimes a level has been attached to one end of the chain, in which case it should be adjusted to indicate horizontal end-positions for a certain pull, this being the pull necessary to overcome the shortening from sag. To hold a plumb-line at the proper mark, with the chain at the right elevation, and stretched the proper amount, requires a steady hand in order that the plumb-bob may hang stationary. This should be near the ground, and when all is ready, it is dropped by the chainman letting go the string. The pin is then stuck and the work proceeds. It is common in this country for the rear chainman to call "stick" when he is ready, and for the head chainman to answer "stuck" when he has set the pin. The rear chainman then pulls his pin and walks on.

There should be eleven pins, marked with strips of colored flannel tied in the rings to assist in finding them in grass or brush. In starting, the rear chainman takes a pin for the initial point, leaving the head chainman with ten pins. When the last pin is stuck, the head chainman calls "out," and waits by this station until the rear chainman comes up and delivers over the ten pins now in his possession. The eleventh pin is in the ground, and serves as the initial point for the second score. Thus only every ten chains need be scored.

Good chaining, therefore, consists in knowing the length of the chain, in true alignment, horizontal and vertical, and in proper stretching, marking, and scoring.

THE STEEL TAPE.

5. **Varieties.**—Steel tapes are now made from one yard to 1000 feet in length, graduated metrically, or in feet and tenths. A pocket steel tape from three to ten feet long should always be carried by the surveyor. A 50-foot tape is best fitted to city surveying where there are appreciable grades. For cities

without grades a 100-foot tape might be found more useful. For measuring base-lines, or for some kinds of mining surveying, a 300 or 500 foot tape is best. These are of small cross-section, being about 0.1 inch wide and 0.02 inch thick. A tape about

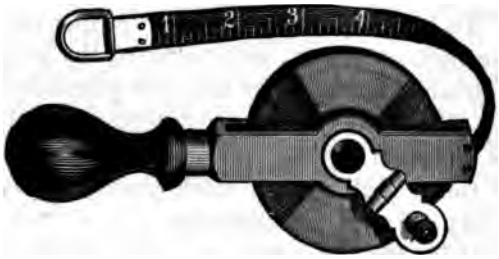

FIG. 2.

0.5 inch wide and 0.02 inch thick (Fig. 2) is perhaps best suited to general surveying.

6. The Use of Steel Tapes.—Steel tape-measures are used just as chains are. They are provided with handles, but the end graduation-marks are usually on the tape itself and not on the handle. They are graduated to order, the graduations being either etched or made on brass sleeves which are fastened on the tape. Their advantages are many. They do not kink, stretch, or wear so as to change their length, so that, with careful handling, they remain of constant length except for temperature. They are used almost exclusively in city and bridge work, and in the measurement of secondary base-lines. The same precautions must be taken in regard to alignment, pull, and marking with the tape, as was described for the chain.*

* For methods of using the steel tape in accurate measurements, see Chapter XIV., Base-Line Measurements.

EXERCISES.

To be worked out on the ground by the use of the chain or tape alone.

7. To chain a line over a hill between two given points, not visible from each other.

Range-poles are set at the given points. Then the two chainmen, each with a range-pole, range themselves in between the two fixed points, near the top of the hill, by successive approximations. The line can then be chained.

8. To chain a line across a valley between two fixed points.

Establish other range-poles by means of a plumb-line held on range between the points.

9. To chain a line between two fixed points when woods intervene, and the true line is not to be cleared out.

Range out a trial line by poles, leaving fixed points. Find the resulting error at the terminus, and move all the points over their proportionate amount. The true line may then be chained.

10. To set a stake in a line perpendicular to a given line at a given point.

All multiples of 3, 4, and 5 are the sides of a right-angled triangle; also any angle in a semicircumference is a right angle.

11. To find where a perpendicular from a given point without a line will meet that line.

Run an inclined line from the given point to the given line. Erect a perpendicular from the given line near the required point, extend it till it intersects the inclined line, and solve by similar triangles.

12. To establish a second point that shall make with a given point a line parallel to a given line.

Diagonals of a parallelogram bisect each other.

13. To determine the horizontal distance from a given point to a visible but inaccessible object.

Use two similar right-angled triangles.

14. To prolong a line beyond an obstacle, in azimuth* and distance.

First Solution: By an equilateral triangle.

Second Solution: By two rectangular offsets on each side of the obstacle.

Third Solution: By similar triangles, as in Fig. 3.

From any point as A run the line AB, fixing the half and three-quarter points at x and y. From any other point as C, run CxD, making $xD = Cx$. From D

---

* The azimuth of a line is the angle it forms with the meridian, and is measured from the south point in the direction S.W. N.E. to 360 degrees. It thus becomes a definite direction when the angle alone is given. Thus the azimuth of 220° corresponds to the compass-bearing of N. 40° E.

run DyE making $DE = 4\ Dy$, fixing the middle point z. From B run BzH, making $zH = Bz$. Then is HE parallel and equal to DB, AC, and CH.

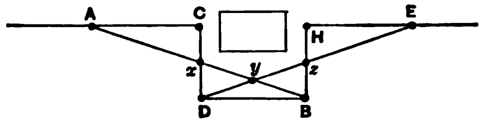

FIG. 3.

Stakes should be set at all the points lettered in the figure. Check: Measure HE and AC. If they are equal the work is correct.

15. To measure a given angle.

Lay off equal distances, b, from the vertex on the two lines, and measure the third side a of the triangle. Then $\tan \frac{1}{2} A = \frac{a}{\sqrt{4b^2 - a^2}}$.

16. To lay out a given angle on the ground.

Reverse the above operation. A is known; assume b and compute a. Then from A measure off $AB = b$. From B and A strike arcs with radii equal to a and b respectively, giving an intersection at C. Then CAB is the required angle. If b is assumed not greater than 0.6 the length of the chain, angles may be laid out up to 90°.

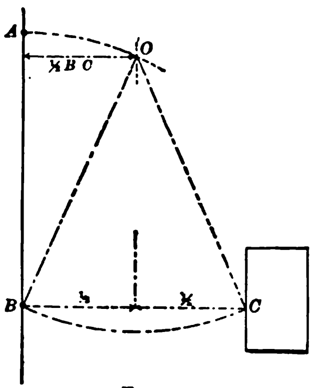

FIG. 3a.

17. To mark a point on a house, bowlder, or other object, near a line of survey, which shall be at right angles to a given point in said line, by means of a chain, tape, or cord.

Let AB be a line of survey. Let C be a house on which a transfer of point B is required at right angles to AB. Then with radius BA swing arc AO, and with $\frac{1}{2}\ BC$ on stick or tape measure to arc from line AB at point O; with radius OB (or BA), from O swing arc touching house in C as required.

CHAPTER II.

INSTRUMENTS FOR DETERMINING DIRECTIONS.

THE COMPASS.

18. **The Surveyor's Compass** consists essentially of a line of sight attached to a horizontal graduated circle, at the centre of which is suspended a magnetic needle free to move, the whole conveniently supported with devices for levelling. Fig.

FIG. 4.

4 shows a very good form of such an instrument. In addition to the above essential features, the instrument here shown has a tangent-screw and vernier-scale at *e* for setting off the declination of the needle; a tangent-scale on the edge of the vertical sight for reading vertical angles, the eye being placed at the sight-disk shown on the opposite standard; and an

auxiliary graduated circle, with vernier, shown on the front part of the plate, for reading angles closer than could be done with the needle. The compass is mounted either on a tripod or on a single support called a Jacob's-staff. It is connected to its support by a ball-and-socket joint, which furnishes a convenient means of levelling.

Although the needle-compass does not give very accurate results, it is one of the most useful of surveying instruments. Its great utility lies in the fact that the needle always points in a known direction, and therefore the direction of any line of sight may be determined by referring it to the needle-bearing. The needle points north in only a few localities; but its *declination from* the north point is readily determined for any region, and then the true azimuth, or bearing of a line, may be found. It has grown to be the universal custom, in finding the direction of a line by the compass, to refer it to *either* the north *or* the south point, according to which one gives an acute angle. Thus, if the bearing is 100° from the south point it is but 80° from the north point, and the direction would be defined as north, 80° east or west, as the case might be: thus no line can have a numerical bearing of more than 90°. In accordance with this custom, all needle-compasses are graduated from both north and south points each way to the east and west points, the north and south points being marked zero, and the east and west points 90°. When the direction of a line is given by this system it is called the *bearing* of the line. When it is simply referred to the position of the needle it is called the *magnetic* bearing. When it is corrected for the declination of the needle, either by setting off the declination on the declination-arc or by correcting the observed reading, it is called the *true* bearing, being then referred to the true meridian.

Because the graduated circle is attached to the line of sight and moves with it, while the needle remains stationary, *E* and

W are placed on the compass-circle in reversed position. Thus when the line of sight is north-east, the north end of the needle points to the left of the north point on the circle, and hence *E* must be put on this side of the meridian line.

In reading the compass, always keep the north end of the circle pointing forward along the line, and read the north end of the needle.

The north end of the needle is usually shaped to a special design, or, if not, it may be distinguished by knowing that the south end is weighted by having a small adjustable brass wire slipped upon it to overcome the tendency the north end has to dip.

ADJUSTMENTS OF THE COMPASS.

19. **The General Principle** of almost all instrumental adjustments is the *Principle of Reversion, whereby the error is doubled and at the same time made apparent.* A thorough mastery of this principle will nearly always enable one to determine the proper method of adjusting all parts of any surveying instrument. It should be a recognized principle in surveying, that no one is competent to handle any instrument who is not able to determine when it is in exact adjustment, to locate the source of the error if not in adjustment, to discuss the effect of any error of adjustment on the work in hand, and to properly adjust all the movable parts. The methods of adjustment should not be committed to memory—any more than should the demonstration of a proposition in geometry. The student in reading the methods of adjustment should *see* that they are correct, just as he sees the correctness of a geometrical demonstration. Having thus had the method and the reason therefor clearly in the mind, he should trust his ability to evolve it again whenever called upon. He thus relies upon the accuracy of his reasoning, rather than on the distinctness of his recollection.

20. To make the Plate perpendicular to the Axis of the Socket.—This must be done by the maker. It is here mentioned because the axis is so likely to get accidentally bent. Instruments made of soft brass must be handled very carefully to prevent such an accident. If this adjustment is found to be very much out, it should be sent to the makers. If much out, it will be shown by the needle, and also by the plate-bubbles.

21. To make the Plane of the Bubbles perpendicular to the Axis of the Socket.—Level it in one position, turn 180°, and correct one-half the movement of each bubble by the adjusting-screw at the end of the bubble-case. Now level up again, and revolve 180°, and the bubbles should remain at the centre. If not, adjust for one half the movement again, and so continue until the bubbles remain in the centre for all positions of the plate.

The student should construct a figure to illustrate this and almost all other adjustments. Thus, in this case, let the figure consist of two lines, one representing the axis of the socket, and the other the axis of the bubble, crossing it. Now if these two lines are not at right angles to each other, when the one is horizontal (as the bubble-axis is when the bubble rests at the centre of its tube) the other is inclined from the vertical. Now with this latter fixed, let the figure be revolved 180° about it (or construct another figure representing such a movement), and it will be seen that the bubble-axis now deviates from the horizontal by *twice* the difference between the angle of the lines and 90°. By now correcting *one half* of this change of direction on the part of the bubble-axis, it will be made perpendicular to the socket-axis. Then by relevelling the instrument, which consists of moving the socket-axis until the bubbles return to the middle of the tubes, the instrument should now revolve in a horizontal plane.

22. To adjust the Pivot to the Centre of the Graduated Circle.—When the two ends of the needle do not read exactly alike it may be due to one or more of three causes: The circle may not be uniformly graduated; the pivot may be bent out of its central position; or the needle may be bent. All our modern instruments are graduated by machinery, so that they have no errors of graduation that could be detected by eye. One or both of the other two causes must therefore exist. If the difference between the two end-readings is constant for all positions of the needle, then the pivot is in the centre of the circle, but the needle is bent. If the difference between the two end-readings is variable for different parts of the circle, then the pivot is bent, and the needle may or may not be straight. To adjust the pivot, therefore, find the position of the needle which gives the *maximum* difference of end-readings, remove the needle, and *bend the pivot at right angles to this position by one half the difference in the extreme variation of end-readings.* Repeat the test, etc. Since the glass cover is removed from the compass-box in making this adjustment, it should be made indoors, to prevent any disturbance from wind.

23. **To straighten the Needle,** set the north end exactly at some graduation-mark, and read the south end. If not 180° apart, bend the needle until they are. This implies that the preceding adjustment has been made, or examined and found correct.

24. **To make the Plane of the Sights normal to the Plane of the Bubbles.**—Carefully level the instrument and bring the plane of the sights upon a suspended plumb-line. If this seems to traverse the farther slit, then that sight is in adjustment. Reverse the compass, and test the other sight in like manner. If either be in error, its base must be re-shaped to make it vertical.

25. **To make the Diameter through the Zero-graduations lie in the Plane of the Sights.**—This should be done by the maker, but it can be tested by stretching two fine hairs vertically in the centres of the slits. The two hairs and the two zero-graduations should then be seen to lie in the same plane. The declination-arc must be set to read zero.

26. To remagnetize the Needle.—Needles sometimes lose their magnetic properties. They must then be remagnetized. To do this take a simple bar-magnet and rub each end of the needle, from centre towards the ends, with the end of the magnet which attracts in each case. In returning the magnet for the next stroke lift it up a foot or so to remove it from the immediate magnetic field, otherwise it would tend to nullify its own action. The needle should be removed from the pivot in this operation, and the work continued until it shows due activity when suspended. An apparently sluggish needle may be due to a blunt pivot. If so, this should be removed, and ground down on an oil-stone.

THE VERNIER.

27. The Vernier is an auxiliary scale used for reading fractional parts of the divisions on the main graduated scale or limb. If we wish to read to tenths of one division on the limb, we make 10 divisions on the vernier correspond to either 9 or 11 divisions on the limb. Then each division on the vernier is one tenth less or greater than a division on the limb. If we wish to read to twentieths or thirtieths of one division on the limb, there must be twenty or thirty divisions on the vernier corresponding to one more or less on the limb.

The zero of the vernier-scale marks the point on the limb whose reading is desired.

Suppose this zero-point corresponds exactly with a division on the limb. The reading is then made wholly on the limb. If a division on the vernier is *less* than a division on the limb, then, by moving the vernier *forward* a trifle, the next *forward* division on the vernier corresponds with *a* division on the limb. (The particular division on the *limb* that may be in coincidence is of no consequence.) On the other hand, if a division on the vernier is *greater* than a division on the limb, then by moving the vernier forward a trifle, the next *backward* division on the vernier comes into coincidence. Thus we have two kinds of verniers, called ***direct*** and ***retrograde*** according as they are read ***forward*** or ***backward*** from the zero-point. Most verniers in use are of the direct kind, but those commonly found on surveyors' compasses for setting off the declination are generally of the retrograde order.

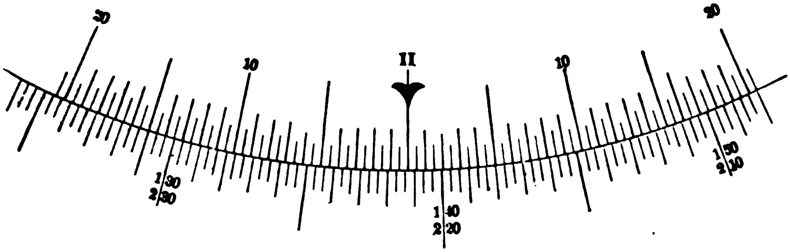

FIG. 5.

In Fig. 5 are shown two direct verniers, such as are used on transits with double graduations. Thus in reading to the right the reading is 138° 45′, but in reading to the left it is 221° 15′. In each case we look along the vernier ***in the direction of the graduation*** for the coincident lines.

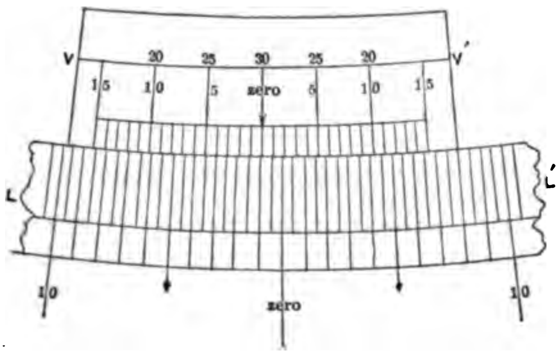

FIG. 6.

In Fig. 6 is shown a special form of retrograde vernier in which the same set of graduation-lines on the vernier serve for either right- or left-hand angles. Here a division of the vernier is larger than a division on the limb, and it must therefore be read backwards. Thus, we see that the zero of the vernier is to the left of the zero of the limb, the angle being 30′ and something more. Starting now toward the right (backwards) on the vernier scale, we reach the end or 15-minute mark, without finding coincident lines; we then skip to the left-hand side of the vernier scale and proceed *towards the right* again until we find coincident lines at the twenty-sixth mark. The reading is therefore $30 + 26 = 56$ minutes. This is the form of vernier usually found on surveyors' compasses for setting off the declination. We have therefore the following

Rules.

First. To find the "smallest reading" of the vernier, divide the value of a division on the limb by the number of divisions in the vernier.

Second. Read forward along the limb to the last graduation preceding the zero of the vernier; then read forward along the vernier if direct, or backward if retrograde, until coincident lines are found. The number of this line on the vernier from the zero-graduation is the number of "smallest-reading" units to be added to the reading made on the limb.

These rules apply to all verniers, whether linear or circular.

THE DECLINATION OF THE NEEDLE.

28. **The Declination* of the Needle** is the horizontal angle it makes with the true meridian. At no place on the earth is this angle a constant. The *change* in this angle is called the *variation* of the declination.

29. **The Daily Variation** in the Declination consists in a

* Formerly called *variation* of the needle, and still so called by navigators and by many surveyors.

swinging of the needle through an arc of about eight minutes daily, the north end having its extreme easterly variation about 8 A.M. and its extreme westerly position about 1.30 P.M. It has its ***mean*** or ***true*** declination about 10.30 A.M. and 8 P.M. It varies with the latitude and with the season, but the following table gives a fair average for the United States. A more extended table may be found in the Report of the U. S. Coast and Geodetic Survey for 1881, Appendix 8.

TABLE OF CORRECTIONS TO REDUCE OBSERVED BEARINGS TO THE DAILY MEAN.

MONTH.	Add to N.E. and S.W. bearings. Subtract from N.W. and S.E. bearings.					Add to N.W. and S.E. bearings. Subtract from N.E. and S.W. bearings.							
	6 A.M.	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 M.	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.
January	1′	1′	2′	2′	1′	0′	2′	3′	3′	2′	1′	1′	0′
April	3	4	4	3	1	1	4	5	5	4	3	2	1
July................	4	5	5	4	1	1	4	5	5	4	3	2	1
October.	1	2	2	2	1	1	3	3	3	2	1	0	0

This table is correct to the nearest minute for Philadelphia, where the observations were made.

30. **The Secular Variation** of the magnetic declination is probably of a periodic character, requiring two or three centuries to complete a cycle. The most extensive set of observations bearing on this subject have been made at Paris, where records of the magnetic declination have been kept for about three and a half centuries. The secular variation for Paris is shown in Fig. 7, and that for Baltimore, Md., in Fig. 8.*

Whether or not either of these curves will return in time to the same extreme limits here given is unknown, as is also the cause of these remarkable changes. The extraordinary variation in the declination at Paris of some 32°, and that at

* These taken from the Coast Survey Report of 1882.

Baltimore of some 5°, show the necessity of paying careful attention to this matter. No reliance should be placed on

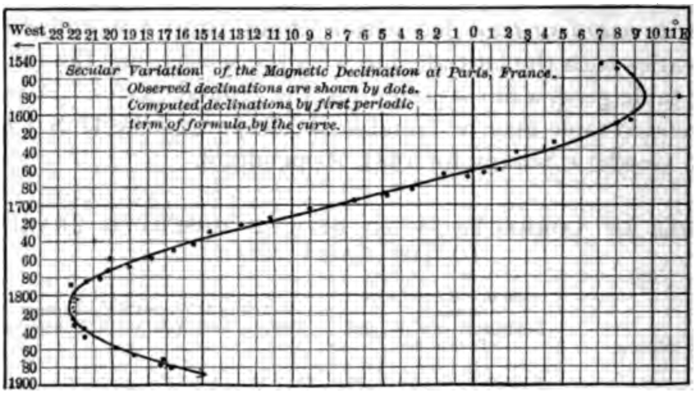

Fig. 7.

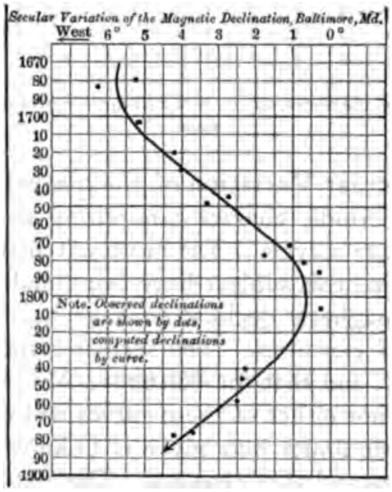

Fig. 8.

old determinations of the declination unless the ***rate of change*** be known, and even then this rate is not likely to be constant a great many years. They also show the necessity of recording the date and the declination of the needle on all plats and records of surveys, with a note stating whether the bearings given were the true or magnetic bearings at the time they were taken.

31. Isogonic Lines are imaginary lines on the earth's surface joining points whose declinations are equal at any given time. The isogonic line joining points having no declination is called the *agonic line*. There is such a line crossing the United States passing just east of Charleston, S. C., and just west of Detroit, Mich. All points east of this line have a western declination, and all points west of it have an eastern declination. The isogonic lines for 1890 for the whole of the United States are shown on Plate I.* It will be noted that where the observations are most thickly distributed, as in Missouri for instance, there the isogonic lines are most crooked; showing that if the declinations were accurately known for all points of this map the isogonic lines would be much more irregular, and would be changed very much in position in many places.

The isogonic lines given on this chart are all moving westward, so that all western declinations are increasing and all eastern declinations are decreasing. They are not all moving at the same rate, however, those in New Brunswick and also those near the eastern boundaries of California and Oregon being about stationary. For many points in the United States and Canada the rate of change in the declination has been observed, and formulæ determined for computing the declination for each point, which formulæ will probably remain good for the next twenty years. The following tables † give this information. In these tables t is the time in calendar years. Thus for July 1, 1885, $t=1885.5$. In the first table all the formulæ have been re-

* Reduced from the U. S. Coast and Geodetic Survey Charts.

† Taken from the U. S. C. and G. Survey Report for 1886.

ferred to one date—Jan. 1, 1850. Here m is used to represent the time in years after 1850, or $m = t - 1850$. Thus, for July 1, 1885, $m = 35.5$.

It will be seen that the change in the declination over the Northern States will average about ***one minute to the mile*** in an east and west direction. A value of the declination found in one end of a county may be some ***forty minutes*** in error in the other end of the same county. This shows that the declination must be known for the exact locality of the survey. In fact, the surveyor can never be sure of his declination until he has observed it for himself for the given time and place. This is best done by means of a transit instrument, and such a method is given in the chapter on Geodetic Surveying. If, however, no transit is at hand, a result sufficiently accurate for compass surveying may be obtained by the compass itself.

AMERICA.

Name of Station.	Latitude.	West Longitude.	The magnetic declination expressed as a function of time in years after 1850 ($m = t - 1850$).
	° ′	° ′	° ° °
Saint John's, Newfoundland	47 34.4	52 41.9	$D = +20.58 + 10.60 \sin(1.4\ m + 62.5)$
Charlottetown, Prince Edward Island	46 14	63 27	$D = +15.95 + 7.78 \sin(1.2\ m + 49.8)$
Halifax, Nova Scotia	44 39.6	63 35.3	$D = +16.21 + 4.34 \sin(1.0\ m + 47.2)$
Quebec, Canada	46 48.4	71 14.5	$D = +14.66 + 3.03 \sin(1.4\ m + 4.6)$
			$+ 0.61 \sin(4.0\ m + 0.3)$
Montreal, Canada	45 30.5	73 34.6	$D = +11.88 + 4.17 \sin(1.5\ m - 18.5)$
			$+ 0.36 \sin(4.9\ m + 19)$
Eastport, Me	44 54.4	66 59.2	$D = +15.14 + 3.90 \sin(1.2\ m + 31.7)$
Portland, Me	43 38.8	70 16.6	$D = +11.26 + 3.16 \sin(1.33\ m + 5.8)$
Burlington, Vt	44 28.5	73 12.0	$D = +10.81 + 3.65 \sin(1.30\ m - 20.5)$
			$+ 0.18 \sin(7.0\ m + 132)$
Hanover, N. H	43 42.3	72 17.1	$D = +9.80 + 4.02 \sin(1.4\ m - 14.1)$
Chesterfield, N. H	42 53.5	72 24	$D = +9.60 + 3.84 \sin(1.35\ m - 16.1)$*
Rutland, Vt	43 36.5	72 55.5	$D = +10.03 + 3.82 \sin(1.5\ m - 24.3)$
Portsmouth N. H	43 04.3	70 42.5	$D = +10.71 + 3.36 \sin(1.44\ m - 7.4)$
Newburyport, Mass	42 48.9	70 49.2	$D = +10.31 + 3.44 \sin(1.4\ m - 4.6)$
Salem, Mass	42 31.9	70 52.5	$D = +9.72 + 3.82 \sin(1.5\ m - 0.7)$*
Boston, Mass	42 21.5	71 03.9	$D = +9.48 + 2.94 \sin(1.3\ m + 3.7)$
Cambridge, Mass	42 22.9	71 07.7	$D = +9.54 + 2.69 \sin(1.30\ m + 7.0)$
			$+ 0.18 \sin(3.2\ m + 44)$
Provincetown, Mass	42 03.1	70 11.3	$D = +9.45 + 3.17 \sin(1.3\ m + 12.8)$*
Nantucket, Mass	41 17.0	70 06.0	$D = +8.61 + 2.83 \sin(1.35\ m + 19.7)$
Providence, R. I	41 50.2	71 23.8	$D = +9.10 + 2.99 \sin(1.45\ m - 3.4)$
			$+ 0.26 \sin(7\ m + 84)$
Williamstown, Mass	42 42.8	73 13.4	$D = +8.84 + 3.13 \sin(1.4\ m - 14.0)$*
Hartford, Conn	41 45.9	72 40.4	$D = +8.06 + 2.90 \sin(1.25\ m - 26.4)$

* Approximate value.

FORMULÆ EXPRESSING THE MAGNETIC DECLINATION AT VARIOUS PLACES IN NORTH AMERICA.—*Continued.*

Name of Station.	Latitude.	West Longitude.	The magnetic declination expressed as a function of time in years after 1850 ($m = t - 1850$).
	° ′	° ′	° ° °
New Haven, Conn.	41 18.5	72 55.7	$D = +\ 7.78 + 3.11 \sin(1.40\,m - 22.1)$
Albany, N. Y.	42 39.2	73 45.8	$D = +\ 8.17 + 3.02 \sin(1.44\,m - 8.3)$
Oxford, N. Y.	42 26.5	75 40.5	$D = +\ 6.19 + 3.24 \sin(1.35\,m - 18.9)$
New York City, N. Y.	40 42.7	74 00.4	$D = +\ 6.61 + 2.40 \sin(1.54\,m - 9.4)$
			$+\ 0.14 \sin(6.3\,m + 64)$
Bethlehem, Pa.	40 36.4	75 23.0	$D = +\ 5.40 + 3.13 \sin(1.55\,m - 38.3)$
Hatborough, Pa.	40 12	75 07	$D = +\ 5.17 + 3.16 \sin(1.54\,m - 16.7)$
			$+\ 0.22 \sin(4.1\,m + 157)$
Philadelphia, Pa.	39 56.9	75 09.0	$D = +\ 5.36 + 3.17 \sin(1.50\,m - 26.1)$
			$+\ 0.19 \sin(4.0\,m + 146)$
Harrisburg, Pa.	40 15.9	76 52.9	$D = +\ 2.93 + 2.98 \sin(1.50\,m + 0.2)$
Huntingdon, Pa.	40 31	78 02	$D = +\ 2.50 + 2.71 \sin(1.5\,m - 4.7)$*
Chambersburg, Pa.	39 55	77 40	$D = +\ 2.79 + 3.10 \sin(1.55\,m - 30.6)$
			$+\ 0.15 \sin(4.6\,m + 124)$
Baltimore, Md.	39 17.8	76 37.0	$D = +\ 3.20 + 2.57 \sin(1.45\,m - 21.2)$
Washington, D. C.	38 53.3	77 00.6	$D = +\ 2.49 + 2.45 \sin(1.45\,m - 15.5)$
			$+\ 0.14 \sin(10\,m + 86)$
Cape Henlopen, Del.	38 46.7	75 05.0	$D = +\ 3.72 + 2.88 \sin(1.4\,m - 21.7)$
Williamsburg, Va.	37 16.2	76 42.4	$D = +\ 2.33 + 2.56 \sin(1.5\,m - 38.1)$
Cape Henry, Va.	36 55.6	76 00.4	$D = +\ 2.48 + 2.22 \sin(1.5\,m - 33.6)$
New Berne, N. C.	35 06	77 02	$D = +\ 0.60 + 2.64 \sin(1.5\,m - 15.4)$*
Charleston, S. C.	32 46.6	79 55.8	$D = -\ 2.14 + 2.77 \sin(1.40\,m - 3.1)$
Savannah, Ga.	32 04.9	81 05.5	$D = -\ 2.92 + 2.23 \sin(1.4\,m - 20.0)$*
Milledgeville, Ga.	33 04.2	83 12	$D = -\ 3.14 + 2.45 \sin(1.5\,m - 39.5)$*
York Factory, British North America	56 59.9	92 26	$D = +\ 4.43 + 14.11 \sin(1.4\,m - 95.6)$
Sault Ste. Marie, Mich.	46 29.9	84 20.1	$D = +\ 1.54 + 2.70 \sin(1.45\,m - 58.5)$

* Approximate value.

FORMULÆ EXPRESSING THE MAGNETIC DECLINATION AT VARIOUS PLACES IN NORTH AMERICA.—*Continued.*

Name of Station.	Latitude.	West Longitude.	The magnetic declination expressed as a function of time in years after 1850 ($m = t - 1850$).
	° ′	° ′	° ° °
Duluth, Minn., and	46 45.5	92 04.5	$D = -7.70 + 2.41 \sin(1.4\ m - 120.0)$*
Superior City, Wis.	46 39.9	92 04.2	
Pierrepont Manor, N. Y.	43 44.5	76 03.0	$D = +5.95 + 3.78 \sin(1.4\ m - 22.2)$
Toronto, Canada	43 39.4	79 23.5	$D = +3.60 + 2.82 \sin(1.4\ m - 44.7)$
			$+ 0.09 \sin(9.3\ m + 136)$
			$+ 0.08 \sin(19\ m + 247)$
Grand Haven, Mich.	43 05.2	86 12.6	$D = -4.95 + 0.0380\ m + 0.00120\ m^2$
Buffalo, N. Y.	42 52.8	78 53.5	$D = +3.66 + 3.47 \sin(1.4\ m - 27.8)$
Detroit, Mich.	42 20.0	83 03.0	$D = -0.97 + 2.21 \sin(1.5\ m - 15.3)$
Erie, Pa.	42 07.8	80 05.4	$D = +2.17 + 2.69 \sin(1.5\ m - 27.3)$
Chicago, Ill.	41 50.0	87 36.7	$D = -6.00 + 0.0274\ m + 0.00074\ m^2$*
Cleveland, Ohio	41 30.3	81 42.0	$D = +0.10 + 2.07 \sin(1.4\ m - 6.2)$
Omaha, Nebr.	41 15.7	95 56.5	$D = -11.62 + 0.0406\ m + 0.00020\ m^2$*
Pittsburgh, Pa	40 27.6	80 00.8	$D = +1.85 + 2.45 \sin(1.45\ m - 28.4)$
Marietta, Ohio	39 25	81 28	$D = +0.02 + 2.89 \sin(1.4\ m - 40.5)$
Cincinnati, Ohio	39 08.6	84 25.3	$D = -2.40 + 2.62 \sin(1.42\ m - 39.8)$
St. Louis, Mo.	38 38.0	90 12.2	$D = -6.04 + 2.60 \sin(1.4\ m - 48.0)$*
Florence, Ala.	34 47.2	87 41.5	$D = -4.25 + 2.33 \sin(1.3\ m - 52.8)$
Mobile, Ala.	30 41.4	88 02.5	$D = -4.38 + 2.69 \sin(1.45\ m - 76.4)$
New Orleans, La.	29 57.2	90 03.9	$D = -5.61 + 2.57 \sin(1.4\ m - 61.9)$
San Antonio, Tex.	29 25.4	98 29.3	$D = -7.40 + 2.88 \sin(1.35\ m - 81.8)$*
Key West, Fla.	24 33.5	81 48.5	$D = -3.70 + 3.16 \sin(1.35\ m - 35.1)$
Havana, Cuba	23 09.3	82 21.5	$D = -4.32 + 2.45 \sin(1.3\ m - 20.8)$*
Kingston, Port Royal, Jamaica	17 55.9	76 50.6	$D = -4.48 + 2.28 \sin(1.1\ m + 10.1)$
Panama, New Granada	8 57.1	79 32.2	$D = -5.67 + 2.16 \sin(1.1\ m - 27.4)$
Acapulco, Mexico	16 50.5	99 52.3	$D = -4.48 + 4.41 \sin(1.0\ m - 85.7)$*

* Approximate value.

FORMULÆ EXPRESSING THE MAGNETIC DECLINATION AT VARIOUS PLACES IN NORTH AMERICA.—*Continued.*

Name of Station.	Latitude.	West Longitude.	The magnetic declination expressed as a function of time in years after 1850 ($m = t - 1850$).
	° ′	° ′	° ° °
Vera Cruz, Mexico	19 11.9	96 08.8	$D = -5.09 + 4.22 \sin(1.2\ m - 63.4)$*
City of Mexico, Mexico	19 26.0	99 11.6	$D = -5.34 + 3.28 \sin(1.0\ m - 87.9)$*
San Blas, Mexico	21 32.5	105 18.4	$D = -6.22 + 3.20 \sin(1.25\ m - 103.2)$
El Paso, Tex.	31 46	106 30	$D = -9.10 + 3.40 \sin(1.3\ m - 108.4)$*
Magdalena Bay, Lower California	24 38.4	112 08.9	$D = -8.87 + 2.20 \sin(1.25\ m - 163.7)$*
San Diego, Cal.	32 42.1	117 14.3	$D = -11.78 + 1.90 \sin(1.15\ m - 151.6)$
Santa Barbara, Cal.	34 24.2	119 43.0	$D = -12.97 + 2.08 \sin(1.3\ m - 146.3)$
Monterey, Cal.	36 36.1	121 53.6	$D = -13.79 + 2.65 \sin(1.10\ m - 156.4)$
San Francisco, Cal.	37 47.5	122 27.3	$D = -13.94 + 2.65 \sin(1.05\ m - 135.5)$
Salt Lake City, Utah	40 46.1	111 53.8	$D = -12.40 + 4.25 \sin(1.4\ m - 121.6)$*
Cape Mendocino, Cal.	40 26.3	124 24.3	$D = -15.25 + 2.45 \sin(1.10\ m - 128.0)$*
Vancouver, Wash.	45 37.5	122 39.7	$D = -17.93 + 3.12 \sin(1.35\ m - 134.1)$*
Wallula, Wash.	46 05	118 55	$D = -17.00 + 3.08 \sin(1.3\ m - 118.1)$*
Cape Disappointment, Wash.	46 16.7	124 02.8	$D = -19.39 + 2.54 \sin(1.25\ m - 158.7)$
Seattle, Wash.	47 35.9	122 20.0	$D = -19.15 + 3.14 \sin(1.4\ m - 136.5)$*
Port Townsend, Wash.	48 07.0	122 44.9	$D = -18.84 + 3.00 \sin(1.45\ m - 122.1)$
Nee-ah Bay, Wash.	48 21.8	124 38.0	$D = -19.83 + 2.91 \sin(1.40\ m - 141.6)$
Nootka, Vancouver Island	49 35.5	126 37.5	$D = -21.25 + 2.74 \sin(1.30\ m - 152.0)$*
Sitka, Alaska	57 02.9	135 19.7	$D = -25.79 + 3.30 \sin(1.30\ m - 104.2)$
Port Mulgrave, Alaska	59 33.7	139 45.9	$D = -24.03 + 7.77 \sin(1.30\ m - 85.8)$
Port Etches, Alaska	60 20.7	146 37.6	$D = -23.71 + 7.89 \sin(1.35\ m - 80.9)$
St. Paul, Kadiak Island	57 48.0	152 21.3	$D = -22.21 + 5.18 \sin(1.35\ m - 72.5)$
Port Clarence, Alaska	65 16	166 50	$D = -18.98 + 7.99 \sin(1.3\ m - 68.4)$*
Chamisso Island, Alaska	66 13	161 49	$D = -23.62 + 7.64 \sin(1.3\ m - 64.0)$*

* Approximate value.

32. Other Variations of the Declination.—In addition to the daily and secular changes in the declination, there are others worthy of mention.

The annual variation is very small, being only about a half-minute of arc from the mean position for the year. It may therefore be neglected.

The lunar inequalities are still smaller, being only about fifteen seconds of arc from the mean position.

Magnetic disturbances are due to what are called magnetic storms. They may occur at any time, and cannot be predicted. They may last a few hours, or even several days. "The following table of the observed disturbances, in a bi-hourly series, at Philadelphia, in the years 1840 to 1845, will give an idea of their relative frequency and magnitude:

Deviations from normal direction.	Number of disturbances.
3′.6 to 10′.8	2189
10′.8 to 18′.1	147
18′.1 to 25′.3	18
25′.3 to 32′.6	3
Beyond......	0

"At Madison, Wis., where the horizontal magnetic intensity is considerably less, very much larger deflections have been noticed. Thus, on October 12, 1877, one of 48′, and on May 28, 1877, one of 1° 24′, were observed." *

The geometric axis of a needle may not coincide with its *magnetic axis*, and hence the readings of two instruments at the same station may differ slightly when both are in adjustment. In this case the declination should be found for each instrument independently.

33. To Find the Declination of the Needle.—The

* From Report of the U. S. Coast and Geodetic Survey for 1882.

method here given is by means of the compass and a plumb-line, and is sufficiently accurate for compass-work. The compass-sights are brought into line with the plumb-line and the pole-star (Polaris), when this is at either eastern or western elongation. This star appears to revolve in an orbit of 1° 18′ radius. Its upper and lower positions are called its upper and lower culminations, and its extreme east and west positions are called its eastern and western elongations, respectively. When it is at elongation it ceases to have a lateral component of motion, and moves vertically upward at eastern and downward at western elongation. If the star be observed at elongation, therefore, the observer's watch may be as much as ten or fifteen minutes in error, without its making any appreciable error in the result. The method of making the observation is as follows:

Suspend a fine plumb-line, such as an ordinary fishing-line, by a heavy weight swinging freely in a vessel of water. The line should be suspended from a rigid point some fifteen or twenty feet from the ground. Care must be taken to see that the line does not stretch so as to allow the weight to touch the bottom of the vessel. Just south of this line set two stakes in the ground in an east and west direction, leaving their tops at an elevation of four or five feet. Nail to these stakes a board on which the compass is to rest. The top of this board should be smooth and level. This compass-support should be as far south of the plumb-line as possible, to enable the pole-star to be seen below the line-support. A sort of wooden box may be provided, in which the compass is rigidly fitted and levelled. Several hundred feet of nearly level ground should be open to the northward, for setting the azimuth-stake. Prepare two stakes, tacks, and lanterns. Find from the table given on page 32 the time of elongation of the star. About twenty minutes before this time, set the compass upon the board, bringing both sights in the plane defined by the plumb-line and star. The line must be illuminated. The star will be found to move slowly east or west, according as it is approaching its eastern or western elongation. When it ceases to move laterally, the compass is carefully levelled, the rear compass-sight brought into the plane of the line and star, and then the forward compass-sight made to coincide with the rear sight and plumb-line. (If the forward sight were tall enough, we could at once bring *both* slits into coincidence with line and star.) Continue to examine rear sight, line, and star, and rear sight, forward sight, and line alternately, until all are found to be in perfect coincidence, the instrument still being level. If this is completed within fifteen minutes of the true local time of elongation, the observation may be considered good; and if it is completed within thirty minutes of the time of elongation, the resulting error in azimuth will be less than one minute of arc. Having completed these observations, remove the plumb-line and set a stake in the line of sight as given by the compass, several hundred feet away. In the top of this stake a tack is to be set exactly on line. For setting this tack, a board may be used, having a vertical slit about $\frac{1}{4}$ inch wide, covered with white cloth or paper, behind which a lamp is held. This slit can then be accurately aligned and the tack set. A small stake with tack is now set just under the compass (or plumb-line), and the work is complete for the night. *Great care must be taken not to disturb the compass after its final setting on the line and star.*

At about ten o'clock on the following day, mount the compass over the south stake. From the north stake lay off a line at right angles to the line joining the two stakes (by compass, optical square, or otherwise) towards the west if eastern elongation, or towards the east if western elongation had been observed. Carefully measure the distance between the two stakes by some standardized unit. From the table of azimuths on page 33 find the azimuth of the star at elongation for the given time and latitude. Multiply the tangent of this angle by the measured distances between the stakes, and carefully lay it off from the north tack, setting a stake and tack. This is now in the meridian with the south point. With the compass in good adjustment, especially as to the bubbles and the verticality of the sights, the observation for declination may now be made. If this be done at about 10.30 A.M., it will give the mean daily declination. Many readings should be made, allowing the needle to settle independently each time. The fractional part of a division on the graduated limb should be read by the declination-vernier, thus enabling the needle to be set exactly at a graduation-mark. If all parts of this work be well done, it will give the declination as accurately as the flag can be set by means of the open sights.

MEAN LOCAL TIME (ASTRONOMICAL, COUNTING FROM NOON) OF THE ELONGATIONS OF POLARIS.

[The table answers directly for the year 1885, and for latitude + 40°.]

Date.		Eastern Elongation.	Western Elongation.	Date.		Eastern Elongation.	Western Elongation.
Jan.	1	$0^h\ 35^m.3$	$12^h\ 24^m.6$	July	1	$12^h\ 39^m.6$	$0^h\ 32^m.8$
"	15	23 36 .1	11 29 .3	"	15	11 44 .7	23 34 .0
Feb.	1	22 29 .0	10 22 .2	Aug.	1	10 38 .2	22 27 .5
"	15	21 33 .7	9 27 .0	"	15	9 43 .3	21 32 .6
Mar.	1	20 38 .5	8 31 .8	Sept.	1	8 36 .7	20 26 .0
"	15	19 43 .4	7 36 .6	"	15	7 41 .7	19 31 .1
Apr.	1	18 36 .4	6 29 .7	Oct.	1	6 38 .9	18 28 .2
"	15	17 41 .4	5 34 .7	"	15	5 43 .9	17 33 .2
May	1	16 38 .6	4 31 .8	Nov.	1	4 37 .0	16 26 .4
"	15	15 43 .7	3 36 .9	"	15	3 41 .9	15 31 .3
June	1	14 37 .1	2 30 .3	Dec.	1	2 38 .9	14 28 .2
"	15	13 42 .2	1 35 .4	"	15	1 43 .6	13 33 .0

I.—AZIMUTHS OF POLARIS WHEN AT ELONGATION FOR ANY YEAR BETWEEN 1895 AND 1910, AND FOR ANY LATITUDE BETWEEN +25° AND +50°.

Lat.	1895.0	1896.0	1897.0	1898.0	1899.0	1900.0	1901.0	1902.0	1903.0	1904.0	1905.0	1906.0	1907.0	1908.0	1909.0	1910.0
°	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
+25	1 22.9	1 22.6	1 22.2	1 21.9	1 21.5	1 21.2	1 20.8	1 20.5	1 20.1	1 19.8	1 19.4	1 19.1	1 18.7	1 18.4	1 18.1	1 17.7
26	23.6	23.2	22.9	22.5	22.2	21.8	21.5	21.1	20.8	20.5	20.1	19.8	19.4	19.1	18.7	18.4
27	24.3	24.0	23.6	23.3	22.9	22.5	22.2	21.9	21.5	21.2	20.8	20.5	20.1	19.8	19.4	19.1
28	25.1	24.7	24.4	24.0	23.7	23.3	23.0	22.6	22.2	21.9	21.6	21.3	20.9	20.5	20.1	19.8
29	25.9	25.5	25.2	24.8	24.5	24.1	23.8	23.4	23.0	22.7	22.4	22.1	21.7	21.3	20.9	20.5
30	26.8	26.4	26.0	25.7	25.3	24.9	24.6	24.2	23.9	23.5	23.1	22.8	22.4	22.1	21.7	21.3
31	27.6	27.3	26.9	26.5	26.2	25.8	25.5	25.1	24.7	24.4	24.0	23.6	23.2	22.9	22.5	22.2
32	28.6	28.2	27.8	27.5	27.1	26.7	26.4	26.0	25.6	25.3	24.9	24.5	24.1	23.8	23.4	23.1
33	29.6	29.2	28.8	28.5	28.1	27.7	27.3	27.0	26.6	26.2	25.9	25.5	25.1	24.7	24.3	24.0
34	30.6	30.3	29.9	29.5	29.1	28.7	28.4	28.0	27.6	27.2	26.9	26.5	26.1	25.7	25.3	25.0
35	31.7	31.3	31.0	30.6	30.2	29.8	29.4	29.0	28.7	28.3	27.9	27.5	27.1	26.8	26.4	26.0
36	32.9	32.5	32.1	31.7	31.3	30.9	30.5	30.1	29.8	29.4	29.0	28.6	28.2	27.9	27.5	27.1
37	34.1	33.7	33.3	32.9	32.5	32.1	31.7	31.3	30.9	30.5	30.1	29.7	29.3	29.0	28.6	28.2
38	35.3	34.9	34.5	34.1	33.7	33.3	33.0	32.6	32.2	31.8	31.4	31.0	30.6	30.2	29.8	29.4
39	36.7	36.3	35.9	35.5	35.1	34.7	34.3	33.9	33.5	33.1	32.7	32.3	31.8	31.4	31.0	30.6
40	38.1	37.7	37.3	36.8	36.4	36.0	35.6	35.2	34.8	34.4	34.0	33.6	33.2	32.8	32.4	32.0
41	39.6	39.2	38.8	38.3	37.9	37.5	37.1	36.7	36.2	35.8	35.4	35.0	34.6	34.2	33.8	33.4
42	41.1	40.7	40.3	39.8	39.4	39.0	38.6	38.2	37.7	37.3	36.9	36.5	36.0	35.6	35.2	34.8
43	42.7	42.3	41.9	41.5	41.0	40.6	40.2	39.8	39.3	38.9	38.5	38.1	37.6	37.2	36.8	36.3
44	44.4	44.0	43.6	43.1	42.7	42.3	41.8	41.4	41.0	40.5	40.1	39.7	39.2	38.8	38.4	37.9
45	46.2	45.8	45.4	44.9	44.5	44.0	43.6	43.2	42.7	42.3	41.8	41.4	40.9	40.5	40.1	39.6
46	48.2	47.7	47.3	46.8	46.4	45.9	45.5	45.0	44.6	44.2	43.7	43.2	42.7	42.3	41.9	41.4
47	50.2	49.7	49.3	48.8	48.3	47.9	47.4	46.9	46.5	46.0	45.6	45.1	44.6	44.2	43.7	43.3
48	52.3	51.9	51.4	50.9	50.4	49.9	49.5	49.0	48.6	48.1	47.7	47.2	46.7	46.3	45.8	45.3
49	54.5	54.1	53.6	53.1	52.6	52.1	51.7	51.2	50.7	50.2	49.8	49.3	48.8	48.4	47.9	47.4
50	1 56.9	1 56.4	1 55.9	1 55.4	1 54.9	1 54.5	1 54.0	1 53.5	1 53.0	1 52.5	1 52.0	1 51.5	1 51.0	1 50.6	1 50.1	1 49.6

If the elongation of Polaris does not come at a suitable time for observing for declination, the upper culmination, which occurs $5^h\ 54^m.6$ after the eastern, or the lower culmination, $6^h\ 03^m.4$ after the western elongation, may be chosen. The objection to this is that the star is then moving at its most rapid rate in azimuth. It is so near the pole, however, that if the observation can be obtained within two minutes of the time of its culmination the resulting error will be less than 1′ of arc. This will then give the true meridian without having to make offsets.*

It must be remembered that the time of elongation given in the table is the *local* time at the place of observation. Inasmuch as *hourly meridian* time is now carried at most points in this country to the complete exclusion of local time, it will be necessary to find the local time from the known meridian or watch time. Thus, all points in the United States east of Pittsburgh use the fifth-hour meridian time (75° w. of Greenwich); from Pittsburgh to Denver, the sixth-hour meridian time (90° w. of Greenwich), etc. To find local time, therefore, the longitude east or west of the given meridian must be found. This can be determined with sufficient accuracy from a map. Thus, if the longitude of the place is 80° w. from Greenwich, it is 5° w. of the fifth-hour meridian, or local time is twenty minutes slower than meridian time at that place If meridian time is used at such a place, the elongation will occur twenty minutes later than given by the table. If the longitude from Washington is given on the map consulted, add it to 77° if west of Washington, and subtract it from 77° if east of Washington, to get longitude from Greenwich.

USE OF THE NEEDLE-COMPASS.

34. The Use of the Needle-compass is confined almost

* For finding the azimuth of Polaris at *any hour*, and hence for making observations for declination of needle at any time, see Art. 381*a* in chapter on Geodetic Surveying.

exclusively to land-surveying, where an error of one in three hundred could be allowed. As the land enhances in value, however, there is an increasing demand for more accurate means of determining areas than the compass and chain afford. The original U. S. land-surveys were all made with the needle, or with the solar, compass and Gunter's chain. Hence all land boundaries in this country have their directions given in compass-bearings, and their lengths in chains of sixty-six feet each.

The compass is used, therefore,—

1. To establish a line of a given bearing.
2. To determine the bearing of an established line.
3. To retrace old lines.

If the true bearing is to be used, the declination of the needle from the meridian must be determined and set off by the vernier.

If the magnetic bearing is used, the declination of the needle at the time the survey was made should be recorded on the plat.

If old lines are to be retraced, the declinations at the times of both surveys must be known.

The needle should be read to the nearest five minutes. This requires reading to sixths of the half-degree spaces, but this can be done with a little practice.

Always lift the needle from the pivot before moving the instrument.

If the needle is sluggish in its movements and settles quickly it has either lost its magnetic force or it has a blunt pivot. In either case it is likely to settle considerably out of its true position. The longer a needle is in settling the more accurate will be its final position. It can be quickly brought very near its true position by checking its motion by means of the lifting screw. In its final settlement, however, it must be left free.

Careful attention to the instrumental adjustments, to local disturbances, and close reading of the needle are all essential to good results with the compass.

35. To set off the Declination, we have only to remember that the declination arc is attached to the line of sight and that the vernier is attached to the graduated circle. If the declination is west, then when the line of sight is north the north end of the needle points to the left of the zero of the graduated circle. In order that it may read zero, or north, the circle must be moved towards the left, or opposite to the hands of a watch. On the other hand, if the declination is east, the circle to which the vernier is attached should be moved with the hands of a watch. This at once enables the observer to set the vernier so that the needle-readings will be the true bearings of the line of sight.

36. Local Attractions may disturb the needle by large or small amounts, and these often come from unknown causes. The observer should have them constantly in mind, and keep all iron bodies at a distance from the instrument when the needle is being read. The glass cover may become electrified from friction, and attract the needle. This can be discharged by touching it with a wet finger, or by breathing upon it. Reading-glasses should not have gutta-percha frames, as these become highly electrified by wiping the lens, and will attract the needle. Such glasses should have brass or German-silver frames. No nickel coverings or ornaments should be near, as this metal has magnetic properties. A steel band in a hat-brim, or buttons containing iron, have been known to cause great disturbance. In cities and towns it is practically impossible to get away from the influence of some local attraction, such as iron or gas pipes in the ground, iron lamp-posts, fences, building-fronts, etc. For this reason the needle should never be used in such places.

In many regions, also, there are large magnetic iron-ore deposits in the ground, which give special values for the declination at each consecutive station occupied. It is practically impossible to use magnetic bearings in such localities.

The test for local attraction in the field-work is to read the bearing of every line from both ends of it. If these are not the same, and no error has been made, there is some local disturbance at one station not found at the other. If there is known to be mineral deposits in the region it may perhaps be laid to that. If not, the preceding station should be occupied again, and the cause of the discrepancy inquired into. If the forward and reverse bearings of all lines agree except the bearings taken from a single station, then it may be assumed there is local attraction at that station.

ELIMINATION OF LOCAL ATTRACTIONS.

37. **To establish a Line of a Given Bearing,** set the compass up at a point on the line, turn off the declination on the declination-arc, and bring the north end of the needle to the given bearing. The line of sight now coincides with the required line, and other points can be set.

38. **To find the True Bearing of a Line,** set the compass up on the line, turn off the declination by the vernier, bring the line of sight to coincide with the line with the south part of the graduated circle towards the observer, and read the north end of the needle. This gives the forward bearing of the line.

39. **To retrace an Old Line,** set the compass over one well-determined point in the line and turn the line of sight upon another such point. Read the north end of the needle. If this reading is not the bearing as given for the line, move the vernier until the north end of the needle comes to the given bearing, when the sights are on line. The reading of the declination-arc will now give the declination to be used in retracing all the other lines of the same survey. If a second well-determined point cannot be seen from the instrument-station, a trial-line will have to be run on an assumed value for the declination, and then the value of the declination used on the first survey computed. Thus, if the trial-line, of length l, comes out a distance x to the *right* of the known point on

the line, the vernier is to be moved *in the direction* of the hands of a watch an angular amount whose tangent is $\frac{x}{l}$. If the trial-line comes out to the *left* of the point, move the vernier in a direction *opposite* to the hands of a watch.

PRISMATIC COMPASS.

40. **The Prismatic Compass** is a hand-instrument provided with a glass prism so adjusted that the needle can be read while taking the sight. A convenient form is shown in Fig. 9, which is carried in the pocket as a watch. The line of

FIG. 9.

sight is established by means of the etched line on the glass cover *S*. It is used in preliminary and reconnoissance work, in clearing out lines, etc.

EXERCISES FOR COMPASS ALONE OR FOR COMPASS AND CHAIN.

41. Run out a line of about a mile in length, on somewhat uneven ground, establishing several stations upon it, using a constant compass-bearing. Then

run back *by the reverse bearings* and note how nearly the points coincide with the former ones. The chain need not be used.

42. Select some half dozen points that enclose an area of about forty acres (one quarter mile square) on uneven ground. Let one party make a compass-and-chain survey of it, obtaining bearing and length of each side. Then let other parties take these field-notes and, all starting from a common point, *run out the lines as given by the field-notes*, setting other stakes at all the remaining corners, each party leaving special marks on their own stakes. Let each party plot their own survey and compare errors of closure.

43. Select five points, three of which are free from local attraction, while two consecutive ones are known to be subject to such disturbance. Make the survey, finding length and forward and reverse bearings of every side. Determine what the true bearing of each course is, and plot to obtain the error of closure.

44. Let a number of parties observe for the declination of the needle, using a common point of support for the plumb-line. Let each party set an independent meridian stake in line with the common point. Note the distance of each stake *from the mean position*, and compute the corresponding angular discrepancies. (March and September are favorable months for making these observations, for then Polaris comes to elongation in the early evening.)

The above problems are intended to impress upon the student the relative errors to which his work is subject.

THE SOLAR COMPASS.

45. The Burt Solar Compass essentially consists first, of a polar axis rigidly attached in the same vertical plane with a terrestrial line of sight, the whole turning about a vertical axis. When this plane coincides with the meridian plane, the polar axis is parallel with the axis of the earth. Second, attached to the polar axis, and revolving about it, is a line of collimation making an angle with the polar axis equal to the angular distance of the sun for the given day and hour from the pole. This latter angle is 90° plus or minus the sun's declination, according as the sun is south or north of the equator. The polar axis must therefore make an angle with the horizon equal to the latitude of the place, and the line of collimation must deviate from a perpendicular to this axis by an angular amount equal to, and in the direction of, the sun's declination. With these angles properly set, and the line of collimation turned upon the sun, the vertical plane through the terrestrial line of sight, and the polar axis must lie in the meridian, for

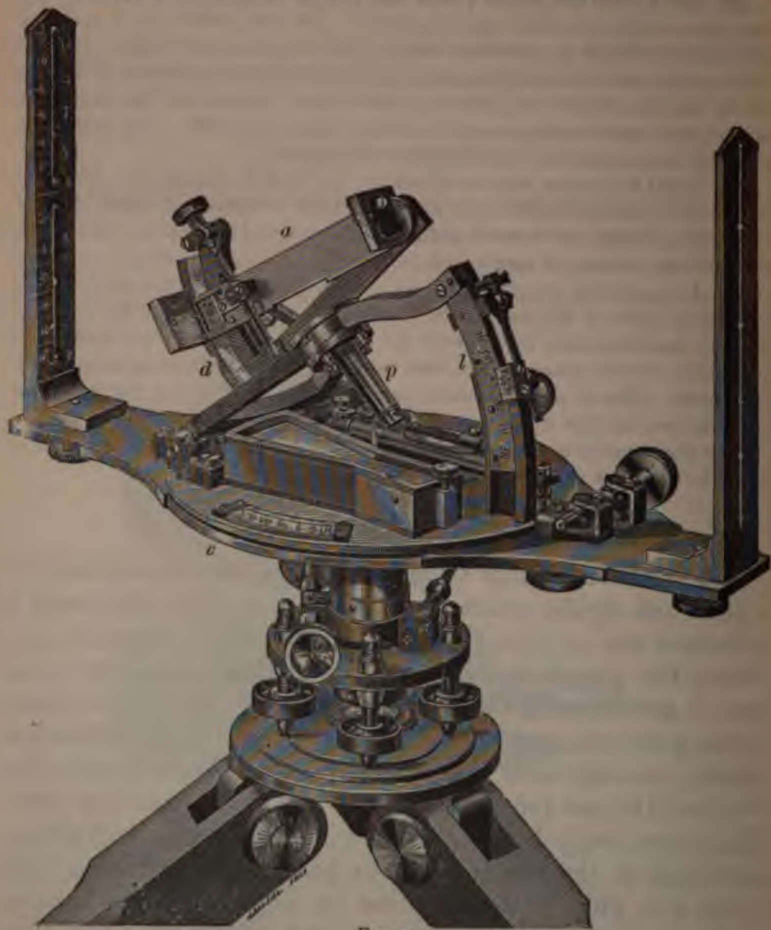

FIG. 10.

otherwise any motion of the line of collimation about its axis would not bring it upon the sun.

In Fig. 10 is shown a cut of this instrument as manufactured by Young & Sons, Philadelphia. The polar axis is shown at *p*, and the terrestrial line of sight is defined by the slits in the vertical sights, the same as in the needle-compass. The line of collimation is defined by a lens at the upper end of the arm *a*, and a silver plate at the lower end, containing graduations with which the image of the sun, as formed by the lens, is made to coincide. The polar axis is given the proper inclination by means of the latitude-arc *l*, and the line of collimation is inclined from a perpendicular to this axis by an amount equal to the sun's declination by means of the declination-arc *d*. When these arcs are properly set, the arm *a* is revolved about the polar axis, and the whole instrument about its vertical axis, until the image of the sun is properly fixed on the lines of the silver plate, when the terrestrial line of sight, as defined by the vertical slits, lies in the true meridian. Any desired bearing may now be turned off by means of the horizontal circle and vernier, shown at *v*. The accuracy with which the meridian is obtained with this instrument depends on the time of day, and on the accuracy with which the latitude- and declination-angles are set off. It is necessary to attend carefully, therefore, to the

ADJUSTMENTS OF THE SOLAR COMPASS.

46. **To make the Plane of the Bubbles perpendicular to the Vertical Axis.**—This is done by reversals about the vertical axis, the same as with the needle-compass.

47. **To adjust the Lines of Collimation.**—The declination-arm *a* has two lines of collimation that should be made parallel. As it is shown in the figure, it is set for a south declination. This is the position it will occupy from Sept. 20 to March 20. When the sun has a north declination, as from March 20 to Sept. 20, the declination-arm is revolved 180° about the polar axis, and a line of collimation established by

a lens and a graduated disk on opposite ends from those previously used. Each end of this arm, therefore, has both a lens and a disk, each set of which establishes a line of collimation. *The second adjustment consists in making these two lines of collimation parallel to each other.* They are made parallel to each other by making both parallel to the faces of the blocks containing the lenses and disks. To effect this, the arm must be detached and laid upon an auxiliary frame which is attached in the place of the arm, and which is called an *adjuster.* With the latitude- and declination-arc set approximately for the given time and place, lay the declination-arm upon the adjuster, and bring the sun's image upon the disk. Now turn the arm carefully bottom side up (not end for end) and see if the sun's image comes between the equatorial lines on the disk.* If not, adjust the disk for one half the displacement, and reverse again for a check. When this disk is adjusted, turn the arm end for end, and adjust the other disk in a similar manner. Having now made both lines of collimation parallel to the edges of the blocks, they are parallel to each other.

48. To make the Declination-arc read Zero when the Line of Collimation is at Right Angles to the Polar Axis.—Set the vernier on the declination-arc to read zero. By any means bring the line of collimation upon the sun. When carefully centred on the disk, revolve the arm 180° quickly about the polar axis, and see if the image now falls exactly on the other disk. If not, move the declination-arm by means of the tangent-screw until the image falls exactly on the disk. Read the declination-arc, loosen the screws in the vernier-plate, and move it back over one half its distance from the zero-reading. Centre the image again, reverse 180°, and test. This adjustment depends on the parallelism of the two lines of collimation. If the vernier-scale is not adjustable,

* It would not be expected to fall between the hour-lines on the disk, since some time has elapsed.

one half the total movement is the index error of the declination-arc, and must be taken into account in all settings on this arc.

The two preceding adjustments should be made near the middle of the day.

49. To adjust the Vernier of the Latitude-arc.—Find the latitude of the place, either from a good map or by a transit-observation. Set up the compass a few minutes before noon, with the true declination set off for the given day and hour. Bring the line of collimation upon the sun, having it clamped in the plane of the sights, or at the twelve-hour angle, and follow it by moving the latitude-arc by means of the tangent-screw, and by turning the instrument on its vertical axis. When the sun has attained its highest altitude, read the latitude-arc. Compare this with the known latitude. Move the vernier on this arc until it reads the true latitude; or, if this cannot be done, the difference is the index error of the latitude-arc. If, however, the latitude used with the instrument be that obtained by it, as above described, then no attention need be paid to this error. This error is only important when the true latitude is used with the instrument in finding the meridian, or where the true latitude of the place is to be found by the instrument. In using the solar compass, therefore, *always use the latitude as given by that instrument by a meridian observation on the sun.**

50. To make the Terrestrial Line of Sight and the Polar Axis lie in the same Vertical Plane.—This should be done by the maker. The vertical plane that is really brought into the meridian by a solar observation is that containing the polar axis, and by as much as the plane of the sights deviates from

* Since the sun may cross the meridian as much as 15 minutes or more before or after mean noon, this observation may have to be taken that much before or after 12 o'clock mean time. It is, however, in all cases, an observation on the sun *at culmination*.

this plane, by so much will all bearings be in error. The best *test* of this adjustment is to establish a true meridian by the transit by observations on a circumpolar star; and then by making many observations on this line, in both forenoon and afternoon, one may determine whether or not the horizontal bearings should have an index-correction applied.

USE OF THE SOLAR COMPASS.

51. **The Solar Compass** is used on land and other surveys where the needle-compass is either too inaccurate, or where, from local attraction, the declination of the needle is too variable to be accurately determined for all points in the survey. Where there is no local attraction, however, and the declination of the needle is well known, the advantages of the solar compass in accuracy are fairly offset by several disadvantages in its use which do not obtain with the needle-compass. Thus, the solar compass should never be used when the sun is less than one hour above the horizon, or less than one hour from noon. Of course it cannot be used in cloudy weather. For such times as these bearings may be obtained by a needle which is always attached, but then the instrument becomes a needle-compass simply. It is also much more trouble, and consumes more time in the field than the needle-compass. But more significant than any of these is the fact that if the adjustments are not carefully attended to, the error in the bearing of a line may be much greater by the solar compass than is likely to be made by the needle-compass, when there is no local attraction. It is possible, however, to do much better work with the solar compass than can be done with the needle-compass.

52. **To find the Declination of the Sun.**—On account of the inclination of the earth's axis to the plane of its orbit, the sun is seen north of the celestial equator in summer, and south

of it in winter. This deviation, north or south of the equator, is called north or south declination, and is measured from any point on the earth's surface in degrees of arc.

On about the 21st of June the sun reaches its most northern declination, and begins slowly to return. Its most southern point is reached about December 21st. In June and December, therefore, the sun is changing its declination most slowly, while at the intervening quadrant-points of the earth's orbit, March and September, it is changing its declination most rapidly, being as much as one minute in arc for one hour in time. It is evident, therefore, that we must regard the declination of the sun as a constantly changing quantity, and, for any given day's work, a table of declinations must be made out for each hour of the day. The American Ephemeris and Nautical Almanac gives the declination of the sun for noon of each day of the year for both Greenwich and Washington. Since the time universally used in this country is so many hours from Greenwich, it is best to use the Greenwich declinations. Since, also, we are five, six, seven, or eight hours west of Greenwich, the declination given in the almanac for Greenwich noon of any day will correspond to the declination here at 7, 6, 5, or 4 o'clock A.M. of the same date, according as Eastern, Central, Mountain, or Western time is used. As this standard time is seldom more than 30 minutes different from local time, and as this could never affect the declination by more than 30 seconds of arc, it will here be considered sufficient to correct the Greenwich declination by the change, as found for the standard time used. Thus, if Central (90th meridian) time is used, the declination given in the almanac is the declination at 6 o'clock A.M. at the place of observation. To this must be added (algebraically) the hourly change in declination, which is also given in the almanac. A table may thus be prepared, giving the declination for each hour of the day.

53. To correct the Declination for Refraction.—All rays of light coming to the earth from exterior bodies are refracted downward, thus causing such bodies to appear higher than they really are. This refraction is zero for normal (vertical) lines, and increases towards the horizon. It varies largely, also, with the special temperature, pressure, and hydrometrical condition of the atmosphere. Tables of refraction give only the mean values, and these may differ largely from the values found for any given time, especially for lines near the horizon. It is for this reason that all astronomical observations made near the horizon are very uncertain. There is but one setting on the solar compass that has reference to the position of the sun in the heavens, and that is the declination. Now, the refraction changes the apparent *altitude* of the body; and by so much as a change in the altitude changes the declination, by so much does the *apparent* declination differ from the *true* declination. Evidently it is the *apparent* declination that should be set off. When the sun is on the meridian, the change in altitude has its full effect in changing the declination, but at other times the change in declination is less than the change in altitude.

The correction to the declination due to refraction is found from the following final equations:*

$$\tan N = \cot \varphi \cos t,$$

$$\tan q = \frac{\sin N}{\cos(\delta + N)} \tan t,$$

$$\tan z = \frac{\cot(\delta + N)}{\cos q},$$

$$d\delta = -dz \cos q,$$

* See Chauvenet's "Spherical Astronomy," vol. i., p. 171, and Doolittle's "Practical Astronomy," p. 159.

where φ = latitude; t = hour angle from the meridian; δ = declination of sun; z = zenith distance of sun; N and q being auxiliary angles to facilitate the computation.

From these equations we may compute the auxiliary angle q, and the zenith distance z, for each hour from noon, for every day of the year. Then from a table of mean refractions, giving the refraction for given altitudes, or zenith distances, which is dz, we may find the corresponding $d\delta$, which is the correction to be applied to the declination for refraction.

In this manner the following table has been computed for the latitude of 40°. For any other latitude the correction is found by multiplying the correction given in the table by the corresponding coefficient, as given in the table "Latitude Coefficients." These coefficients were obtained by plotting the ratios of the actual refraction at the different latitudes to that at latitude 40°, for each hour from 7 A.M. to 5 P.M. and for the various declinations. It was found that this ratio was almost a constant, except for very low altitudes, where the inherent uncertainties of an observation are very large, from the actual refraction varying so largely from the mean, as given in the tables. A mean value of this ratio was chosen, therefore, which enables the corrections at other latitudes to be found in terms of those in latitude 40° without material error. These ratios are given in the Table of Latitude Coefficients.

EXAMPLE.

Let it be required to prepare a table of declination settings for a point whose latitude is 38° 30′, which lies in the "Central Time Belt," and for April 5, 1890.

Since the time is 6 h. earlier than that at Greenwich, the declination given in the Ephemeris for Greenwich mean noon (6° 9′ 57″) is the declination for the given place at 6 A.M. If the point were in the "Eastern Time Belt" it would be the declination at 7 A.M., etc. Suppose it is desired to prepare declination settings from 7 A.M. to 5 P.M. From the table of

TABLE OF REFRACTION CORRECTIONS TO BE APPLIED TO THE DECLINATIONS.

DATE.	REFRACTION CORRECTION. Latitude 40°.	DATE.	REFRACTION CORRECTION. Latitude 40°.	DATE.	REFRACTION CORRECTION. Latitude 40°.	DATE.	REFRACTION CORRECTION. Latitude 40°.
Jan. 1 2 3	*1 h. 1′ 58″ 2 2 16 3 3 04 4 6 23	Feb. 13 14 15 16 17	1 h. 1′ 16″ 2 1 25 3 1 48 4 2 47 5 8 39	Mar. 30 31 April. 1 2 3	1 h. 42″ 2 47 3 57 4 1′ 19 5 2 18	May. 14 15 16 17 18	1 h. 23″ 2 27 3 34 4 49 5 1′ 18
4 5 6 7 8	1 1 54 2 2 11 3 2 59 4 6 01	18 19 20 21 22	1 1 12 2 1 20 3 1 40 4 2 31 5 6 49	4 5 6 7 8	1 39 2 44 3 54 4 1 14 5 2 08	19 20 21 22 23	1 22 2 26 3 33 4 47 5 1 15
9 10 11 12 13	1 1 51 2 2 07 3 2 51 4 5 40	23 24 25 26 27	1 1 07 2 1 15 3 1 33 4 2 18 5 5 29	9 10 11 12 13	1 36 2 41 3 51 4 1 10 5 1 58	24 25 26 27 28	1 21 2 25 3 32 4 46 5 1 13
14 15 16 17 18	1 1 46 2 2 01 3 2 40 4 5 00	28 Mar. 1 2 3 4	1 1 03 2 1 10 3 1 27 4 2 06 5 4 39	14 15 16 17 18	1 34 2 38 3 48 4 1 06 5 1 49	29 30 31 June. 1 2	1 20 2 24 3 31 4 44 5 1 11
19 20 21 22 23	1 1 42 2 1 56 3 2 31 4 4 35	5 6 7 8 9	1 0 59 2 1 06 3 1 21 4 1 56 5 4 04	19 20 21 22 23	1 32 2 36 3 45 4 1 02 5 1 42	3 4 5 6 7	1 19 2 23 3 30 4 43 5 1 10
24 25 26 27 28	1 1 37 2 1 50 3 2 22 4 4 07	10 11 12 13 14	1 55 2 1 02 3 1 15 4 1 47 5 3 34	24 25 26 27 28	1 30 2 34 3 42 4 58 5 1 36	8 9 10 11 12	1 18 2 22 3 29 4 43 5 1 09
29 30 31 Feb. 1 2	1 1 32 2 1 44 3 2 13 4 3 41	15 16 17 18 19	1 52 2 58 3 1 10 4 1 39 5 3 08	29 30 May. 1 2 3	1 28 2 32 3 39 4 55 5 1 30	13 14 15 16 17	1 18 2 22 3 29 4 42 5 1 08
3 4 5 6 7	1 1 26 2 1 37 3 2 04 4 3 21	20 21 22 23 24	1 48 2 54 3 1 05 4 1 32 5 2 51	4 5 6 7 8	1 26 2 30 3 37 4 53 5 1 26	18 19 20 21 22	1 18 2 22 3 29 4 42 5 1 08
8 9 10 11 12	1 1 21 2 1 31 3 1 56 4 3 04	25 26 27 28 29	1 45 2 50 3 1 01 4 1 25 5 2 34	9 10 11 12 13	1 25 2 29 3 36 4 51 5 1 22	23 24 25 26 27	1 18 2 22 3 29 4 42 5 1 08

* The hours are counted each way from noon. Thus 9 A.M. and 3 P.M. would correspond to the 3d hour in the table.

DATE.	REFRACTION CORRECTION. Latitude 40°.	DATE.	REFRACTION CORRECTION. Latitude 40°.	DATE.	REFRACTION CORRECTION. Latitude 40°.	DATE.	REFRACTION CORRECTION. Latitude 40°.
June. 28 29 30 July. 1 2	1 h. 18″ 2 22 3 29 4 43 5 1′ 09	Aug. 17 18 19 20 21	1 h. 32″ 2 36 3 45 4 1′ 02 5 1 42	Oct. 6 7 8 9 10	1 h. 1′ 03″ 2 1 10 3 1 27 4 2 06 5 4 39	Nov. 20 21 22 23 24	1 h. 1′ 46′ 2 2 01 3 2 40 4 4 59
3 4 5 6 7	1 19 2 23 3 30 4 43 5 1 10	22 23 24 25 26	1 34 2 38 3 48 4 1 06 5 1 49	11 12 13 14 15	1 1 07 2 1 15 3 1 33 4 2 18 5 5 29	25 26 27 28 29	1 1 50 2 2 06 3 2 49 4 5 33
8 9 10 11 12	1 20 2 24 3 31 4 44 5 1 11	27 28 29 30 31	1 36 2 41 3 51 4 1 10 5 1 58	16 17 18 19 20	1 1 12 2 1 20 3 1 40 4 2 31 5 6 49	30 Dec. 1 2 3 4	1 1 54 2 2 11 3 2 59 4 6 01
13 14 15 16 17	1 21 2 25 3 32 4 46 5 1 13	Sept. 1 2 3 4 5	1 39 2 44 3 54 4 1 14 5 2 08	21 22 23 24 25	1 1 16 2 1 25 3 1 48 4 2 47 5 8 39	5 6 7 8 9	1 1 58 2 2 16 3 3 04 4 6 23
18 19 20 21 22	1 22 2 26 3 33 4 47 5 1 15	6 7 8 9 10	1 42 2 47 3 57 4 1 19 5 2 18	26 27 28 29 30	1 1 21 2 1 31 3 1 56 4 3 04 5 11 01	10 11 12 13 14	1 2 00 2 2 19 3 3 09 4 6 38
23 24 25 26 27	1 23 2 27 3 34 4 49 5 1 18	11 12 13 14 15	1 48 2 50 3 1 01 4 1 25 5 2 34	31 Nov. 1 2 3 4	1 1 26 2 1 37 3 2 04 4 3 21 5 13 57	15 16 17 18 19	1 2 01 2 2 20 3 3 11 4 6 47
28 29 30 31 Aug. 1	1 25 2 29 3 36 4 51 5 1 22	16 17 18 19 20	1 48 2 54 3 1 05 4 1 32 5 2 51	5 6 7 8 9	1 1 32 2 1 44 3 2 13 4 3 41	20 21 22 23 24	1 2 01 2 2 20 3 3 11 4 6 49
2 3 4 5 6	1 26 2 30 3 37 4 53 5 1 26	21 22 23 24 25	1 52 2 58 3 1 10 4 1 39 5 3 08	10 11 12 13 14	1 1 37 2 1 50 3 2 22 4 4 07	25 26 27 28 29	1 2 00 2 2 19 3 3 09 4 6 13
7 8 9 10 11	1 28 2 32 3 39 4 55 5 1 30	26 27 28 29 30	1 55 2 1 02 3 1 15 4 1 47 5 3 34	15 16 17 18 19	1 1 42 2 1 56 3 2 31 4 4 35	30 31	
12 13 14 15 16	1 30 2 34 3 42 4 58 5 1 36	Oct. 1 2 3 4 5	1 59 2 1 06 3 1 21 4 1 56 5 4 04				

TABLE OF LATITUDE COEFFICIENTS.

Latitude.	Coefficient.	Latitude.	Coefficient.	Latitude.	Coefficient.
15°	.30	30°	.65	45°	1.20
16	.32	31	.68	46	1.24
17	.34	32	.71	47	1.29
18	.36	33	.75	48	1.33
19	.38	34	.78	49	1.38
20	.40	35	.82	50	1.42
21	.42	36	.85	51	1.47
22	.44	37	.89	52	1.53
23	.46	38	.92	53	1.58
24	.48	39	.96	54	1.64
25	.50	40	1.00	55	1.70
26	.53	41	1.04	56	1.76
27	.56	42	1.08	57	1.82
28	.59	43	1.12	58	1.88
29	.62	44	1.16	59	1.94

NOTE.—For any other latitude than 40° the refraction corrections given in the preceding table are to be multiplied by the coefficients given in this table to obtain the true refraction corrections for that latitude.

latitude corrections we find that the refraction corrections will be .94 of those given in the table. The following table of declination settings may now be made out:

Hour.	Declination.	Refr. Cor.	Setting.	Hour.	Declination.	Refr. Cor.	Setting.
7	+ 6° 10′ 54″	+ 2′ 00″	+ 6° 12′ 54″	1	+ 6° 16′ 35″	+ 37″	+ 6° 17′ 12″
8	6 11 51	+ 1 10	6 13 01	2	6 17 31	+ 41	6 18 12
9	6 12 47	+ 51	6 13 38	3	6 18 28	+ 51	6 19 19
10	6 13 44	+ 41	6 14 25	4	6 19 25	+ 1′ 10″	6 20 35
11	6 14 41	+ 37	6 15 18	5	6 20 22	+ 2 00	6 22 22

From March 20th to September 20th the declination is positive, while from September 20th to March 20th it is negative. From December 20th to June 20th the hourly correction is positive, while from June 20th to December 20th it is negative. The refraction correction is always positive. Particular attention must be given to all these signs in making out the table of declination settings.

54. Errors in Azimuth due to Errors in the Declination and Latitude Angles.—The spherical triangle involved in an observation by the solar compass is shown in Fig. 11, where P is the pole, Z the zenith, and S the sun. Then

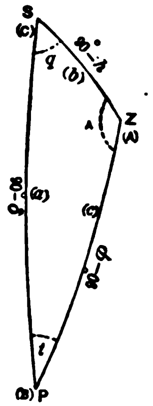

FIG. 11.

the angle at $P = t$, the hour-angle from the meridian;
" " $Z = A$, the azimuth from the north point;
" " $S = q$, the variable or parallactic angle.

Also, the arc $PZ =$ the co-latitude $= 90° - \phi$;
" $PS =$ the co-declination $= 90° - \delta$;
" $ZS =$ the co-altitude, or zenith distance $= 90° - h$.

Taking the parenthetical notation of the figure, we have, from spherical trigonometry,

$$\cos (a) = \cos (c) \cos (b) + \sin (c) \sin (b) \cos (A).$$

But in terms of δ, ϕ, h, and A, this becomes

$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos A. \quad . \quad . \quad . \quad (1)$$

In a similar manner, from two other fundamental equations of the spherical triangle, we may write

$$\cos \delta \cos t = \cos \phi \sin h - \sin \phi \cos h \cos A; \qquad (2)$$
$$\cos \delta \sin t = \cos h \sin A. \qquad (3)$$

If we differentiate equation (1) with reference to A and δ, and then with reference to A and ϕ, we obtain, after some reductions by the aid of equations (2) and (3)

$$*dA_\delta = -\frac{d\,\delta}{\cos\phi\sin t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$dA_\phi = +\frac{d\phi}{\cos\phi\tan t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, if the change (or error) in δ and ϕ be taken as 1 minute of arc, or, in other words, if the settings for declination or latitude be erroneous by that amount, either from errors in the instrumental adjustments or otherwise, then equations (4) and (5) show what is the *error due to this cause* in the azimuth, or in the direction of the meridian, as found. In the following table, values of dA_δ and dA_ϕ are given for various values of ϕ and t (latitude and hour-angle). In this table no attention is paid to signs, as it is intended mainly to show the size of the errors to which the work is liable from inaccurate settings for declination and latitude; the values may, however, be used as corrections to the observed azimuths from such inaccuracies by observing the instructions in the appended note.

* dA_δ signifies the change in A due to a small change, $d\delta$, in δ, the other functions involved in equation (1) remaining constant. Similarly for dA, when ϕ alone changes. The derivation of equations (4) and (5) involves a knowledge of the infinitesimal calculus.

TABLE OF ERRORS IN AZIMUTH (BY SOLAR COMPASS) FOR 1' ERROR IN DECLINATION OR LATITUDE.

Hour.	For 1' Error in Declination.			For 1' Error in Latitude.		
	Lat. 30°	Lat. 40°	Lat. 50°	Lat. 30°	Lat. 40°	Lat. 50°
11.30 A.M. / 12.30 P.M.	8'.85	10'.0	12'.9	8'.77	9'.92	11'.8
11 A.M. / 1 P.M.	4.46	5.05	6.01	4.33	4.87	5.80
10 A.M. / 2 P.M.	2.31	2.61	3.11	2.00	2.26	2.70
9 A.M. / 3 P.M.	1.63	1.85	2.20	1.15	1.30	1.56
8 A.M. / 4 P.M.	1.34	1.51	1.80	0.67	0.75	0.90
7 A.M. / 5 P.M.	1.20	1.35	1.61	0.31	0.35	0.37
6 A.M. / 6 P.M.	1.15	1.30	1.56	0.00	0.00	0.00

NOTE.—Azimuths observed with erroneous declination or co-latitude may be corrected by this table by observing that for the line of collimation set *too high*, the azimuth of any line *from the south point* in the direction S.W.N E. is found *too small* in the *forenoon* and *too large* in the *afternoon* by the tabular amounts for each minute of error in the altitude of the solar line of sight. The reverse is true for this line set too low.

Several important conclusions may be drawn from this table and from equations (4) and (5).

First—That the solar compass should never be used between 11 A.M. and 1 P.M., and preferably not between 10 A.M. and 2 P.M., if the best results are desired.

Second—That at 6 o'clock A.M. and P.M., when the line of collimation lies in a plane at right angles to the plane of the meridian, no small change in the latitude-arc will affect the accuracy of the result.

Third—From equations (4) and (5), we see that the errors from declination and from latitude have opposite signs, or that errors from like erroneous settings in declination and co-latitude have the same sign. Therefore, if the declination-angle be erroneously set off, and the co-latitude-angle be also affected by *an equal error in the opposite direction*, then the two resulting errors in azimuth will tend to compensate. From the table it may be seen that for the same latitude and hour-angle they would nearly balance each other numerically. If, therefore, the declination-angle be affected by an error, ***and the co-latitude of the place then found by a meridian observation with the compass, the error of the declination will appear in the resulting co-latitude, with the opposite sign.*** In this way any constant error in the declination-angle may be nearly eliminated.

Fourth—The best times of day for using the solar compass are from 7 to 10 A.M. and from 2 to 5 P.M. So far as the instrumental errors are concerned, the greater the hour-angle the better the observation; but when the sun is near the horizon, the uncertainties in the refraction may cause unknown errors of considerable size.

Fifth—That for a given error in the setting for declination or latitude the resulting error in azimuth will have opposite signs in forenoon and afternoon. For, in equations (4) and (5), the hour-angle, t, has different signs before and after noon; and therefore *sin* t and *tan* t change sign, thus changing the sign of the expression. If, therefore, a 10-o'clock azimuth is in error 5′ in one direction from erroneous settings, a 2-o'clock observation with the same instrument should give an azimuth 5′ in error in the opposite direction.

55. **Solar Attachments** are appliances fitted upon transit-instruments for the purpose of finding the meridian, the same as is done by the solar compass. The principles of construction and use are the same as those of the solar compass, the application of these principles being quite various, however, giving rise to several forms of attachments, some of which will be discussed in connection with the transit. Their adjustments and limitations are nearly the same as those here given.

EXERCISES WITH THE SOLAR COMPASS.

56. Determine a true meridian line by an observation on a circumpolar star or otherwise, by either the compass or transit. Set the solar compass up on one point of this line with a target set at another point on the established meridian. Having carefully adjusted the compass, set the declination-arc to the right angle for the given day and hour, corrected for refraction, and make a meridian observation on the sun for latitude. If the true latitude of the place is known, the difference will be the index error of the latitude-arc. Leave the latitude-arc set at the readings obtained by the meridian observation (whether it is the true latitude or not), and make a series of determinations of the meridian by the compass at various times of day. These will usually be in error from the true meridian by small amounts. Determine the size of these errors by turning upon the target and reading the horizontal circle. Record these errors, with the time of day and name of observer. Each student should make a series of such observations, determining for himself the errors to which the work is liable. The same meridian may be used for all, after it has been properly checked by duplicate observations.

57. Set the latitude- or declination-angle say 3′ from its true value, and observe at various hours of the day, and see if the resulting errors in azimuth are about three times those given in the table. Note that these resulting errors are in opposite directions and equal in amount in fore- and after-noon observations.

58. With the solar compass on the meridian as before, select a series of points, six or more, which are fixed and plainly visible through the slits. Find the bearing to each of these points by a separate observation on the sun in each case, *paying no attention to the target on the true meridian.* Remove the solar compass and let another student, ignorant of the first bearings, set the ordinary needle-compass over the same point. Bring the line of sight upon the target and make the needle read south by moving the vernier on the declination-arc. In other words, set off the declination of the needle. The bearings given by the needle compass should now agree with those obtained by the solar compass. Read upon the series of selected points, obtaining the bearings to the nearest five minutes. Let a third student take a transit (or the solar compass would do) and find the true bearings of the selected points with reference to the established meridian. Compare results and so obtain some data for determining the relative accuracy of the solar and the needle-compass. The mean of two azimuths by the solar compass taken on the same line at equal intervals from noon should be the true azimuth of the line if the instrument has not changed its adjustments in the mean time. This is the way to find the true azimuth of a line by the solar compass.

59. Run a line over a series of points (six or more) in the forenoon by the solar compass, and determine the bearings. In the afternoon run it back again, *using the bearings obtained in the forenoon*, set other stakes where the points are not coincident with the old ones, and note the residual discrepancy at the close of the work. Divide this by twice the length of the line, and this is the error of closure *due to erroneous bearings*. The chain may be used on the first running of the line, but on the retracing the stakes may be set opposite the first ones, if not coincident. The object is to determine how much of the error of closure in surveying may be attributed to erroneous bearings.

Do the same with the needle compass and compare results. The points need not be in line, nor need they enclose an area.

CHAPTER III.

INSTRUMENTS FOR DETERMINING HORIZONTAL LINES.

PLUMB-LINE AND BUBBLE.

60. The Plumb-line and the Bubble-tube are at once the most simple, universal, and essential of all appliances used in surveying and astronomical work. Without them neither the zenith nor the horizon could be effectually determined, and the determination of altitudes and of horizontal lines and planes would be out of the question. Even azimuths, bearings, and horizontal angles require that the circle by which they are obtained shall be brought into a horizontal position.

The direction of the plumb-line is by definition a vertical line, pointing to the zenith, and a plane at right angles to this line is *for that point* a horizontal plane. As no two plumb-lines can be parallel, so no two planes, respectively horizontal at two different positions on the earth's surface, can be parallel.

Parallel horizontal planes can only be planes at different elevations, all horizontal for a single position on the earth's surface.

A level surface is a surface (not a plane) which is at every point perpendicular to a plumb-line at that point. If the earth were covered with a fluid in a quiescent state, the surface of this fluid would be a level surface. This surface would not be a true oblate spheroid, but would in places vary several hundred feet from such a mean spheroidal surface. This is owing to the fact that the earth is not a homogeneous body, thus causing the centre of mass to deviate from the centre of form. Owing also to much irregularity in the distribution of the mass, with respect to the form of the earth, there are many irregular *deviations of the plumb-line** from any one point. A level surface follows all such deviations.

A bubble-tube is a round glass tube bent or ground so that its inside upper surface is circular on a longitudinal section. This is nearly filled with ether, the remaining space being occupied with ether-vapor, which forms the bubble. This tube is usually graduated to assist in determining the exact position of the bubble in the tube. If the tube has been ground to a perfect circular longitudinal section, then ***a longitudinal line tangent to this inner surface at the centre of the air-bubble is a level line***, in whatever part of the tube the bubble may lie. If this were not a level line, the centre of gravity of the bubble would not occupy its highest possible position and would move until it did. Since the position of the centre of a bubble in a tube is determined by reading the position of its ends and taking the mean, it is necessary that the arc shall be of uniform curvature—that is, circular.

A line tangent to the inner surface of the bubble-tube at its centre, as defined by the graduations (or another line parallel to it) is called *the axis of the bubble*. When the bubble is in the centre of the tube, therefore, its axis is horizontal.

Proposition I. If a bubble-tube be rigidly attached to a frame, and if this frame be reversed on two supports lying in the vertical plane through the bubble-axis, the supporting points are level when the bubble occupies ***the same portion*** of the tube in both positions of frame, whether this be the centre

* In the northern portion of the United States, in the vicinity of the Great Lakes, deviations of the plumb-line (Clarke's Spheroid being used) have been found as great as 10 or 12 seconds of arc. See Primary Triangulation of the U. S. Lake Survey.

or not; providing, of course, that the points of support on the frame were identical in the two cases.

For, the tangent horizontal lines being identical in the two positions of the bubble, the vertical distances from this line to the points of support must be equal, otherwise the direct and reversed positions would not give identical tangent lines. The points of support are therefore in a horizontal line.

Proposition II. If a bubble-tube be revolved about an axis in such a way that *the bubble keeps a constant position in the tube*, the axis of revolution is vertical.

For, since the bubble-tube maintains a constant inclination to the horizon (this inclination being zero when the bubble is in the centre), the plane of motion can have no vertical component, and, therefore, the axis of revolution must be vertical.

Cor. 1. Similarly we may say that if a bubble-tube be revolved 180° about an axis, and if the bubble have the same reading in the two positions, then the plane of revolution has no vertical component in the direction of the bubble-axis, and therefore the axis of revolution lies in a vertical plane at right angles to the bubble-axis. If the same test be made for any other two horizontal positions 180° apart (preferably 90° from first position) and the bubble have the same reading in the two cases, then the axis of revolution lies in a vertical plane at right angles to these new positions of bubble-axis, and therefore it lies at the intersection of these two vertical planes, or it is vertical. If two bubble-tubes (not parallel to each other and preferably at right angles) be rigidly attached to a frame that revolves about an axis, and if each bubble has the *same* reading in two positions of frame 180° apart, the axis of revolution is vertical, even though the two bubbles do not read alike nor either is at the middle of its tube.

Cor. 2. In all cases where a bubble-tube has been shifted 180° in the same supports, or axis, the angular difference between the two positions of the bubble is twice the angular deviation of the supports from a horizontal, or of the axis from a vertical.

61. The Accurate Measurement of Small Vertical Angles is accomplished by means of the bubble with greater readiness and precision than by any other device known. For this purpose the bubble-tube should be ground accurately to the arc of a circle with a long radius, and uniformly graduated. Then a given bubble-movement in any part of the tube corresponds to a known angular change, when the angular value of a movement of one division in the graduated scale has been determined. These graduations are usually made on the top of the glass tube. To measure a small angle by means of the bubble, read the two ends of the bubble to divisions and tenths, and take the one half-difference of end readings.* Shift the bubble a given amount and read both ends again, taking one half the difference. The difference of these two results in divisions of the scale, multiplied by the angular value of one division on the scale, is the vertical angle through which the tube was shifted.

62. The Angular Value of One Division of the Bubble may be found in various ways.

(*a*) *By a telescopic line of sight.* Attach the bubble-tube rigidly to a mounted telescope, putting the bubble-axis in the plane of the telescope. Measure off a convenient base-line on level ground of from 200 to 500 feet. Set the telescope at one end of this base, and hold a rod vertically at the other. Bring the bubble near one end of its tube by moving the telescope vertically, and read the two ends. Read the height of the cross-wires on the rod. Bring the bubble near the other end of tube and read both the bubble and rod. Repeat many times. Reduce the work by taking the half-difference of the two end

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* Bubbles are read from the middle outwards towards the ends. Then the half-difference of end readings is the distance of the centre of the bubble from the centre of the scale.

readings in each case, thus giving the distance of the centre of the bubble from the centre of tube for each position. Take the mean of these results for each set of end readings separately. If these mean results were for opposite ends of the tube, add them together and this gives the *average movement of bubble.* Similarly take the mean of the upper readings and the mean of the lower readings on the rod, and take the difference, and this is the *average movement of the line of sight.* Calling the bubble-movement in divisions of scale D, the movement on the rod, in feet, R, and the length of the base, in feet, B, we would have, in seconds of arc,

$$\text{angular value of 1 div. of bubble} = \frac{R}{BD \sin 1''}.\ *$$

(*b*) *By a large vertical circle.* Mount the bubble rigidly upon the circle, having its axis parallel to the plane of the circle. Move the bubble from end to end of tube, as before, reading the corresponding angular changes directly upon the circle. Divide the mean angular movement by the mean movement of bubble.

This requires a large circle with micrometer attachments, such as is used on astronomical instruments.

(*c*) *By a level trier.* This consists of a beam hinged at one end and moved vertically by means of a micrometer screw at the other. The bubble-tube is placed upon the beam, and the bubble moved back and forth by means of the screw, each revolution of which gives a known angular movement to the beam.

63. **General Considerations.**—A bubble is sensitive directly as the length of the radius of curvature, or indirectly as its rate of curvature. It is also sensitive in proportion to its length, a long bubble† settling much more quickly and ac-

* Log sin 1" =

† This refers to the length of the air-bubble itself, and not to the glass tube.

curately than a short one. Some bubble-tubes have a chamber at one end connected with the main space by a small hole through the bottom of the dividing partition. This enables the length of the bubble to be under control. As ether expands and contracts very largely with temperature, the bubble is apt to be too long in winter and too short in summer if the chamber is not used. The bubble-tube should not be rigidly confined by metallic fastenings about its centre, if the value of one division is significant, as the changes of temperature will change its curvature. Bubble-tubes, or level-vials as they are often called, may be sealed by glass stoppers set in a glue made by dissolving isinglass in hot water, and covering with gold-beater's skin set with the same glue, the whole varnished over when dry.

THE ENGINEER'S LEVEL.

64. **The Engineer's Level** consists of a telescopic horizontal line of sight joined to a spirit-level, the whole properly supported and revolving on a vertical axis. Such an instrument is shown in Fig. 12. The vertical parts of the frame which support the telescope are called wyes, and the cylindrical bearings on the telescope-tube are called the pivot-rings. The telescope can be lifted out of the wyes by loosening the clips over the rings, these being held by the small pins attached to strings and shown in the cut. A clamp and tangent-screw are connected with the axis for holding it to a given pointing or for moving it horizontally while clamped. The attached bubble enables the line of sight in the telescope to be brought into a horizontal position.

The construction of the instrument is best shown by the sectional view given in Fig. 13.

The objective is a compound lens, the two parts having different refractive powers in order that the image may be flat. A simple lens gives a spherical image. The image is formed

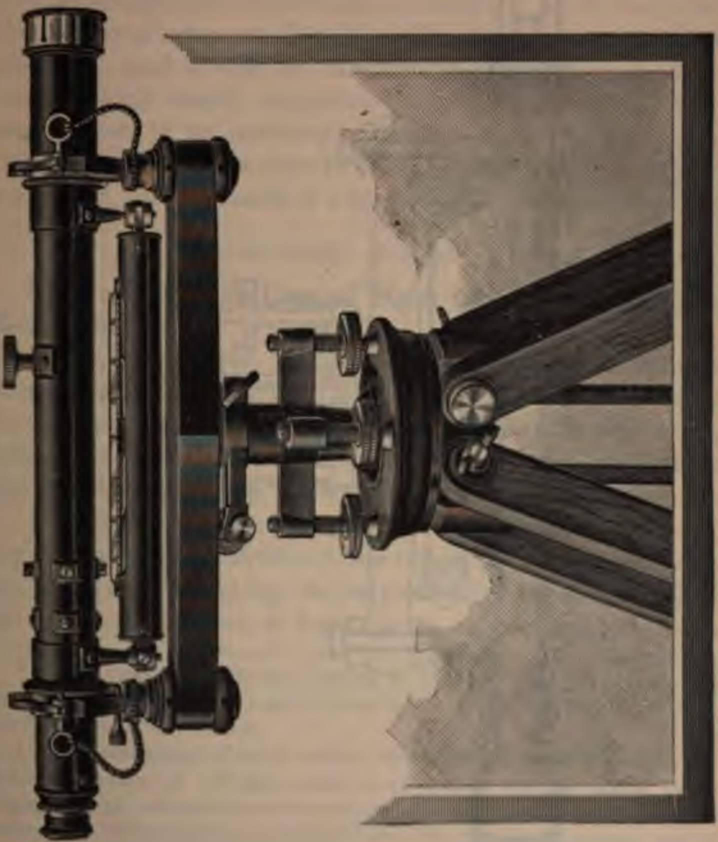

FIG. 12.

at the plane of the cross-wires, which are attached to the reticule held in place by the capstan-screws shown in the cut. The

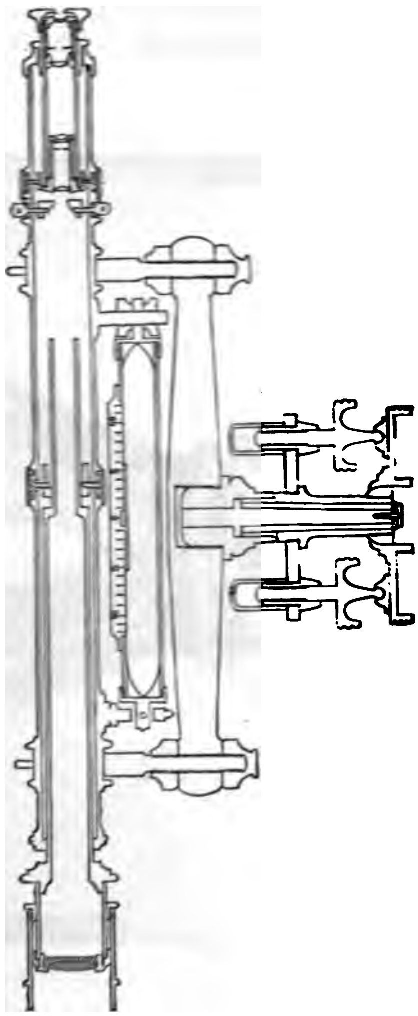

FIG. [illegible]

line of sight is the line joining the two corresponding points in object and image with which the intersection of the cross-wires coincides.* Evidently this line of sight may lie anywhere in the field of view within the limits of movement of the reticule. The line of collimation is simply the true position of the line of sight. The eye-piece serves only to magnify the image, and sometimes to invert it, as is the case in the sectional view of Fig. 13. The image itself is always inverted; and if this be examined by an eye-piece of two lenses, which simply magnifies but does not invert, the object is seen in an inverted position. If four lenses are used in the eye-piece, it re-inverts the image so that the object is seen erect. This results in a loss of light and of distinctness.

ADJUSTMENTS OF THE LEVEL.

65. To make the Line of Sight parallel to the Axis of the Bubble.

First, or Indirect, Method.—This method rests on the proposition that if two lines are parallel to a third line, they are parallel to each other. This method is indirect, but the manipulations are readily performed. It is the usual method, and is frequently given as two separate adjustments.

First, bring the line of sight to coincide with the centres of the pivot-rings by revolving the telescope, bottom side up, in the wyes, and adjusting the reticule until the intersection of the wires remains on a fixed point of the image.† If the

* More correctly, it is the line joining the *inner principal point* of the objective with that point of the image covered by the intersection of the cross-wires. See Fig. 61, and note to same.

† The optical axis of a lens is the line joining the centres of the true spherical surfaces bounding it. If this axis is not coincident with the axis of the telescope, or rings, owing to an erroneous adjustment of the objective slide by the screws near the centre of the telescope tube, Fig. 13, or the improper setting of the lens in its case, then the image will be shifted laterally a small amount equal to the lateral deviation of the two "principal points" of the lens from each other. In this case the image itself will appear to rotate as the telescope is revolved. If now the centre of the cross-wires be moved so as to remain on a fixed portion of the image, it no longer occupies the axis of the telescope, but the line of sight is now *parallel* to this axis, so that this adjustment still accomplishes all that is

5

instrument gives an erect view of the object, there is one inversion between the wires and the eye, and therefore the reticule must be moved in the direction of and one half the amount of its apparent displacement. If the view is inverted, there is no inversion between wires and eye, and therefore its apparent is its true displacement.

Second, make the axis of the bubble-tube parallel to the bottoms of the rings by reversing the telescope end for end in the wyes and adjusting the bubble until it remains in the centre of the tube for the two positions. The telescope should be removed and replaced with great care so as not to disturb the relative elevation of the wyes by any jar or shock. The axis, of course, should be clamped to prevent any horizontal motion in making either part of this adjustment.

This method is based on an assumption which may or may not be true. It is that the pivot-rings are of the same size, and therefore the lines joining their centres and bottoms are parallel.

To find the relative size of the pivot-rings, use a striding-level resting on the two pivot-rings and read in reversed positions. Then change the rings in their supports and read the level again in reversed positions. To reduce the notes, the value of one division of the striding-level must be known.*

The objective is always properly centred and adjusted when the instrument leaves the maker's hands; but it is apt to become loose in its frame, and this frame also loosens in the telescope-tube. If the glass is loose in its frame, unscrew it from the telescope-tube and screw up the tightening band

desired. Or, the objective may have its optical axis coincident with that of the telescope and the optical axis of the eye-piece *not parallel* to that of the objective, and this will cause the image and wires to appear to rotate together—when the telescope is revolved. This need cause no error in the work, but should be adjusted by the screws shown just back of the capstan screws, Fig. 13.

* See adjustments in Precise Levelling, Chap. XIV.

from the rear side. *Do not take the glasses apart under any circumstances*, for they are ground for a given relative position and would not be true for any other. A loose objective is a fatal defect in a levelling-instrument and must be constantly guarded against.

Second, or Direct, Method.—This consists in adjusting the bubble directly to the line of sight, whether this be in the centre of the pivot-rings or not. It is sometimes called the "peg adjustment." Drive two pegs on nearly level ground about 200 feet apart. Set the level about eight or ten inches from one of them, or so that when the rod is held upon it in a vertical position the eye end of the telescope will swing about a half inch from its face. Turn the eye end of the telescope upon the graduated face of the rod, the bubble being in the middle of its tube; look through the object end and set a pencil-point on the rod at the centre of the small field of view, which should be from $\frac{1}{8}$ to $\frac{1}{4}$ inch in diameter. Read the elevation of this point, which we will call a. Hold the same rod on the distant peg and, with the bubble in the middle, set the target on the line of sight, and call this reading b. Now carry the instrument to the distant peg, set it near it, read the elevation of the instrument as before, which reading we will call a'; carry the rod to the first peg and set the target on the line of sight, giving the reading b'. If the line of sight had been parallel to the axis of the bubble in each case, it would have been horizontal when the bubble was in the middle of the tube, and hence the difference between the a and b readings in each case would have been the difference of elevation of the pegs.* We should therefore have had

$$a - b = b' - a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

*This assumption neglects the effect of the earth's curvature. This is eight inches to one mile, and is proportional to the square of the distance. For 200 feet it would be about 0.001 of a foot, and twice this, or 0.002 of a foot, is the error made in the above assumption.

If the line of sight was not parallel to the axis of the bubble, however, then the differences of elevation of the two pegs, as obtained by the two sets of observations, are not equal, and we should have

$$(a - b) - (b' - a') = d \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now d is twice the deviation of the line of sight from the bubble-axis for the given distance. (Let the student construct a figure and show this.) If, therefore, the target be moved up or down as the case may be, a distance equal to $\frac{1}{2}d$, then the line of sight may be brought to this position by the levelling-screws, and the bubble adjusted to bring it to the middle, or else the instrument may be left undisturbed with the bubble in the middle, and the line of sight adjusted to read upon the target by moving the reticule. The significant fact is that by moving the target $\frac{1}{2}d$ from its last position a true horizontal line is established, and either the bubble or the line of sight can be adjusted to it after the other has been brought into a horizontal position by means of the levelling-screws. Equation (2) may be written

$$(a + a') - (b + b') = d; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which it may be seen at once that the line of sight inclines down when d is positive, and up when d is negative. We may therefore have for setting the target the following

Rule: Add together the two heights of instrument and the two rod readings, subtract the latter from the former, and take one half the remainder. Move the target by this amount from the b' reading, up when positive and down when negative. It is then in a horizontal line with the cross-wires of the instrument.

It will be noted that no distances are measured in the above method as is usually prescribed in peg adjustments. After adjusting either the line of sight or the bubble at the second peg, return to the first peg, read height of instrument again, and then read the rod on the second peg for a check. See if this new value of $(a - b)$ agrees with the adjusted value of $(b' - a')$. If not, adjust again.

This method is independent of the relative size of the pivot-rings and of the condition of the objective. (The objective must have a *fixed* condition or no adjustment is worth anything.) Although the essential relation of parallelism is obtained between the line of sight and the bubble, it must not be expected that the telescope can be reversed in the wyes or revolved 180° about its axis without both these auxiliary adjustments appearing to be in error. For inasmuch as these two lines have been made parallel without reference to the axis of the telescope or to the bottoms of the rings, they probably are not parallel to either of these. If the first method is used and the adjustment made, it should stand the test of the second (the necessary assumptions being true), but if adjusted by the second method it should not be expected to stand the test of the first method. At the same time the second method is absolute, while the first is based on assumptions that are often untrue. This adjustment should be examined every day in actual practice.

66. To bring the Bubble-axis into the Vertical Plane through the Axis of the Telescope.—Turn the telescope slightly back and forth in the wyes, and note the action of the bubble. If it remains in the centre the adjustment is correct. If not, move one end of the bubble by means of the lateral adjusting-screws. If this adjustment is very much in error it should be made approximately right before going on with the preceding adjustment.

67. To make the Axis of the Wyes perpendicular to the Vertical Axis of the Instrument.—This is to enable the telescope to be revolved horizontally without re-levelling. Level the instrument in one position. Revolve 180° horizontally, and correct one half the movement of the bubble by the wye-adjustment and the other half by the levelling-screws. Repeat for a check.

68. Relative Importance of Adjustments.—The first adjustment is by far the most important. The second can only enter in the work when the telescope is revolved slightly from its true position in the wyes. Most modern levels have some device for holding the telescope in its proper position when in use. This position is such as brings the horizontal wire truly horizontal. The last adjustment given is only a matter of convenience. It saves stopping to relevel after revolving the telescope. It does not affect the accuracy of the work appreciably. It is absolutely essential, however, that the line of sight should be truly horizontal when the bubble is in the middle of the tube, or reads zero, and this makes the first adjustment here given of such vital consequence.

69. Focussing and Parallax.—The eye-piece serves to give a distinct and magnified view of the image. It also inverts the image in all instruments where the object is seen in an erect position. Since the magnifying power of the eye-piece is large, its focal range of distinct vision is very small, depending on its magnifying power. With the ordinary field-instruments it is about one sixteenth of an inch. Both the image, as formed by the objective, and the cross-wires, should lie in the focus of the eye-piece. They should therefore lie in the same plane. Now the image may be moved back and forth by moving the objective in or out, but the plane of the cross-wires is fixed. If the two are brought into the same plane, therefore, the image must be brought upon the wires. To accomplish this, first focus the eye-piece on the wires so that they appear most distinct. In doing this there should be no image visible, so that either the objective is thrown out of focus or the telescope is turned to the sky. The eye-piece is most accurately focussed by finding its inner and outer limits for distinct vision of the wires, and then setting it at the mean position. The objective may now be moved until the image also comes into focus. This will have to be done for each pointing if the distances are different. If the image is not brought into exact coincidence with the cross-hairs, these will seem to move slightly on the image as the eye is moved behind the eye-piece. This angular displacement of the wires on the image is called *parallax*, and can only occur when they are not in the same plane. It is removed by refocussing the objective, thus moving the image, until there is no perceptible relative movement of wires and image as the eye is shifted, when they are practically in coincidence. If there is parallax, the reading may be in error by its maximum angular amount. If the eye were always held at the centre of the eye-piece there would be no parallax, and it is to accomplish this that the eye-piece is covered by a shield with a small hole in its centre. Still, the slight movement of the eye thus allowed is sufficient to cause some parallactic error if the wires and image are not practically coincident. When the eye-piece is once adjusted to distinct vision on the cross-wires it requires no further attention so long as the instrument is used by the same person. Another person, having eyes of a different focal range, would have to readjust the eye-piece. The eye-piece adjustment, therefore, is personal, and is made once for all for a given individual; while the objective adjustment depends on the distance of the object from the instrument, is made for each pointing, and is considered perfect when the parallax is removed.*

* This discussion is worded for an erecting telescope, where the objective moves. In an inverting instrument the eye-piece and reticule may move *together* in the telescope while the objective remains fixed. Here the image takes different positions in the telescope-tube, as the distances vary, and the cross-wires are moved to suit. There is also a motion of the eye-piece with reference to the wires, and this is the eye-piece adjustment; while the movement of both together is what is called the objective adjustment in the above discussion.

FIG. 13*a*.

69a. Architect's Compass Level.—Fig. 13*a* is a cut of a cheap but very useful instrument known as an Architect's Compass Level.* It is a combination of a level and a needle compass, and is used for laying out buildings, running ditches, street grades, and especially for obtaining both a plan and profile of a line by once running it. A great deal of work was done on the Mississippi River Survey with an improvised instrument of this kind, in running trans-alluvial level lines from bluff to bluff across the bottom lands subject to overflow. A similar instrument, without the compass-box, but having the graduated circle, reading by vernier to five minutes of arc, is manufactured by several instrument-makers. It is called an architect's level, and is very generally used by architects and by surveyors in rural practice. These instruments cost only about one-half as much as the standard engineer's level. They have no clamp and tangent screws, but this is not a serious objection.

* Manufactured by Queen & Co., of Philadelphia, Pa.

THE LEVELLING-ROD.

70. **The Levelling-rod** is used to measure the vertical distance from the line of sight down to the turning-point or bench-mark. There are two general classes, Self-reading, or Speaking, and Target Rods.

A Self-reading, or Speaking, Rod is one so graduated as to enable the observer to note at once the reading of the point which lies in the line of sight, this reading being in all cases the distance to the bottom of the rod. The rod-man here has nothing to do but to hold the rod vertical. The observer notes and records the reading.

A Target-rod is furnished with a sliding target moved by the rod-man in response to signals from the observer until it accurately coincides with the line of sight. Its position is then read with great accuracy by means of a vernier scale.

Fig. 14 is one form of self-reading rod which is also fitted with a target. This is called the Philadelphia rod. Fig. 14*a* is the New York rod, and is not self-reading. It is the standard target-rod used in this country. The one here shown is in three sections, whereas those in common use are in two parts only.

It is necessary that the rod be held vertical when in use, and on sloping ground, or when the wind is blowing, it is difficult to do this. To insure a vertical rod, therefore, especially in the plane of the line of sight, two level-bubbles are sometimes attached, such as shown in the accompanying cuts. When not in use they can be removed and folded up as shown.

Another method of attaining the same end is by means of Thompson's Levelling Target, shown in Fig. 15*a*. This target

Fig. 14.

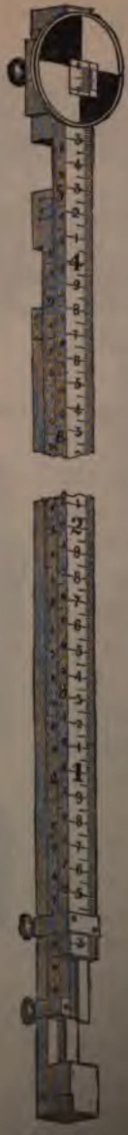

Fig. 14a.

Fig. 15.

is bent at right angles, and so lies against two faces of the rod. If held so that both faces show, the middle dividing line will appear as a broken line when the rod is not vertical.

Most targets slide on the rod, and have a clamp screw and springs. When the rod is wet, the target is apt to stick and move with a jerking motion. The target shown in Fig. 15 is mounted on rollers in order to obviate this difficulty.*

Various patterns of self-reading rods are used. For rough work a twelve- or fourteen-foot rod, 2 inches wide and 1¼ inches thick, painted and fitted with an iron or brass shoe at bottom, graduated to hundredths of a foot, will be found very efficient. The graduations should be so distinct that they can be read through the telescope at a distance of five or six hundred feet.

THE USE OF THE LEVEL.

71. The Level is used—

(*a*) To find the relative elevation of points a considerable distance apart.

(*b*) To obtain the profile of a line.

(*c*) To establish a grade.

These objects may be more or less intermingled in any given piece of work. Whatever may be the ultimate object of the work, however, the immediate object for any given setting of the instrument is to find how much higher or lower a certain forward, or unknown, point is than a certain other back, or known, point. Thus, the rod being held on the known point, the line of sight is turned upon it and the rod-reading gives at once the height of instrument above that point. If the rod be now held on the forward, or unknown, point, and the line of sight turned upon it, this rod-reading gives the distance of that point below the line of sight. The reading on the known point is called the *back-sight*, and that on the unknown point is called the *fore-sight*. If the elevation of the known point be given, we find the elevation of the line of sight by adding

* Both these targets are manufactured by Keufel & Esser, New York.

the rod-reading at that point. By subtracting from this elevation the reading on the unknown point, the elevation of that point is obtained. Thus we have found the relative elevations of the two points by referring them both to the horizontal plane through the instrument. Since the back-sight reading gives the elevation of the instrument, and since this is always greater than the elevation of that point, it follows that the back-sight reading is essentially positive. For a similar reason the fore-sight reading is essentially negative, since any point on which the rod is held is lower than the line of sight. It will also be seen that there can be but one back-sight (unless the height of the instrument is to be found from readings on several known points, and the mean taken), while there can be any number of fore-sights from one instrument position. Thus, the height of the instrument having been determined, the elevations of any number of points, in any direction, may be determined by referring them all to the horizontal plane through the instrument, whose elevation has been obtained by the single back-sight reading. It is also important to remember that the terms "back-sight" and "fore-sight" have no reference to directions or points of the compass, but they do have a rational significance when we think of the work proceeding from the known point to the unknown point or points. Thus, we refer *back* to the known point for height of instrument, and then transfer this knowledge *forward* to the points whose elevations we wish to find.

DIFFERENTIAL LEVELLING.

72. Differential Levelling consists in finding the difference of elevation of points a considerable distance apart. The 'evation of the first point being known or assumed, the differ- ' elevation between this and any other point is found d algebraically, thus giving the elevation of the second The "plane of reference" is the surface of zero-eleva- alled the "datum plane." This is not really a *plane* but a *level* surface, according to the definition given in art. 60. It is, however, universally denominated the "plane of reference," "datum plane," or simply "datum." The problem, then, is to find the difference of elevation between two distant points. If the points were near together and had not too great a difference of elevation, a single setting of the instrument would be sufficient. If they are too far apart for this, either in distance or in elevation, then more than one setting of the instrument must be made. In this case the intervening points occupied by the rod are called *turning-points*, the terminal points being called *bench-marks*. The successive differences of elevation of these turning-points is determined by setting the level equally distant from them, and so they serve to divide up the total distance between terminal points into a series of short spaces, each of which can be covered by a single setting of the instrument. The successive differences of elevation of turning-points being found, their algebraic sum would be the difference of elevation of the terminal points, or bench-marks. But since all the back-sights are essentially positive and all the fore-sights are essentially negative, we may at once add all the back-sights together and all the fore-sights together, and take the difference of the sums. This is the difference of elevation between terminal points, and has the sign of the larger sum, the back-sights being positive and the fore-sights negative. This difference of elevation added algebraically to the elevation of the initial point gives the elevation of the final point. Evidently the route travelled in passing from one bench-mark to another is of no consequence so long as the true difference of elevation is obtained.

73. **Length of Sights.**—Where the ground is nearly level it is desirable to make the length of sights (distance from instrument to rod) as long as practicable, in order to increase the rate of progress. For the best work this distance may be from 100 to 300 feet, according to the state of the atmosphere. When the air and ground differ greatly in temperature there

result innumerable little upward and downward currents of air, the upward being warmer than the downward currents. The warmer air is more rarefied than the colder, and thus a ray of light passing from the rod to the instrument passes alternately through denser and rarer media, each change producing a slight refraction of the ray. This causes a peculiar tremulous condition of the image in the telescope, so that it is difficult to determine just what part of it is covered by the cross-hairs. At such times the air is said to be "trembling" or "dancing" or "unsteady." It always occurs more or less in clear weather, owing to the earth then being hotter than the air, and it varies with the quality of the soil, cinders or gravel being very bad. When the air is in this condition the length of sights should be shortened.

The back and fore sights for any setting of the instrument should always be equal in length. Levelling is the only kind of field-surveying wherein the instrumental errors may be thoroughly eliminated without duplicating the observations. This may be done in levelling by making the back and fore sights of equal length. For, since the *difference* between back and fore sights is always the quantity used, it follows that if both are too large or too small by the same amount, the *difference* will be unchanged. If, when the bubble is in the middle of its tube, the line of sight is inclined upwards by a given small angle, then it has this relation to the horizontal on both fore and back sights, and *if the lengths of sights were equal* the fore and back rod-readings were equally in error. It is therefore very desirable that these sights should be made of equal length. Moreover, the effect of the earth's curvature is eliminated by so doing, however long the sights may be. There are other kinds of errors that are not eliminated by this means, but those that are eliminated are of sufficient importance to warrant great care to secure equal sights for each setting. If it is impossible to do this at any time, the inequality should be balanced off at the next one or two settings, by making them unequal in the opposite direction by the same amount. The equality of sights can be determined by pacing with sufficient accuracy.

74. **Bench-marks** are fixed points of more or less permanent character whose elevations are determined and recorded for future reference. The general and particular location of a bench-mark should be so distinctly described that any one could find it from its description. Whenever the work is temporarily interrupted a temporary bench-mark is set, such as a substantial stake driven into the ground, or a spike in the root of a tree. The prime requisite of a good bench-mark is that it shall not change its elevation during the period in which it is to be used. If this period is not more than two or three years, a spike driven in the spreading root of a tree near the trunk and well above ground will serve. The wood should be trimmed away from it so as to leave a projecting spur that will not be overgrown. The tree itself should then be marked by notching or otherwise, and carefully located in the description.

If the mark is to serve for from five to fifty years, stone or brick structures or natural rock should be selected. The water-tables, or corners of stone steps, of buildings, copings of foundation and retaining walls, piers and abutments of bridges, or copper bolts leaded in natural rock may serve. If artificial structures are chosen, those should be selected which have probably settled to a fixed position, and for this reason old structures are preferable to new ones.

When stakes are used for temporary benches it is often advisable to set two or even three for a check. In this case the *mean elevation* is the elevation used. In starting from such a series of benches there would be as many back-sights for the first setting of the instrument as there were benches, the mean of which, added to the mean elevation of the benches, would give the height of instrument. In running a continuous line of levels it is advisable to set a bench-mark at least as often as one to the mile.

75. The Record in differential levelling is very simple. The bubble always being put in the middle of the tube, and the rod-positions chosen equally distant from the instrument, the bubble-reading and the length of sights may be omitted from the record, unless some knowledge of the distance run is desired, when the length of sights may be inserted.

FORM OF RECORD FOR DIFFERENTIAL LEVELLING.

No. of Station.	Back-sights.	Fore-sights.	Elevation of Mean Benches.	Remarks.
	3.426 3.878		96.301	B. S. on B. M. 31 " " " 31 a
1 2	3.652 4.517	4.879 3.472		
3	3.216	4.361 4.873	94.718	F. S. on B. M 32 " " " 32 a
		4 617		
	+11.385	−12.968 +11.385		
		− 1.583		

It will be seen that the mean of the readings on the two bench-marks was used in each case. The back-sights being essentially positive and the fore-sights essentially negative, these signs are prefixed to the sums, and the algebraic sum of these gives the elevation of the forward above or below the rear benches. This added to the elevation of the initial point gives the elevation of the final point. These points are the mean elevation of two bench-marks in the example given.

76. The Field-work should be done with great care if the best results are to be obtained. The instrument should be adjusted every day, especially the parallelism of bubble-axis and line of sight. The instrument and rod should both be set in firm ground. An iron pin, about one inch square at top, six to eight inches long, and tapering to a point, should be used for the turning-point. A rope or leather noose should be passed through an eye at top to serve as a handle. To hold the rod upright the rodman should stand squarely behind it and keep it balanced on the pin. When the target is set and clamped the rodman reads it and records it on a paper he carries for the purpose. He then carries it to the observer, if it was a back-sight reading, or he awaits the coming of the observer if it was a fore-sight reading, when the observer also reads it and records it in his note-book. The rodman then calls off his reading, and the observer notes its agreement with his recorded reading. In this way two wholly independent readings are obtained and any erroneous reading corrected. Errors of one foot or one tenth are not very uncommon in reading target rods. The rodman should be especially careful to protect the turning-point from all disturbances between the forward and back readings upon it. The observer must not only obtain an accurate bisection on the target, but he must know that the bubble is accurately in the centre of the tube when this bisection is obtained. When the observer walks forward to set his instrument he counts his paces, and takes as long a sight as the nature of the ground or the condition of the atmosphere will allow. When the rodman comes up he counts his paces to the instrument and then goes *the same distance* in advance. Thus the observer controls the length of sights, making them whatever he likes; and it is the business of the rodman to see that the back- and fore-sight for every instrument-station are equal.

PROFILE LEVELLING.

77. **In Profile Levelling** the object is to obtain a profile of the surface of the ground on certain established lines. Here both the distances from, and the elevation above, some fixed initial point must be obtained. When the line is laid out stakes are usually driven every hundred feet, these positions

being obtained by a chain or tape. It is now the business of the leveller to obtain the elevation of the ground at each of these stakes, and at as many other intermediate points as may be necessary to enable him to draw a fairly accurate profile of the ground. The 100-foot stakes are usually numbered, and these numbers are entered on the level record. The intermediate points are called pluses. Thus, a point 40 feet beyond the twenty-fifth 100-foot stake is called 25 + 40, being really 2540 feet from the initial point. It is evident that no plus-distance can be more than 100 feet, and these are usually paced by the rodman. The intermediate points are selected with reference to their value in determining the profile. These are points where the slope changes, being mostly maximum and minimum points, or the tops of ridges and bottoms of hollows. Turning-points are selected at proper distances, depending on the accuracy required, and these may or may not be points in the line whose profile is desired. The levelling-instrument also is not set on line, if it is found more convenient to set it off the line.

In profile levelling, since absolute elevations with reference to the datum-plane are to be obtained from every instrument-position, it is necessary to find the height of instrument above datum for every setting, and from this height of instrument, obtained by a single back-sight reading on the last turning-point, the elevations of any number of points are found by subtracting the readings upon them.

78. **The Record** in profile levelling is much more elaborate than in differential levelling. The following form is considered very convenient for profile-work where the line has been laid out and 100-foot stakes set:*

* This sample page was contributed to *Engineering News* in June, 1879, and the form of record is credited to Mr. E. S. Walters, a railroad engineer of large experience.

GUATEMALA AND HONOLULU RAILROAD. Feb. 30, 1876.

B. S.	El. of T. P. and B. M.	F. S.	H. I.	I. S.	S. E.	Sta.	Remarks.
10.552	195.497		206.049			B. M.	
				9.32	196.73	188 + 44	
				11.41	194.64	189	
				7.01	199.04	190	
				2.07	203.98	+ 30	
				1.62	204.43	191	
				0.38	205.67	+ 20	
				0.82	205.13	192	
0.515	202.797	3.252	203.312			T. P.	
				3.10	200.21	193	
				2.70	200.61	+ 50	
	198.048			5.264		B. M.	
				8.20	195.11	194	
				9.35	193.96	195	
3.411	194.840	8.472	198.251			T. P.	
				4.28	193.97	196	
				5.06	193.19	197	
				7.20	191.05	+ 60	
				10.60	187.65	+ 65	
				7.00	191.25	198	
				5.46	192.79	199	
9.527	195.083	3.168	204.610			T. P.	
				10.25	194.36	+ 35	
				8.62	195.99	200	
24.005		14.892		6.04	198.57	201	
14.892							
9.113							

In the above headings, B. S. denotes back-sight; F. S., fore-sight; I. S., intermediate sight; H. I., height of instrument; T. P., turning-point; B. M., bench-mark; S. E., surface-elevation; Sta., station.

It will be noted that there is but one back-sight and one height of instrument for each setting. The back-sight and fore-sight readings from the same instrument-station are not found here on the same line, as in differential levelling, but the fore- and back-readings on the same turning-point are on the same line. Thus, the rod was first read on the bench-mark whose elevation was known to be 195.497 feet above datum. The reading on this bench was 10.552, thus giving a height of instrument of 206.049. This is marked B. M. in the station column, and evidently has but one reading upon it in starting the work from it. A series of intermediate sights are then taken at various 100-foot stakes and pluses, the readings on which, when subtracted from the H. I., give the surface-elevations at those points. When the work has progressed as far in front of the instrument as the B. M. was back of it, a turning-point is set, and the reading upon it recorded in the column of fore-sights. This reading was 3.252, which, subtracted from the H. I. 206.049, gives 202.797 as the elevation of the turning-point. The instrument is now moved forward and a back-sight reading taken upon this T. P. of 0.515, which added to 202.797 gives 203.312 as the new H. I. At this setting a new bench was established by taking an intermediate sight upon it of 5.264, and writing the elevation in the B. M. column instead of in the S. E. column. The readings on bench-marks and turning-points are made to thousandths, while the intermediate sights for surface-elevation are read only to hundredths of a foot. The last height of instrument is checked by adding the back-sights and fore-sights, taking the difference and applying it to the elevation of the initial point with its proper sign, remembering that back-sights are positive and fore-sights negative. The profile is now constructed by the data found in the S. E. and Sta. columns, these being adjacent to each other. One of the great merits of this form of record is that wherever it is necessary to combine any two numbers by addition or subtraction, they are found in adjacent columns. In constructing the profile, some kind of profile or cross-section paper is used, and the horizontal scale made much smaller than the vertical. Thus, if the horizontal scale were 400 feet to the inch, the vertical scale might be 10 or 20 feet to the inch.

LEVELLING FOR FIXING A GRADE.

79. **In fixing a grade** the profile may be obtained and the grade marked upon it. The vertical distance between the surface-line and the grade-line, at any point is the depth of cut or fill at that point, and this may be marked on the line stakes at once, without the aid of the level or rod, if only the centre depths are desired, as in the case of a ditch or trench. If the sides are to have a required slope, however, the level and rod are necessary to fix the horizontal distance of the limiting or "slope" stakes from the centre stakes whenever the ground is not strictly a level surface. This operation is called "cross-sectioning," and is described in Chapter XIII., on Determination of Volumes.

If the grade be known before the profile is determined, together with the absolute elevation of the initial point, as is sometimes the case with ditches and trenches for pipe lines or sewers, then the depth of cut (or fill) may be at once determined and marked on the line stakes when the profile is taken. The form of record might be the same as given above for profile levelling, with the addition of two columns after the "Station" column, one being Elevation of Grade, and the other Cut or Fill. The elevation of grade would be found for each profile point by adding if an up, and subtracting if a down, grade, the differences of elevation corresponding to the successive distances in the profile. The difference between the corresponding "surface-elevation" and "elevation of grade" would be the cut or fill at each point, which could be at once taken out and marked on the line stake.

THE HAND-LEVEL.

80. **Locke's Hand-level** is a very convenient little instrument for rough work, such as is done on reconnaissance expeditions. It consists of a telescope with a bubble attached in

such a way that the position of the bubble is seen by looking through the telescope. A horizontal line of sight is thus readily determined. It is supposed to be adjusted once for all.

FIG. 16.

EXERCISES WITH THE LEVEL.

81. Adjust the bubble to the line of sight by the first, or indirect, method, and then test it by the second, or direct, method. If this second method does not show it to be in adjustment, where does the error lie?

82. Cause the line of sight and bubble-axis to make a considerable angle with each other (that is, put it badly out of adjustment in this particular), and level around a block or two, closing on the starting-point, being careful to make back and fore sights as nearly equal as possible. Of course the final elevation of the point should agree with the assumed initial elevation. The difference of these elevations is the error of closure of the level polygon. If the back and fore sights were exactly equal this should be zero, notwithstanding the erroneous adjustment.

83. Put the instrument in accurate adjustment, and level over the same polygon as before, making the back and fore sights quite unequal, and note the error of closure. If the instrument were in exact adjustment and there were no errors of observation, should the error of closure be zero?

84. Range out a line on uneven ground about a half-mile in length, and set stakes every hundred feet. Let each student determine the profile independently. When all have finished, let them copy their profiles on the same piece of tracing-cloth, starting at a common point. The vertical scale should be large, so as to scatter the several profile lines sufficiently on the tracing. Each profile should be in a different color or character of line.

85. Select a line on nearly level ground, about a half-mile in length. Establish a substantial bench-mark at each end. Let each student determine the difference of elevation of these benches twice, running forward and back. See if the results are affected by the direction in which the line is run.

If each student could do this several times some evidence would be obtained as to there being such a thing as "personal equation" in levelling; that is, each person tending to always obtain results too high or too low. Why is it improbable that there could be any personal equation in levelling?

CHAPTER IV.

INSTRUMENTS FOR MEASURING ANGLES.

THE TRANSIT.

86. The Engineer's Transit is the most useful and universal of all surveying-instruments. Besides measuring horizontal and vertical angles it will read distances by means of stadia wires, determine bearings by means of the magnetic needle, do the work of a solar compass by means of a special attachment, and do levelling by means of a bubble attached to the telescope. It is therefore competent to perform all the kinds of service rendered by any of the instruments heretofore described, and is sometimes called the "universal instrument." A cut of this instrument is shown in Fig. 17. Fig. 18 is a sectional view through the axis of a transit of different manufacture.

The telescope, needle-circle, and vernier plates are rigidly attached to the inner spindle which turns in the socket *C*, Fig. 18. This portion of the instrument is called the *alidade*, as it is the part to which the line of sight is attached. The socket *C* carries the horizontal limb, shown at *B*, and may itself revolve in the outer socket attached to the levelling-head. Either or both of these connections may be made rigid by means of proper clamping devices. If the horizontal limb *B* be clamped rigidly to the levelling-head and the alidade spindle be allowed to revolve, then horizontal angles may be read by noting the vernier-readings on the fixed horizontal limb for the different pointings of telescope. If the horizontal limb itself be set and clamped so that one of the verniers reads zero

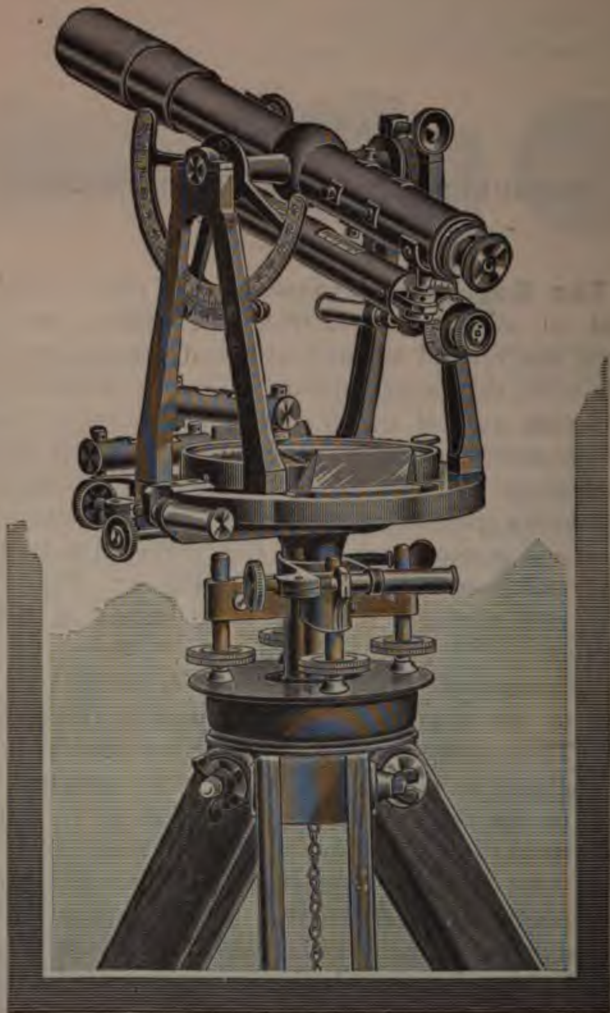

FIG. 17.

when the telescope is on the meridian, then for any other pointing of the telescope the reading of this same vernier gives the true azimuth of the line. It is necessary, therefore, to have two independent movements of telescope and horizontal limb on the same vertical axis. The magnetic needle is shown at *N*. The plumb-line is attached at *P*; this should always be in the vertical line passing through the centre of the graduated horizontal circle. This will be the case when

FIG. 18.

it is attached directly to the axis itself, for this must always be made vertical.

The limb is graduated from zero to 360°, and sometimes with a second set of figures to 90° or 180°. There are two verniers reading on the horizontal limb 180° apart. Both the instruments shown in Figs. 17 and 18 have shifting centres, enabling the final adjustment of the instrument over a point to be made by moving it on the tripod-head. The telescope is shorter than those used in levelling-instruments in order that it may be revolved on its horizontal axis without having the standards too high. It is called a transit instrument on account of this movement, which is similar to that of an astronomical transit used for observing the passage (transit) of stars across any portion of the celestial meridian. When the telescope is too long to be revolved in this way the instrument is called a *theodolite*. This is the only essential difference between them.* The "plain transit" has neither a vertical circle nor a bubble attached to the telescope.

ADJUSTMENTS OF THE TRANSIT.

87. **The Adjustments of the Engineer's Transit** are such as to cause (1) the instrument to revolve in a horizontal plane about a vertical axis, (2) the line of collimation to generate a vertical plane through the instrument-axis when the telescope is revolved on its horizontal axis, (3) the axis of the telescope-bubble to be parallel to the line of collimation, thus enabling the instrument to do levelling, and (4) the vernier on the vertical circle so adjusted that its readings shall be the true altitude of the line of collimation. These four results are attained by making the following five adjustments:

88. **First. To make the Plane of the Plate-bubbles perpendicular to the Vertical Axis.**—This adjustment is the same as with the compass. (One of the plate-bubbles is usually set on one pair of standards.) Bring both bubbles to the centre, revolve 180°, correct one half the movement on the levelling-screws and the other half by raising or lowering the adjustable end of the bubble-tube. Each bubble should be brought parallel to a set of opposite levelling-screws in making this adjustment, so that the correcting for one bubble does not throw the other out. When either bubble will maintain a fixed position in its tube as the instrument is revolved horizontally, the axis of revolution is vertical. One bubble is

* The first engineer's transit instrument was made by Wm. J. Young (now Young & Sons), Philadelphia, 1831. All American engineer's altitude-azimuth instruments are now made to revolve in this way.

therefore sufficient for making this axis vertical, but two are somewhat more convenient, especially for indicating when the axis has become inclined from unequal settling or expansion while in use.

89. Second. To make the Line of Sight perpendicular to the Horizontal Axis of the Telescope.*—When this is done, the line of sight will generate *a plane* when the telescope is revolved on its horizontal axis. If the line of sight is not perpendicular to the horizontal axis, it generates the surface of a cone when the telescope is revolved, the axis of the cone being the axis of revolution, and the apex being at the intersection of the line of sight with this axis.

Set the instrument on nearly level ground, where a view can be had in opposite directions. Set the line of sight on a definite point a few hundred feet away. Revolve the telescope and set another point in the opposite direction. Revolve the alidade until the line of sight comes upon the first point. Revolve the telescope again and fix a third point on the line of sight beside the second point set. Measure off one-fourth the distance between these two points from the last point set, and bring the line of sight to this position by moving the reticule laterally. This movement of the reticule is direct in an erecting instrument and reversed in an inverting instrument.

The student should illustrate the correctness of this method by means of a figure. The four pointings were the intersections of a diametral horizontal plane with the surfaces of the the two cones generated. These cones were pointed in opposite directions, but had one element in common, being the two pointings to the first point. The two opposite elements diverged by four times the difference between the semi-angle of the cone (subtended by the line of sight and the axis of rotation) and 90°.

90. Third. To make the Horizontal Axis of the Tele-

* This is called the *Adjustment for Collimation*, since it consists in bringing the line of sight into coincidence with the line of collimation, which is simply the true position for the line of sight.

scope perpendicular to the Axis of the Instrument.—When this is done the former is horizontal when the latter is vertical, and, the second adjustment having been made, the line of sight will generate a *vertical* plane when the telescope is revolved.

Set the instrument firmly and level it carefully. Suspend a plumb-line some 20 or 30 feet long, some 15 or 20 feet from the instrument. The weight should rest in a pail of water and the string should be hung from a rigid support. There should be no wind, and the cord should be small and smooth. A small fish-line is very good. Care must be exercised that the weight does not touch the bottom of the pail from the stretching of the cord. Set the line of sight carefully on the cord at top, the plate-bubbles indicating a strictly vertical instrument-axis. Clamp both horizontal motions and bring the telescope to read on the bottom portion of the cord. The cord is apt to swing to and fro slightly, but its mean position can be chosen. If the line of sight does not correspond to this mean position, raise or lower the adjustable end of the horizontal axis until this test shows the line of sight to revolve in a vertical plane. Constant attention must be given to the plate-bubbles to see that they do not indicate an inclined vertical axis.

Or, two points nearly in a vertical line may be used, as the top and bottom of the vertical corner of a building. Set on the top point and revolve to the bottom point. Note the relation of the line of sight to this point. Revolve 180° about both vertical and horizontal axes, and set again on the top point. Lower the telescope again and read on the bottom point. If the telescope-axis of revolution is horizontal, the second pointing at bottom should coincide with the first. If not, adjust for one half the difference between these two bottom readings.

It will be noted that the second and third adjustments are necessary to the accomplishment of the second result cited in art. 87.

91. Fourth. To make the Axis of the Telescope-bubble parallel to the Line of Sight.—This adjustment is performed by means of the "peg-adjustment," as described in art. 65, p. 65, second method. The height of the instrument may now be measured to the centre of the horizontal axis if it be found more convenient than sighting backwards through the telescope. When this adjustment is made the instrument is competent to do levelling the same as the levelling-instrument. The telescope is not quite so stable, however, in the transit because it is mounted on an axis instead of in two rigid wyes.

92. Fifth. To make the Vernier of the Vertical Circle read Zero when the Line of Sight is Horizontal.—Having made the axis of the telescope-bubble parallel to the line of sight, bring this into the centre of its tube, and adjust the vernier of the vertical circle till it reads zero on the limb. If this vernier is not adjustable, the reading in this position is its index error. The line of sight might still be adjusted to the vernier by moving the reticule, and then adjusting the bubble to the line of sight. To do this use the "peg-adjustment" as described in art. 65, making the vertical circle read zero each time, and paying no attention to the telescope-bubble. Correct the line of sight by $\frac{1}{2}$ d. as given by Eq. (2), p. 66, by moving the reticule, and this should give a horizontal pointing for a zero-reading of the vertical circle. Then adjust the bubble to this reading by bringing it to the centre of the tube by means of the vertical motion at one end of the bubble-tube. If the reticule is disturbed after making the second adjustment, that adjustment should be tested again to see if it had been disturbed.

93. Relative Importance of the Adjustments.—The first adjustment is important in all horizontal and vertical angular measurements. In measuring vertical angles the error may be the full amount of the deviation of the vertical axis from the vertical, and in measuring horizontal angles something very much less than this.

The second adjustment is more important in the running of a straight line by revolving the telescope than in any other kind of work, for here the error in the continuation of the line is twice the error of adjustment. It is also important in measuring horizontal angles between points not in the same horizontal plane.

The third adjustment is most important in the measurement of horizontal angles between points not in the same horizontal plane, as in the determination of the azimuth of a line by an observation on a circumpolar star.

The fourth and fifth adjustments are important only in levelling operations, either by reading the vertical angle or by the use of the bubble.

INSTRUMENTAL CONDITIONS AFFECTING THE ACCURATE MEASUREMENT OF HORIZONTAL ANGLES.*

94. **Eccentricity.**—This is of two kinds: (1) *eccentricity of centres*, and (2) *eccentricity of verniers*. If the axis of the conical outer socket *C*, Fig. 18, is not exactly in the centre of the graduated limb *B*, then when the telescope with the vernier plates *V* are revolved in this socket, the verniers will have an eccentric motion with reference to the graduated limb. If the line joining the zeros of the verniers passes through the axis of the socket, it is evident that there is but one position of these verniers which will give readings on the limb 180° apart, and that is when both centres lie in this diametral line. For all other positions of the verniers, one of them will read as much too large as the other does too small; so that if the mean

* For extended discussions of this subject, see Bauernfeind's "Vermessungskunde," § 144, vol. i., and Jordan's "Handbuch der Vermessungskunde," § 88, vol. i. Also translations from these, by Prof. Eisenmann, in *Journal of the Association of Engineering Societies*, vol. iv. p. 196.

of the two vernier-readings be taken, this error from eccentricity would be eliminated.

Eccentricity of verniers is due to their zeros not falling on a diametral line through the axis of the spindle; in other words, they are not 180° apart. This involves no error in measuring horizontal angles. It is convenient, however, to have the verniers read exactly 180° apart. In any case, reading of both verniers and taking the mean eliminates all errors from eccentricity. An eccentricity of centres of *one one-thousandth of an inch* would cause a maximum error of 1′-08″ on a six-inch circle if but one vernier were read. It is not unusual for an instrument to have an eccentricity of centres of several times this amount, either from wear or from faulty construction, or both. The necessity for reading both verniers in all good work is therefore apparent.

95. **Inclination of Vertical Axis.**—The horizontal angle between points at different elevations is obtained by measuring the horizontal angle subtended by two vertical planes passing through these points and the point of observation. These vertical planes are the planes described by the line of sight as the telescope is revolved. By this means the points may be said to be projected vertically on the horizontal plane and then the angle measured. If the vertical axis of the instrument is somewhat inclined, these projecting planes are not vertical, neither do they have the same inclination to the horizon on different parts of the limb. The projecting planes through two points will therefore neither be vertical nor equally inclined to the horizon. The measured horizontal angle thus obtained will therefore be in error. The vertical axis is always inclined when the plate-bubbles are not in adjustment or when they do not show a level position.

If the axis be inclined 5′ from the vertical, and readings be taken on points 60° apart, one being 10° above and the other 10° below the horizon, the maximun error from this source would be about 1′. If the inclination in this case were 1°, the maximum error would be 18′. This shows the importance of keeping the plate-levels in adjustment and of watching them during the progress of the work to see that they remain in the centre.

96. Inclination of Horizontal Axis of Telescope.—This causes the plane generated by the line of sight to be inclined from the vertical as much as the axis of revolution is from the horizontal. The projecting planes are therefore all equally inclined, and the resulting error in horizontal angle is a function of the difference of elevation of the two points. If one point is 10° above and the other 10° below the horizon, and if the inclination of the axis is 5′, the resulting error in the measurement of the horizontal angle is 1′–45″. This error is not a function of the size of the horizontal angle, and would be the same for two points in the same vertical plane, the instrument indicating a horizontal angle of 1′ 45″ between them for the case here chosen. In making the adjustment of the horizontal axis by means of the plumb-line, if the line be 15 feet distant and suspended 15 feet above the instrument, then the pointing to the top will have an altitude of 45°. In this case the angular error made in bisecting the plumb-line will be the angular divergence of the axis of rotation from the horizontal. If the combined error of the two bisections be 0.05 in., the angular error in the adjustment will be 1′. The adjustment may readily be made closer than this.

Errors from this source are eliminated by revolving the telescope and reading the same angle in the reversed position. The mean of the two values will be independent of this error. If many measurements are made of one angle, there should be an equal number with telescope direct and reversed.

The student should show by a figure how this elimination is effected by the reversal of the telescope.

97. The Line of Sight not being Perpendicular to the Horizontal Axis.—This causes the projecting planes to be conical surfaces, which become vertical on the horizon. Since the error of collimation is necessarily a small angle, thus causing the conical surface to be very nearly a plane, and since this surface is vertical on the horizon, the resulting error in measuring horizontal angles is very small unless the difference in the elevations of the points is very great. If the points are distant, as they always are in the accurate measurement of horizontal angles, then their angular elevation is necessarily small, so that this source of error is insignificant in this kind of work. When straight lines are prolonged by reversing the telescope, however, this adjustment becomes very important, for the error then enters the work with twice its angular amount. It is eliminated by revolving the alidade until the line of sight, with telescope reversed, falls again on the rear point, and again revolving the telescope. The point now falls as far on one side of the true position as it before did on the other. The middle point lies therefore in the line prolonged.

Let the student illustrate by diagram.

THE USE OF THE TRANSIT.

98. To measure a Horizontal Angle.—Having centred the instrument over the vertex of the angle required, take a pointing to one of the points and clamp both alidade and limb. Make the final bisection by means of either tangent-screw. Read the two verniers, and record them, calling one the reading of vernier *A* and the other of vernier *B*. Loosen the alidade clamp and turn upon the second point, clamp, and set by the upper tangent-screw. Read both verniers again. Correct the readings of vernier *A* by half the difference between the *A* and *B* readings in each case. The difference between these corrected readings is the value of the angle.

Be careful not to disturb the lower clamp- or tangent-screw after reading on the first point. If there are two abutting tangent-screws for the lower plate, be sure that both are snug, otherwise there may be some play here which would allow the limb to shift its position, in which case the true angle would not be obtained. If there is but a single tangent-screw working against a spring on the other side of the armature, as shown in Fig. 16, then there can be no lost motion unless the friction on the axis is greater than the spring can overcome, which should never be the case.

Do not set the clamp-screws too tightly, as it strains and wears out the instrument unnecessarily. A very gentle pressure is usually sufficient to prevent slipping. This caution applies equally well to all levelling-, adjusting-, and connecting-screws in the instrument. The young observer is generally inclined to set them up hard, as he would in heavy iron-work. It must be remembered that brass is a soft material, easily distorted and worn, and that the parts should be strained as little as possible to insure against movement in ordinary handling.

The subject of measurement of horizontal angles is further discussed in Chapter XIV., on Geodetic Surveying.

99. **To measure a Vertical Angle.**—Vertical angles are usually referred to the horizon, and are angles of elevation or depression above that plane. If the vernier on the vertical circle has been properly adjusted (or its index error determined in case it is not adjustable and the line of sight has not been adjusted to it), then the altitude of a point is obtained at once by turning the line of sight upon it and reading the vertical angle. Special attention must here be given to the bubble parallel to the vertical circle, for it is on this bubble that the accuracy of the result wholly depends. If there is but one vernier, it is designed to read both ways, as is shown in Figs. 5 or 6, p. 19. In this case errors of eccentricity cannot be eliminated.

To eliminate errors of adjustment of the plate-bubbles and of the vernier on the vertical circle, revolve the alidade 180°, re-level, read the vertical angle again with telescope in a reversed position, and take the mean. This can only be done in case the vertical limb is a complete circle. In many instruments it is but a half-circle or less, in which case this elimination cannot be made. The accuracy of the adjustments alone can then be relied on, and these must be frequently tested. If the plate-bubble parallel to the vertical circle, the telescope-bubble, and the vernier of the vertical circle have all been once accurately adjusted, then when these bubbles are brought to a zero-reading the vertical circle should also read zero. This test can always be readily applied, and, though not an absolute check, it is a very good one, inasmuch as two of these three adjustments would have to be out by the same amount and in the same direction to still agree with the third.

100. To run out a Straight Line.—The transit-instrument is especially adapted to the prolongation of straight lines, as long tangents on railroads, and yet it requires the most careful work and much repetition to run a line that approximates very closely to a straight line.

Having determined the direction which the line is to take from the initial point, set accurately over this point, turn the telescope in the given direction, and set a second point at a convenient distance. These two points now determine the line, and it remains to prolong it indefinitely over such uneven ground as may lie in its course. The line, when established, is to be the trace of a vertical plane through the first two points on the surface of the ground. If the line of sight always revolved in a vertical plane, and no errors were made in handling the instrument and in setting the points, the problem would be easiiy solved, but we may safely say that the surface generated by the line of sight never is a vertical plane. (The adjustments being never absolutely correct.)

This surface is a cone whose axis is not strictly horizontal, for both the horizontal and vertical axes are somewhat inclined from their true positions. It remains then so to make the observations that all these errors of adjustment will be eliminated. The following programme accomplishes this:

(1) Set accurately over the forward point, putting one pair of levelling-screws in the line.

(2) Clamp the horizontal limb in any position.

(3) Level carefully, and turn upon the rear point.

(4) Relevel for the bubble that lies across the line.

(5) Make the bisection on the rear point, revolve the telescope, and set a point in advance. This may be a tack in a stake set with great care by making the bisection on a pencil held vertically on the stake.

(6) Unclamp the alidade and revolve it about the vertical axis till the telescope comes on the rear point.

(7) Relevel for the cross bubble again.

(8) Make the bisection on the rear point, revolve the telescope again, and set a second point in advance beside the first one. The mean of these two positions should lie in the vertical plane through the two established points, whatever may be its elevation, and regardless of small errors in the instrumental adjustments. For the reversals of the telescope and alidade eliminated the errors of collimation and horizontal axis, while the relevelling eliminated the error due to the error of adjustment of the plate-bubble. If this bubble were out of adjustment the vertical axis inclined as much to one side for the first setting as it did to the other side for the second setting.

This operation may be repeated for a check, or to further eliminate errors of observation. The instrumental errors are wholly eliminated by one set of observations, as above given. It will be noted that this method is independent of the graduation of the limb. The only assumptions are that the instrument and its adjustments are rigid during the reversal of the telescope, and that the pivots of the horizontal axis are true cylinders.

101. **Traversing.**—A traverse, in surveying, is a series of consecutive courses whose lengths and bearings, or azimuths, have been determined. When a compass is used the bearing of each course is determined by the needle independently of that of the preceding course. When a transit is used and the needle not read, the graduated circle of the instrument is always oriented, or brought into the meridian, by taking a back-sight to the preceding station. If the azimuth* of the first course is known with reference to the meridian, the azimuth of all subsequent courses may be at once determined by properly orienting the limb of the instrument at the successive stations. Thus, if the south point has a zero azimuth the limb of the instrument should be oriented at each station, so that when the telescope points south vernier A shall read zero.

The *forward azimuth* of a line is its angular deviation from the south point when measured at the rear station forward along the line.

The *back azimuth* of a line is its angular deviation from the south point at the forward station when measured from that station back along the line.

The forward and back azimuth differ by 180° plus or minus the convergence of the meridians at the two extremities of the line. If this line is north and south it lies in the meridian, and hence its forward and back azimuth differ by 180°. When the course has an easterly or westerly component, or, in other words, when its extremities have different longitudes, the divergence of the line from the meridian at one end differs from its divergence from the meridian at the other by as much

* In this treatise azimuth is always reckoned from the south point in the direction S.W.N.E. to 360°. The bearing of the line is thus given by its numerical value alone, without the aid of letters.

as these meridians differ from parallelism. This is inappreciable on short lines, and hence in traversing the forward and back azimuth will be considered as differing by 180°.

The field-work proceeds as follows, so far as the transit is concerned. Let it be assumed that from the initial point *A* of the survey the true azimuth to some other point *Z* is given. Let the stations be *A*, *B*, *C*, etc.

Set vernier *A* to read the known azimuth *AZ*. With the alidade and limb clamped together, turn the telescope on *Z* and clamp the limb, setting carefully by means of the lower tangent-screw. If the alidade be now loosened and vernier *A* made to read zero, the telescope would point south. Turn the telescope on *B* by moving the alidade alone, and the reading of vernier *A* gives the forward azimuth of the line *AB*. Move the instrument to *B* and set vernier *A* to read the back azimuth of *AB*, which is found by adding 180° to or subtracting it from the forward azimuth, according as this was less or more than 180°. With alidade and limb clamped at this reading, turn upon *A*, clamp the limb and unclamp the alidade, and the instrument is again properly oriented for reading directly the true azimuth of any line from this station, as the line *BC*, for instance. In this manner a traverse may be run with the transit, the field-notes showing the true azimuth of each course without reduction. The lengths of the courses may be found in any manner desired.

If preferred, the telescope may be revolved on its horizontal axis and vernier *A* left with its forward reading, for orienting. Then revolve the telescope back to its normal position and proceed with the work.*

* For a method of computing the coördinates of the courses, and the use of the traverse table, see chapter on Land Surveying.

THE SOLAR ATTACHMENT.

102. **The Solar Attachment** is a device to be fastened to the telescope axis of a transit-instrument, thus making a combination that will do the work of a solar compass. One form of this device is shown in Fig. 19.* The various spherical functions concerned in the problem are also represented in this figure by their several great circles. The polar axis, declination-arc, and collimation-arm are the same here as in the solar compass. The latitude-arc is here replaced by the vertical circle of the transit, and the telescope gives the line of sight. The adjustments and working of this attachment are so nearly identical with those of the solar compass that they will not be repeated here. If the student has mastered the *principles* involved in the use of the solar compass he will have no difficulty in using the attachment.

Various forms of solar attachments have been invented, the most recent and perhaps the most efficient of which is that shown in Fig. 20, invented by G. N. Saegmuller in 1881. It is manufactured by Fauth & Co., Washington, D. C., and by Keuffel & Esser, New York. It consists simply of an auxiliary telescope with bubble attached, having two motions at right angles to each other. These motions are horizontal and vertical when the main telescope, to which the attachment is rigidly fastened, is horizontal. If the main telescope be put in the meridian and elevated into the plane of the celestial equator, however, then the vertical axis of the attachment also lies in the meridian but points to the pole. It therefore becomes a polar axis about which the auxiliary telescope may revolve. If this telescopic line of sight be at right angles to the polar axis, it will generate an equatorial plane. If the line of sight be inclined to this plane by an amount equal to and in the direction of the sun's declination, then when revolved on its polar axis it

* From Gurley's Catalogue.

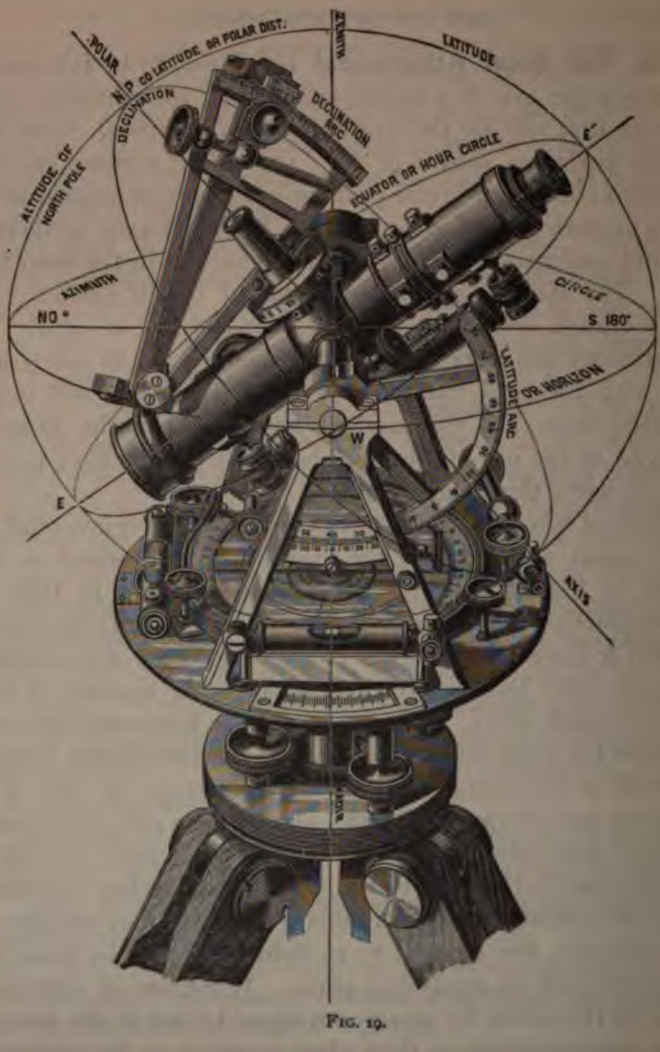
POLAR
N P
CO LATITUDE OR POLAR DIST.
ZENITH
LATITUDE
DECLINATION
DECLINATION ARC
ALTITUDE OF NORTH POLE
EQUATOR OR HOUR CIRCLE
AZIMUTH
CIRCLE
N 0°
S 180°
LATITUDE ARC
OR HORIZON
W
E
AXIS

FIG. 19.

Fig. 20.

would follow the sun's path in the heavens for the given day, provided the sun did not change its declination during the day. It only remains, therefore, to show how the latitude and declination angles may be set off in order that the competency of this instrument to do the work of the solar compass may become apparent.

To set off the declination-angle, turn the main telescope down or up according as the declination is north or south, and set the declination-angle on the vertical circle. Bring the small telescope into the plane of the large one and revolve it about its horizontal axis until its bubble comes to the centre of its tube. The angle formed by the two telescopic lines of sight is the declination-angle. Revolve the main telescope until it has an altitude equal to the co-latitude of the place, and clamp it in this position. With the vertical motions of both telescopes clamped, and their lateral motions free, if the line of sight of the small telescope can be brought upon the sun the main telescope must lie in the meridian. The vertical circle of the transit is thus seen to do the work of both the latitude and declination arcs of the solar compass.

103. Adjustments of the Saegmuller Attachment.—*First.* All the adjustments of the transit must be as perfect as possible, but especially the plate and telescope bubbles, the vernier of the vertical circle, and the transverse axis of the telescope.

Second. To make the Polar Axis perpendicular to the Plane of the Line of Collimation and Horizontal Axis of the Main Telescope.—Carefully level the instrument and bring the telescope-bubble to the middle of its tube. The line of sight and horizontal axis of this telescope should now be horizontal, so that the polar axis is to be made vertical. To test this, revolve the auxiliary telescope about the polar axis, and see if the bubble on the small telescope maintains a constant position. If not, correct half the movement by means of the adjusting, screws at the base of the small disk, and the other half by revolving the auxiliary telescope. These adjusting-screws are exactly analogous to the levelling-screws of the main instrument.

Third. To make the Line of Sight of the Small Telescope parallel to the Axis of the Attached Bubble.—Make the large telescope horizontal by bringing its attached bubble to the middle of its tube. Bring the small telescope in the same plane and make it also horizontal by means of its bubble, clamping its vertical motion. Measure the vertical distance between the axes of the two telescopes, and lay off this distance on a piece of paper by two plain horizontal lines. Set this paper up at a convenient distance from the instrument, and on about the same level. Bring the line of sight of the large telescope on the lower mark, and see if that of the small telescope falls on the upper mark. If not, adjust its reticule until its line of sight come on the upper mark. Revolve back to the horizontal to see if both bubbles again come to the middle simultaneously.

When this adjustment is completed, there should be five lines in the instrument parallel to each other when instrument and telescopes are level,—viz., the axes of the two telescope-bubbles and of the plate-bubble on the standards, and the two lines of sight,—and, in addition, the vernier on the vertical circle, should read zero.

The seven adjustments (five of the transit and two of the attachment) must all be carefully made and frequently tested if the best results are desired. When this is done, this attachment will give the meridian to the nearest minute of arc, if observations be taken when the sun is more than one hour from the horizon and two hours from the meridian. The advantages of the Saegmuller attachment consist mainly in having a telescopic line of sight, and in the use of the vertical limb of the transit for setting off the declination and co-latitude. The effect of small errors in the latitude and declination angles, such as may be due to errors in the adjustments, is shown by the table, art. 54, p. 51.

THE GRADIENTER ATTACHMENT.

104. The Gradienter is a tangent-screw with a micrometer-head attached to the horizontal axis of the telescope for the purpose of turning off vertical angles that are expressed in terms of its tangent as so many feet to the hundred. Such a device is shown in Fig. 17. In railroad work, the grade or slope is expressed in this manner, as 26.4 feet per mile, or as 0.5 foot per 100 feet. The micrometer-head is graduated so that one revolution raises or lowers the telescope by 1 foot or 0.5 foot in 100 feet. It is divided into 100 or 50 parts, so that each division on the head is equivalent to 0.01 foot in 100 feet. This attachment is found very convenient in railroad work. It is also of general utility in obtaining approximate distances. On level ground the distance is read directly, but on sloping ground the rod is still held vertical, and the distance read is too great. The true horizontal distance may be found by multiplying the distance read by the factors for horizontal distance given in table V.* Thus, if one revolution of the screw raises the line of sight 1 foot at a distance of 100 feet, and if at a certain unknown distance one revolution of screw caused the line to pass over 5.5 feet on the rod, then the distance was 550 feet if the ground was horizontal. If the rod-readings had a mean vertical angle of 15°, the horizontal distance was $550 \times 93.3 = 513$ feet.

CARE OF THE TRANSIT.

105. The Transit should be protected from rain and dust as much as possible. A silk gossamer water-proof bag should be carried by the observer to be used for this purpose. If water gets inside the telescope, remove the eye-piece and let it dry out. If moisture collects between the two parts of the objec-

* This table is for reduction of stadia measurements, and is explained in the chapter on Topographical Surveying, Art. 205.

tive remove it, and dry it with a gentle heat over a stove or lamp, but *do not separate the glasses.* If dust settles on the wires it may be blown off by removing both objective and eye-piece and blowing gently through the tube. Dust should be removed from the glasses by a camel's-hair brush, which should always be carried for the purpose. A clean handkerchief may be used with a gentle pressure to prevent scratching in case the dust is gritty. Use alcohol for cleansing greasy or badly soiled glasses. No part exposed to dust should be oiled, as this serves to retain all the dust that may fall on it. The centres should be cleaned occasionally with chamois skin, and oiled by a very little pure watch-oil. In the absence of watch-oil plumbago will be found to serve. A soft lead-pencil may be scraped and a little rubbed on the spindles with the finger. The tripod legs should have no lost motion either at the head or in their iron shoes. If the legs are split, as in Fig. 17, and fastened by thumb-nuts, these should be loosened when the instrument is carried and tightened again after setting. They may thus be made very tight and rigid while the instrument is in use without danger of breaking the bolts in closing the legs, which is very liable to result if the screws are not loosened. For a method of putting in new cross-wires see chapter on Topographical Surveying, Art. 207.

EXERCISES WITH THE TRANSIT.

106. Establish three stations forming a triangle. Measure the three horizontal angles and see if their sum is 180°.

107. Prolong a line in azimuth and distance by carrying both around an imaginary obstruction, and then check the azimuth by a back-sight and the distance by measurement. Thus, let A and B be two points establishing a line. The problem is to establish two other points, C and D, in the continuation of the line AB, with an imaginary obstruction to both sight and measurement between B and C. The distance BC is also to be obtained.

The equilateral triangle will be found most efficient.

108. Find both the distance to and the height of an inaccessible steeple, chimney, smokestack, or tree.

Measure a base-line such that its two extremities make with the given object approximately an isosceles triangle (it is desirable that no angle of the triangle should be less than 30° nor more than 120°). The top of the object only need be visible from the two ends of the base. Measure both the horizontal and vertical angles at the extremities of the base-line subtended by the other two points of the triangle. Let A and B be the extremities of the base and P the point whose distance and elevation are required. We then have for horizontal angles

$$\text{Sin } P : \sin A :: AB : BP;$$

also

$$\sin P : \sin B :: AB : AP.$$

In reading the vertical angles to the base-stations the reading should be taken on a point as high above the ground (or peg) as the telescope is above the peg over which it is set. The difference in the elevations of the two pegs is then obtained. The vertical angle to the point P is taken to the summit, and height of instrument added in each case to find its elevation above peg. If A be the lower of the two base-stations and if I_A and I_B be the heights of instrument (line of sight) above the peg in the two cases, and if V_A, V_B, V_P and V_P' be the vertical angles read to the corresponding points, we may write:

$$\text{Elevation of } B \text{ above } A = AB \tan V_B;$$
$$\text{" \quad " } P \text{ " \quad } A = AP \tan V_P.$$

Also, from the vertical angles taken at B, we have:

$$\text{Elevation of } A \text{ below } B = AB \tan V_A;$$
$$\text{" \quad " } P \text{ above } B = BP \tan V_P'.$$

We now have a check on both the relative elevations and on the distances AP and BP. Assuming the elevation of A to be zero, we have:

$$\text{Elevation of } P \text{ above } A = AP \tan V_P = AB \tan V_B + BP \tan V_P'.$$

This equality will not result unless the observations were well taken, the computations accurately made, and the instrument carefully adjusted. The adjustments mainly involved here are the plate-bubbles and the vernier on the vertical circle. If the points are a considerable distance apart, as over a half-mile, the elevations obtained by reading the vertical angles are appreciably too great, on account of the earth's curvature. This may be taken as eight inches for one mile and proportional to the square of the distance. Or, we may write:

Elevation correction on long sights, in inches,* $= - 8$ (distance in miles)^2.

If the distances are all less than about half a mile, no attention need be paid to this correction in this problem.

* For a full discussion of this subject see chap. XIV.

109. Find the height of a tree or house above the ground, on a distant hill, without going to the immediate locality.

110. Find the horizontal length and bearing of a line joining two visible but inaccessible objects. Use the magnetic bearing if the true bearing of the base-line is not known.

111. Find the horizontal length and bearing of a line joining two inaccessible points both of which cannot be seen from any one position.

Let A and B be the inaccessible points. Measure a base CD such that A is seen from C and B from D. Auxiliary bases and triangles may be used to find the lengths of AC and BD. Knowing AC and CD and the included angle, compute AD in bearing and distance. The angle ADB may now be found, which, with the adjacent sides AD and BD known, enables the side AB to be found in bearing and distance.

112. With the transit badly out of level, or with horizontal axis of the telescope thrown considerably out of the horizontal, measure the horizontal angle between two objects having very different angular elevations. Do this with both telescope normal and telescope reversed, and note the difference in the values of the angle obtained in the two cases.

113. Select a series of points on uneven ground, enclosing an area, and occupy them successively with the transit, obtaining the traverse angles. That is, knowing or assuming the azimuth of the first line, obtain the azimuths of the other connecting lines, or courses, with reference to this one, returning to the first point and obtaining the azimuth of the first course as carried around by the traversed line. This should agree with the original azimuth of this course. The distances need not be measured for this check.

114. Lay out a straight line on uneven ground by the method given in Art. 100, occupying from six to ten stations. Return over the same line and establish a second series of points, paying no attention to the first series, and then note the discrepancies on the several stakes. In returning, the two final points of the first line become the initial points of the second, this return line being a prolongation of the line joining these two points. If these deviate ever so little, therefore, from the true line, the discrepancy will increase towards the initial point.

Similar exercises to those given for the solar compass may be assigned for the solar attachment.

THE SEXTANT.

115. **The Sextant** is the most convenient and accurate hand-instrument yet devised for measuring angles, whether horizontal, vertical, or inclined. It is called a sextant because its limb includes but a 60° arc of the circle. It will measure angles, however, to 120°. It is held in the hand, measures an angle by a single observation, and will give very accurate results even when the observer has a very unstable support, as

FIG. 21.

on board ship. It is exclusively used in observations at sea, and is always used in surveying where angles are to be measured from a boat, as in locating soundings, buoys, etc., as well as in reconnoissance work, explorations, and preliminary surveys. It has been in use since about 1730.

The accompanying cut shows a common form of this instrument as manufactured by Fauth & Co., Washington. The limb has a $7\frac{1}{2}$-inch radius, and reads to 10 seconds of arc.

There is a mirror *M* (Fig. 22), called the ***Index Glass***, rigidly attached to the movable arm *MA*, which carries a vernier reading on the graduated limb *CD*. There is another mirror, *I*, called the ***Horizon Glass***, rigidly attached to the frame of the instrument, and a telescope pointing into this mirror, also rigidly attached. This mirror is silvered on its lower half, but clear on its upper half. A ray of light coming from *H* passes

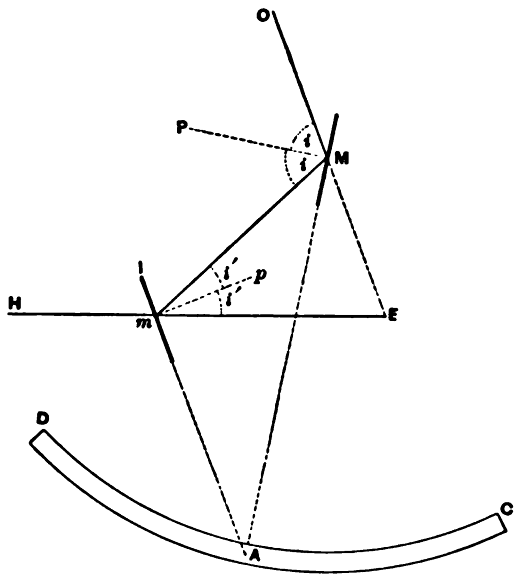

FIG. 22.

through the clear portion of the mirror *I* on through the telescope to the eye at *E*. Also, a ray from an object at *O* strikes the mirror *M*, is reflected to *m*, and then through the telescope to *E*. Through one half of the objective come the rays from *H*, and through the other half the rays from *O*, each of which sets of rays forms a perfect image. By moving the arm *MA* it is evident these images will appear to move over each other, and for one position only will they appear to coincide. The bringing of the two images into exact coincidence is what the observation consists in, and however unsteady the motion of the observer may be, he can occasionally see both images at once, and so by a series of approximations he may finally put the arm in its true position for exact superposed images. The angle subtended by the two objects is then read off on the limb.

116. The Theory of the Sextant rests on the optical principle that "if a ray of light suffers two successive reflections in the same plane by two plane mirrors, the angle between the first and last directions of the ray is twice the angle of the mirrors."

To prove this, let OM and mE be the first and last positions of the ray, the latter making with the former produced the angle E. The angle of the mirrors is the angle A. The angles of incidence and reflection at the two mirrors are the angles i and i', PM, and pm being the normals.

We may now write:

$$\begin{aligned} \text{Angle } E &= OMm - MmE \\ &= 2(i - i'); \\ \text{angle } A &= ImM - mMA \\ &= (90^\circ - i') - (90^\circ - i) \\ &= i - i'; \end{aligned}$$

Therefore $E = 2A$. Q. E. D.

When the mirrors are brought into parallel planes, the angle A becomes zero, whence E also is zero, or the rays OM and Hm are parallel. This gives the position of the arm for the zero-reading of the vernier. The limb is graduated from this point towards the left in such a way that a 60° arc of the circle will read to 120°. That is, a movement of 1° on the arc really measures an angle of 2° in the incident rays, so it must be graduated as two degrees instead of one. The very large radius enables this to be done without difficulty.

ADJUSTMENTS OF THE SEXTANT.

117. **To make the Index Glass perpendicular to the Plane of the Sextant.**—Bring the vernier to read about 30° and examine the arc and its image in the index glass to see if they form a continuous curve. If the glass is not perpendicular to the plane of the arc, the image will appear above or below the arc, according as the mirror leans forward or backward. It is adjusted by slips of thin paper under the projecting points and corners of the frame.

118. **To make the Horizon Glass Parallel to the Index Glass for a Zero-reading of the Vernier.**—Set the vernier to read zero and see if the direct and reflected images of a well-defined distant object, as a star, come into exact coincidence. If not, adjust the horizon glass until they do. If this adjustment cannot be made, bring the objects into coincidence, or even with each other so far as the motion of the arm is concerned, and read the vernier. This is the index error of the instrument and is to be applied to all angles read. The better class of instruments all allow the horizon glass to be adjusted. This adjustment is generally given as two, but it is best considered as one. If made parallel to the index glass after that has been adjusted, it must be perpendicular to the plane of the instrument.

119. **To make the Line of Sight of the Telescope parallel to the Plane of the Sextant.**—The reticule in the sextant carries four wires forming a square in the centre of the field. The centre of this square is in the line of collimation of the instrument.

Rest the sextant on a plane surface, pointing the telescope upon a well-defined point some twenty feet distant. Place two objects of equal height upon the extremities of the limb that will serve to establish a line of sight parallel to the limb. Two lead-pencils of same diameter will serve, but they had best be of such height as to make this line of sight even with that of the telescope. If both lines of sight come upon the same point to within a half-inch or so at a distance of 20 feet, the resulting maximum error in the measurement of an angle will be only about 1″.

THE USE OF THE SEXTANT.

120. To measure an Angle with the sextant, bring its plane into the plane of the two objects. Turn the direct line of sight upon the fainter object, which may require the instrument to be held face downwards, and bring the two images into coincidence. The reading of the limb is the angle required. It must be remembered that the angles measured by the sextant are *the true angles subtended by the two objects at the point of observation*, and not the vertical or horizontal projection of these angles, as is the case with the transit. The true vertex of the measured angle is at E, Fig. 21. It is evident the position of E is dependent on the size of the angle, being at a great distance back of the instrument for a very small angle. The instrument should therefore not be used for measuring very small angles except as between objects a very great distance off. The sextant is seldom or never used for measuring angles where the position of the instrument (or the vertex of the angle) needs to be known with great accuracy.

EXERCISES FOR THE SEXTANT.

121. Measure the altitude of the sun or a star at its culmination by bringing the direct image, reflected from the surface of mercury held in a flat dish on the ground, into coincidence with the image reflected from the index glass. Half the observed angle is the altitude of the body. The altitude of a terrestrial object may be obtained in the same manner, in which case the vessel of mercury should rest on an elevated stand; the sextant could then be brought near to it and the angular divergence of the two incident rays to the mercury surface and index glass reduced to an inappreciable quantity.

If the observation of a heavenly body be made on the meridian and the declination of the body at the time of observation be known, the latitude of the place is readily found.

121*a*. Measure the angle subtended by two moving bodies, as of two men walking the street in the same direction, or of two boats on the water. (This is to illustrate the capacity of the sextant, for none but a reflecting instrument bringing two converging lines of sight into coincidence is competent to do this.)

The exercises given in Arts. 106, 108, 109, and 110 for the transit may also serve for the sextant. Further applications of the sextant in locating soundings are given in chap. X.

122. Wood's Double Sextant.—The accompanying cut illustrates a new instrument of great utility in locating sound-

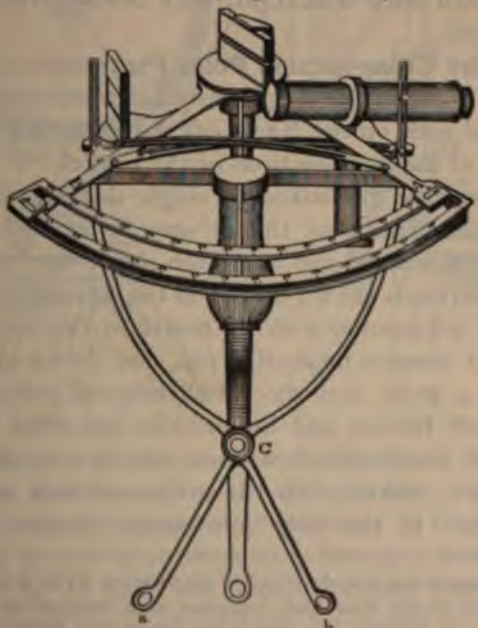

FIG. 23.

ings, especially in running water. By its use a single observer may readily observe simultaneously the two angles necessary to locate his position. It is simply two sextants combined in one. There are two entire limbs graduated in opposite directions, and two movable mirrors. There will, of course, be three images brought into simultaneous juxtaposition when properly set for both angles. The instrument is the invention of Mr. G. W. Wood, who has had many years' experience in locating soundings on the United States Coast and Mississippi River Surveys. The instrument is likely to be universally used in locating soundings in running water.* Its use saves one observer in such cases, for the boat cannot be stopped for making the observation. Without it two observers are necessary, each with a sextant, they reading the two angles simultaneously.

122a. The Cross-section Polar Protractor.—The accompanying cuts (Figs. 24 and 25) illustrate an instrument recently invented and used on the New York Aqueduct for taking polar coördinates of the cross-sections of the tunnel. It consists of a plain circular disk, graduated to single degrees, and mounted on a tripod in such a way that it may be levelled up and also have a vertical motion and a motion about the vertical axis.† The construction is shown clearly in the figures.

In use it is mounted with its centre in the axis of the tunnel. A light wooden measuring rod, not shown in the figures, tapering to a point and shod with brass of sufficient length, and graduated to feet and hundredths, lies upon the wooden arm or rest, which revolves upon the face of the disk, and slides out to a contact with the surface at such points as are to be taken. If the only information desired is whether

* This instrument was first described in *Engineering News* of June 28, 1890.

† Those used on the New York Aqueduct were designed by F. W. Watkins and Alfred Craven, and were manufactured by Heller and Brightly, Philadelphia. See description in "Trans. Am. Soc. Civ. Eng'rs," 1890, and *Engineering News* July 26, 1890.

or not the excavation is sufficient, or beyond the established lines, then the rod is set to the proper radius, and if it swings clear, the fact is determined. If a true copy of the actual cross-section is desired, then the rod is brought into contact with the significant points in the cross-section (mostly the

FIG. 24.

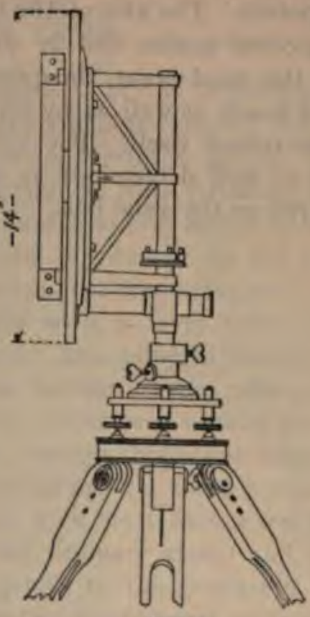

FIG. 25.

points of greatest projection and depression of the surface), and the angle and distance read and recorded. In the instrument here shown, the graduation increases in both directions from the top to 180° at the bottom. Perhaps a better arrangement would be to have the angles increase continuously to 360°. The work could be plotted by means of such a protractor as shown in Figs. 64 or 66, Chapter VIII. The points being plotted, they should be joined by a free-hand line and the area determined by the planimeter.

If the cross-section contains one or more marks from which the axis of the tunnel may be found, as an alignment mark and a bench mark (which may be one and the same), then the instrument may be set up at random on this section, and these fixed marks pointed in and plotted, along with the cross-section points. The axis of the tunnel can then be laid off from the plotted marks, and by drawing in the established lines from this axial point, the question of clearance may be determined nearly as well as by setting the instrument in the axis of the tunnel itself. The actual cross-section and area are quite as well determined as if the instrument were carefully centered on the axial line.

CHAPTER V.

THE PLANE TABLE.

123. The Plane Table consists of a drawing-board properly mounted on which rests an alidade carrying a line of sight rigidly attached to a plain ruler with a fiducial edge. The line of sight is usually determined by a telescope, as in Fig. 26. This telescope has no lateral motion with respect to the ruler, but both may be moved at pleasure on the table. The telescope has a vertical motion on a transverse axis, as in the transit. It is also provided with a level tube, either detachable or permanently fixed. The table is levelled by means of one round or two cross bubbles on the ruler of the alidade. The line of sight of the telescope is usually parallel to the fiducial edge of the ruler, though this is not essential. It is only necessary that they should make a fixed horizontal angle with each other. The table itself must have a free horizontal angular movement and the ordinary clamp and slow-motion screw. The table corresponds to the graduated limb in the transit, the alidades in the two instruments performing similar duties. Instead, however, of reading off certain horizontal angles, as is done with the transit, and afterwards plotting them on paper, the directions of the various pointings are at once drawn on the paper which is mounted on the top of the table, no angles being read. The true relative positions of certain points in the landscape are thus transferred directly to the drawing-paper to any desired scale. The magnetic bearing of any line may be determined by means of the *declinator*, which is a small box carrying a needle which can swing some ten degrees either side of the zero-line. The zero-line

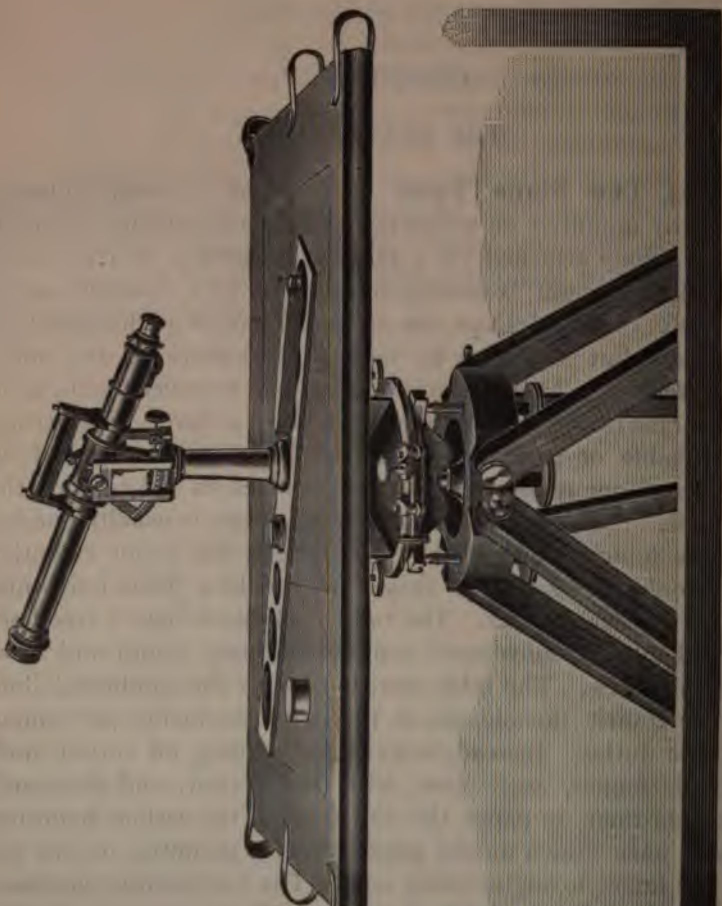

FIG. 26.

being parallel to one edge of the box, the magnetic meridian may be at once marked down on any portion of the map, and the bearing of any intersecting line determined by means of a protractor. The instrument has been long and extensively used for mapping purposes, and is still the only instrument used for the "filling-in" of the topographical charts of the U. S. Coast and Geodetic Survey. An extended account of the instrument and the field methods in use on that service may be found in Appendix 13 of the Report of the U. S. Coast and Geodetic Survey for 1880. The following discussion is partly from that source.

ADJUSTMENTS OF THE ALIDADE.

124. To make the Axes of the Plate-bubbles parallel to the Plane of the Table.—Level the table with the alidade in any position, noting the readings of the bubbles. Mark the exact position of the alidade on the table, take it up carefully, and, reversing it end for end, replace it by the same marks. If the bubbles now have the same readings as before, with reference to the table they are parallel to the plane of the table. If not, adjust the bubbles for one half the movement and try again.

125. To cause the Line of Sight to revolve in a Vertical Plane.—This adjustment is the same as in the transit. It need not be made with such extreme accuracy, however, and the plumb-line test is sufficient. With the instrument carefully levelled, cause the line of sight to follow a plumb-line through as great an arc as convenient. If the line of sight deviates from the plumb-line raise or lower one end of the transverse axis of the telescope, until it will follow it with sufficient exactness.

126. To cause the Telescope-bubble and the Vernier on the Vertical Arc to read Zero when the Line of Sight is Horizontal.—This adjustment is also the same as in the transit. The methods given for the transit may be used with the plane table, or a sea horizon may be used as establishing a horizontal line, or a levelling-instrument may be set up beside the plane table having the telescopes at the same elevation, and both lines of sight turned upon the same point in the horizontal plane as determined by the level. The bubble and vernier are then both adjusted to this position of telescope.

This adjustment is important if elevations are to be determined either by vertical angles or by horizontal lines of sight. If only geographical position is sought this adjustment may be neglected.

THE USE OF THE PLANE TABLE.

127. In using the Plane Table at least two points on the ground, over which the table may be set, must be plotted on the paper to the scale of the map before the work of locating other points can begin. This requires that the distance between these points shall be known, which distance becomes the base-line for all locations on that sheet. Any error in the measurement or plotting of this line produces a like proportional error in all other lines on the map.

The plane table is set over one of these plotted points, the fiducial edge of the ruler brought into coincidence with the two points, and the table revolved until the line of sight comes on the distant point. The table is now clamped and carefully set by the slow-motion screw in this position, when it is said to be *oriented*, or *in position*.

In Figs. 27 and 28, let T, $T_{\prime}'$ $T_{\prime}''$ $T_{\prime}'''$ represent the plane-table sheet and the points a and p the original plotted points. The corresponding points on the ground are A and P, the latter being covered by p in Fig. 27, and the former by a in Fig. 28. In Fig. 27, the plotted point p is centred over the point P, the ruler made to coincide with ap, and the telescope made to read on A by shifting the table. For plotting the directions of

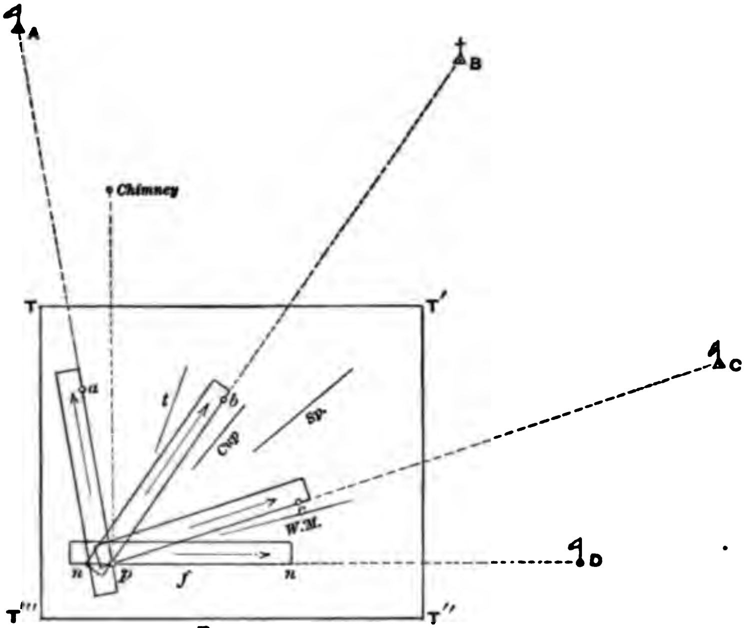
A
B
Chimney
T
T'
C
a
t
b
Cup
Sp.
W.M.
D
n
p
f
n
T'''
T''

FIG. 27.

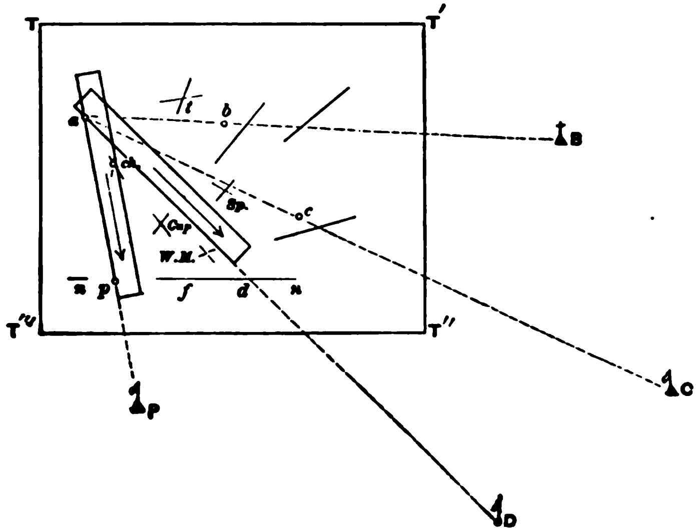
T
T'
t
b
a
B
ch.
Sp.
c
Cup
W.M.
n
p
f
d
n
T'''
T''
C
P
D

FIG. 28.

other objects on the ground, the alidade is made *to revolve about* *p* just as the transit revolves about its centre. A needle is sometimes stuck at this point, and the ruler caused to press against it in all pointings, but this defaces the sheet. Other pointings are now made to *B*, *C*, and *D*, which may be used as stations, and also to a chimney (*ch.*), a tree (*t.*), a cupola (*cup.*), a spire (*sp.*), and a windmill (*w.m.*). Short lines are drawn at the estimated distance from *p*, and these marked with letters, as in the figure, or by numbers, and a key to the numbers kept in the sketch- or note-book.

The table is now removed to *A*, the other known point, and set with the point *a* on the plot over the point *A* on the ground, when the table is approximately oriented. The ruler is now set as shown in Fig. 28, coinciding with *a* and *p*, but pointing towards *p*. The table is then swung in azimuth until the line of sight falls on *P*, when it is clamped. It is now oriented * for this station, and pointings are taken on all the objects sighted from *P*, and on such others as may be sighted from subsequent stations, the alidade now revolving about the point *a* on the paper. The intersections on the plot of the two pointings taken to the same object from *A* and *P* will evidently be the true position on the plot for those points with reference to, and to the scale of, the line *ap*. These intersections are shown in Fig. 28.

It is evident that if other points, as *D* or *C*, be now occupied, the table oriented on either *A* or *P*, and pointings taken on any of the objects sighted from both *A* and *P*, the third or fourth line drawn to the several objects should intersect the first two in a common point. This furnishes a check on the work, and should be taken for all important points. It is preferable also to have more than two points on the sheet pre-

* It will be noted that this process of orienting the plane table is practically identical with that by which the limb of the transit is oriented in traversing (art. 101).

viously determined. Thus, if *B* were also known and plotted at *b*, when the table had been oriented on any other station, and a pointing taken to *B*, the fiducial edge of the ruler should have passed through *b*.

As fast as intersections are obtained and points located the accompanying details should be drawn in on the map to the proper scale. If distances are read by means of stadia wires on a rod held at the various points (see chap. VIII), then a single pointing may locate an object, the distance being taken off from a scale of equal parts, and the point at once plotted on the proper direction-line. It is now common to do this in all plane-table surveying.

128. Location by Resection.—This consists in *locating the points occupied* by pointings to known and plotted points. The simple case is where a single pointing has been taken to this point from some known point, and a line drawn through it on the sheet. It is not known what point on this line represents the plotted position of this station. The setting of the instrument can therefore be but approximate, but near enough for all purposes. The table can be oriented as before, there being one pointing and corresponding line from a known point. A station is then selected, a pointing to which is as nearly 90 degrees from the orienting line as possible, and the alidade so placed that while the telescope sights the object the fiducial edge of the ruler passes through the plot of the same on the sheet. The intersection of this edge with the former line to this station gives the station's true position on the sheet. This latter operation is called *resection*. Another resection from any other determined point may be made for a check.

129. To find the Position of an Unknown Point by Resection on Three Known Points.—This is known as the *Three-point Problem*, and occurs also in the use of the sextant in locating soundings. It is fully discussed in that connection (see Art. 228), so that only a mechanical solution suitable for the problem in hand will be given here. It is understood there are three known points, *A*, *B*, and *C*, plotted in *a*, *b*, and *c* on the map. The table is set up over any given point (not in the circumference of a circle through *A*, *B*, and *C*). Fasten a piece of tracing-paper, or linen, on the board, and mark on it a point *p* for the station *P* occupied. Level the table, but of course it cannot be oriented. Take pointings to *A*, *B*, and *C*, and draw lines on the tracing-paper from *p* towards *a*, *b*, and *c*, long enough to cover these distances when drawn to scale. Remove the alidade and shift the tracing-paper until the three lines drawn may be made to coincide exactly with the three plotted points *a*, *b*, and *c*. The point *p* is then the true position of this point on the sheet. This being pricked through, the table may now be oriented and the work proceed as usual.

130. To find the Position of an Unknown Point by Resection on Two Known Points.—This is called the Two-point Problem, and but one of several solutions will be given. It is evident that if the table could be properly oriented over the required point, its position on the sheet could be at once found by resection on the two known points. The table may be oriented in the following manner: Let *A* and *B* be the known points plotted in *a* and *b* on the sheet. Let *C* be the unknown point whose position *c* on the sheet is desired. Select a fourth point *D*, which may be occupied, and so placed that intersections from *C* and *D* on *A* and *B* will give angles between 30 and 120 degrees. Fasten a piece of tracing linen or paper on the board, marking a point *d'* at random. Set up over *D*, orienting the table as nearly as may be by the needle or otherwise. Draw lines from *d'* towards *A*, *B*, and *C*. Mark off on the latter the estimated distance to *C*, to scale, calling this point *C'*. Set up over *C*, with *c'* over the station, orienting on *D* by the line *c'd'*. This brings the table parallel to its former position at D. From c' draw lines to A and B, intersecting the corresponding lines drawn from d' in a' and b'. We now have a quadrilateral $a'b'c'd'$ similar to the quadrilateral formed by the true positions of the plotted points $abcd$, but it differs in size, since the distance $c'd'$ was assumed, and also in position (azimuth), since the table was not properly oriented at either station. Remove the alidade, and shift the tracing until the line $a'b'$ coincides with a and b on the sheet. Replace the alidade on the tracing, bringing it into coincidence with $c'a'$, $c'b'$, or $c'd'$, and *revolve the table on its axis* until the line of sight comes upon A, B, or D, as the case may be. The table is now oriented, when the true position of c may be readily found by resecting from a and b, which, when pricked through, gives its position on the sheet.

The student may show how the same result could have been obtained without the aid of tracing-paper.

If the fourth point D may be taken *in range* with A and B, the table may be properly oriented on this range, and a line drawn towards C from any point on this range line on the plot. Then C is occupied, and the table again properly oriented by this line just drawn, when the true position of c may be found by resecting from a and b, as before.

In general, if the table can be properly oriented over any unknown point from which sights may be taken to two or more known (plotted) points, the position of this unknown point is at once found by resection from the known points.

The student would do well to look upon the table and the attached plot as analogous to the graduated horizontal limb in the transit. The principles and methods of orienting are precisely similar, the pointings differing only in this, that with the transit the horizontal angle, referred to the meridian, is read off, recorded, and afterwards plotted, while with the plane table this bearing is immediately drawn upon the sheet.

131. The Measurement of Distances by Stadia.—This method of determining short distances is now generally used in connection with the plane table. It is fully discussed in chapter VIII., where the principles of its action and its use with the transit are given at length. The same principles, field methods, and tables apply to its use with the plane table, with such modifications as one accustomed to the use of the plane table would readily introduce. When used in this way it enables a point to be plotted from a single pointing, it being located by polar coördinates (azimuth and distance), instead of by intersections.

EXERCISES WITH THE PLANE TABLE.

132. Make a plane table survey of the college campus, measuring the length of one side for a base.

133. Having located several points on the sheet by *intersections*, occupy them and check their location by *resection*.

134. Locate a point (not plotted) by resection on three known points (art. 129).

135. Locate a point (not plotted) by resection on two known points, first taking the auxiliary point D not in line with AB, and then by taking it in line with AB. This gives a check on the position of the point, and shows the advantages of the second method when it is feasible.

CHAPTER VI.

ADDITIONAL INSTRUMENTS USED IN SURVEYING AND PLOTTING.

THE ANEROID BAROMETER.

136. The Aneroid Barometer consists of a circular metallic box, hermetically sealed, one side being covered by a corrugated plate. The air is mostly removed, enough only being left in to compensate the diminished stiffness of the corrugated cover at higher temperatures.

FIG. 29.

This cover rises or falls as the outer pressure is less or greater, and this slight motion is greatly multiplied and transmitted to an index pointer moving over a scale on the outer face. The motion of the index is compared with a standard mercurial barometer and the scale graduated accordingly. Inasmuch as all barometric tables are prepared for mercurial barometers, wherein the atmospheric pressure is recorded in inches of mercury, the aneroid barometer is graduated so that its readings are identical with those of the mercurial column.

Figure 29 shows a form of the aneroid designed for elevations to 4000 feet above or to 2000 feet below sea-level. It has a vernier attachment and is read with a magnifying-glass to single feet of elevation. It must not be supposed, however, that elevations can be determined with anything like this degree of accuracy by any kind of barometer. The barometer simply indicates the pressure at the given time and place, but for the same place the pressure varies greatly from various causes. All barometric changes, therefore, cannot be attributed to a change in elevation, when the barometer is carried about from place to place.

If two barometers are used simultaneously, which have been duly compared with each other, one at a fixed point of known elevation and the other carried about from point to point in the same locality, as on a reconnoissance, then the two sets of readings will give very close approximations to the differences of elevation. If the difference of elevation between distant points is desired, then long series of readings should be taken to eliminate local changes of pressure. The aneroid barometer is better adapted to surveys than the mercurial, since it may be transported and handled with greater ease and less danger. It is not so absolute a test of pressure, however, and is only used by exploring or reconnoissance parties. For fixed stations the mercurial barometer is to be preferred. It has been found from experience that the small aneroids of $1\frac{3}{4}$ to $2\frac{1}{2}$ inches diameter give as accurate results as the larger ones.

137. Barometric Formulæ.—In the following derivation of the fundamental barometric formula the calculus is used, so that the student will have to take portions of it on trust until he has studied that branch of mathematics. All that follows Eq. (4) he can read.

Let H = height of the "homogeneous atmosphere"* in lat. 45°.

h = corresponding height of the mercurial column.

δ = the relative density of the "homogeneous atmosphere" with reference to mercury.

z = difference of elevation between two points, with barometric readings of h' and h_1, at the higher and at the lower point respectively.

Then from the equilibrium between the pressures of the mercurial column and atmosphere we have:

$$h = \delta H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, for a small change in elevation, dz, the corresponding change in the height of the mercurial column would be

$$dh = \delta dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in (2) the value of δ as given by (1), we have:

$$dh = \frac{h}{H} dz;$$

or,

$$dz = H\frac{dh}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating (3) between the limits h' and h_1 we have:

$$z = H\int_{h'}^{h_1}\frac{dh}{h} = H \log_e \frac{h_1}{h'} \quad . \quad . \quad . \quad . \quad (4)$$

* "Homogeneous atmosphere" signifies a purely imaginary condition wherein the atmosphere is supposed to be of uniform density from sea-level to such upper limit as may be necessary to give the observed pressure at the observed temperature.

where the logarithm is in the Napierian system. Dividing by the modulus of the common system to adapt it to computation by the ordinary tables, we have:

$$z = 2.30258 H \log_a \frac{h_1}{h'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If H_0 be the height of the homogeneous atmosphere at a temperature of 32° F., and if h_0 be the height of the mercurial column at sea-level at same temperature, and if g_m and g_a be the specific gravities of mercury and air respectively, then, evidently,

$$H_0 g_a = h_0 g_m,$$

or,

$$H_0 = \frac{h_0 g_m}{g_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From experiment we have:

$$h_0 = 29.92 \quad \text{inches,}$$
$$g_m = 13.596$$
$$g_a = 0.001239$$

whence

$$H_0 = 26{,}220 \text{ feet.}$$

This is on the assumption that gravity is constant to this height above sea-level. When this is corrected for variable gravity we have:

$$H_0 = 26{,}284 \text{ feet.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Equation (7) gives the height of the homogeneous atmosphere at a temperature of 32° F. But since the volume of a gas under constant pressure varies directly as the temperature, and since the coefficient of expansion of air is 0.002034 for 1° F., we have for the height of the homogeneous atmosphere at any temperature:

$$H = H_0 \left[1 + 0.002034 (t - 32°)\right] \quad . \quad . \quad . \quad (8)$$

If the temperature chosen be the mean of the temperatures at the two points of observation, as t' and t_1 for the upper and lower points respectively, then we should have:

$$H = H_0 \left[1 + 0.002034\left(\frac{t'+t_1}{2} - 32\right)\right]$$
$$= 26{,}284\,[1 + 0.001017\,(t' + t_1 - 64)] \quad . \quad . \quad (9)$$

Substituting this value of H in Eq. (5) we obtain:

$$z = 60{,}520\,[1 + 0.001017\,(t' + t_1 - 64)] \log \frac{h_1}{h'} . \quad . \quad (10)$$

If we wish to refer this equation to approximate sea-level (height of mercurial column of 30 inches) and to a mean temperature of the two stations of 50° F., we may write:

$$\log \frac{h_1}{h'} = \log \frac{\frac{30}{h'}}{\frac{30}{h_1}} = \log \frac{30}{h'} - \log \frac{30}{h_1}.$$

Also, when $t' + t_1 = 100°$, we have

$$t' + t_1 - 64 = 36°.$$

Substituting these equivalents in eq. (10), we obtain

$$z = 60520\,(1 + 0.001017 \times 36)\left(\log \frac{30}{h'} - \log \frac{30}{h_1}\right),$$

$$z = 62737 \log \frac{30}{h'} - 62737 \log \frac{30}{h_1} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In this equation, the two terms of the second member represent the elevations of the upper and lower points respectively, above a plane corresponding to a barometric pressure of 30 inches and for a mean temperature of the two positions of 50° F.

Table I. is computed from this equation, the arguments being the readings of the barometer, h' and h_1 at the upper and lower stations respectively, the tabular results being elevations above an approximate sea-level. The difference between the two tabular results gives the difference of elevation of the two points, for a mean temperature of 50° and no allowance made for the amount of aqueous vapor in the air. For other temperatures, and for the effect of the humidity (which is not observed, but the average conditions assumed to exist), a certain correction needs to be applied, which correction is not an absolute amount, but is always a *certain proportion* of the difference of elevation as obtained from eq. (11) or table I. If the two elevations taken from the table be called A' and A_1, and the correction for temperature and humidity be C, we would have

$$z = (A' - A_1)(1 + C). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It is seen, therefore, that C is a coefficient which, when multiplied into the result obtained from table I., gives the correction to be applied to that result. The values of C are given in table II. for various values of $t' + t_1$.

The following example will illustrate the use of the tables:

TABLE I. BAROMETRIC ELEVATION.*

Containing $A = 62737 \log \frac{30}{h}$. Argument, h.

h.	A.	Dif. for .01.	h.	A.	Dif. for .01.	h.	A.	Dif. for .01.
Inches.	*Feet.*	*Feet.*	*Inches.*	*Feet.*	*Feet.*	*Inches.*	*Feet.*	*Feet.*
11.0	27,336	−24.6	14.0	20,765	−19.5	17.0	15,476	−16.0
11.1	27,090	24.4	14.1	20,570	19.3	17.1	15,316	15.9
11.2	26,846	24.2	14.2	20,377	19.1	17.2	15,157	15.8
11.3	26,604	24.0	14.3	20,186	18.9	17.3	14,999	15.7
11.4	26,364	23.8	14.4	19,997	18.8	17.4	14,842	15.6
11.5	26,126	23.6	14.5	19,809	18.6	17.5	14,686	15.5
11.6	25,890	23.4	14.6	19,623	18.6	17.6	14,531	15.4
11.7	25,656	23.2	14.7	19,437	18.5	17.7	14,377	15.4
11.8	25,424	23.0	14.8	19,252	18.4	17.8	14,223	15.3
11.9	25,194	22.8	14.9	19,068	18.2	17.9	14,070	15.2
12.0	24,966	22.6	15.0	18,886	18.1	18.0	13,918	15.1
12.1	24,740	22.4	15.1	18,705	18.0	18.1	13,767	15.0
12.2	24,516	22.2	15.2	18,525	17.9	18.2	13,617	14.9
12.3	24,294	22.1	15.3	18,346	17.8	18.3	13,468	14.9
12.4	24,073	21.9	15.4	18,168	17.6	18.4	13,319	14.7
12.5	23,854	21.7	15.5	17,992	17.5	18.5	13,172	14.7
12.6	23,637	21.6	15.6	17,817	17.4	18.6	13,025	14.6
12.7	23,421	21.4	15.7	17,643	17.3	18.7	12,879	14.6
12.8	23,207	21.2	15.8	17,470	17.2	18.8	12,733	14.4
12.9	22,995	21.0	15.9	17,298	17.1	18.9	12,589	14.4
13.0	22,785	20.9	16.0	17,127	16.9	19.0	12,445	14.3
13.1	22,576	20.8	16.1	16,958	16.9	19.1	12,302	14.2
13.2	22,368	20.6	16.2	16,789	16.8	19.2	12,160	14.2
13.3	22,162	20.4	16.3	16,621	16.7	19.3	12,018	14.1
13.4	21,958	20.1	16.4	16,454	16.6	19.4	11,877	14.0
13.5	21,757	20.0	16.5	16,288	16.4	19.5	11,737	13.9
13.6	21,557	19.9	16.6	16,124	16.3	19.6	11,598	13.9
13.7	21,358	19.8	16.7	15 961	16.3	19.7	11,459	13.8
13.8	21,160	19.8	16.8	15,798	16.2	19.8	11,321	13.7
13.9	20,962	−19.7	16.9	15,636	−16.0	19.9	11,184	−13.7
14.0	20,765		17.0	15,476		20.0	11,047	

* This table taken from Appendix 10, Report U. S. Coast and Geodetic Survey, 1881.

TABLE I. Barometric Elevation.—*Continued.*

Containing $A = 62737 \log \frac{30}{h}$. Argument, h.

h.	A.	Dif. for .01.	h.	A.	Dif. for .01.	h.	A.	Dif. for .01.
Inches.	*Feet.*	*Feet.*	*Inches.*	*Feet.*	*Feet.*	*Inches.*	*Feet.*	*Feet.*
20.0	11,047	−13.6	23.0	7,239	−11.8	26.0	3,899	−10.5
20.1	10,911	13.5	23.1	7,121	11.7	26.1	3,794	10.4
20.2	10,776	13.4	23.2	7,004	11.7	26.2	3,690	10.4
20.3	10,642	13.4	23.3	6,887	11.7	26.3	3,586	10.3
20.4	10,508	13.3	23.4	6,770	11.6	26.4	3,483	10.3
20.5	10,375	13.3	23.5	6,654	11.6	26.5	3,380	10.3
20.6	10,242	13.2	23.6	6,538	11.5	26.6	3,277	10.2
20.7	10,110	13.1	23.7	6,423	11.5	26.7	3,175	10.2
20.8	9,979	13.1	23.8	6,308	11.4	26.8	3,073	10.1
20.9	9,848	13.0	23.9	6,194	11.4	26.9	2,972	10.1
21.0	9,718	12.9	24.0	6,080	11.3	27.0	2,871	10.1
21.1	9,589	12.9	24.1	5,967	11.3	27.1	2,770	10.0
21.2	9,460	12.8	24.2	5,854	11.3	27.2	2,670	10.0
21.3	9,332	12.8	24.3	5,741	11.2	27.3	2,570	10.0
21.4	9,204	12.7	24.4	5,629	11.1	27.4	2,470	9.9
21.5	9,077	12.6	24.5	5,518	11.1	27.5	2,371	9.9
21.6	8,951	12.6	24.6	5,407	11.1	27.6	2,272	9.9
21.7	8,825	12.5	24.7	5,296	11.0	27.7	2,173	9.8
21.8	8,700	12.5	24.8	5,186	10.9	27.8	2,075	9.8
21.9	8,575	12.4	24.9	5,077	10.9	27.9	1,977	9.7
22.0	8,451	12.4	25.0	4,968	10.9	28.0	1,880	9.7
22.1	8,327	12.3	25.1	4,859	10.8	28.1	1,783	9.7
22.2	8,204	12.2	25.2	4,751	10.8	28.2	1,686	9.7
22.3	8,082	12.2	25.3	4,643	10.8	28.3	1,589	9.6
22.4	7,960	12.2	25.4	4,535	10.7	28.4	1,493	9.6
22.5	7,838	12.1	25.5	4,428	10.7	28.5	1,397	9.5
22.6	7,717	12.0	25.6	4,321	10.6	28.6	1,302	9.5
22.7	7,597	12.0	25.7	4,215	10.6	28.7	1,207	9.5
22.8	7,477	11.9	25.8	4,109	10.5	28.8	1,112	9.4
22.9	7,358	−11.9	25.9	4,004	−10.5	28.9	1,018	−9.4
23.0	7,239		26.0	3,899		29.0	924	

TABLE I. BAROMETRIC ELEVATIONS.—*Continued.*

Containing $A = 62737 \log \frac{30}{h}$. Argument, h.

h.	A.	Dif. for .01.	h.	A.	Dif. for .01.	h.	A.	Dif. for .01.
Inches.	*Feet.*	*Feet.*	*Inches.*	*Feet.*	*Feet.*	*Inches.*	*Feet.*	*Feet.*
29.0	924	−9.4	29.7	274	−9.2	30.4	361	−9.0
29.1	830	9.4	29.8	182	9.1	30.5	451	8.9
29.2	736	9.3	29.9	91	9.1	30.6	540	8.9
29.3	643	9.3	30.0	00	9.1	30.7	629	8.8
29.4	550	9.2	30.1	−91	9.0	30.8	717	8.8
29.5	458	9.2	30.2	181	9.0	30.9	805	−8.8
29.6	366	−9.2	30.3	271	−9.0	31.0	−893	
29.7	274		30.4	361				

TABLE II.

CORRECTION COEFFICIENTS TO BAROMETRIC ELEVATIONS FOR TEMPERATURE AND HUMIDITY.*

$t_1 + t'$.	C.	$t_1 + t'$.	C.	$t_1 + t'$.	C.
0°	−0.1025	60	−0.0380	120	+0.0262
5	− .0970	65	− .0326	125	+ .0315
10	− .0915	70	− .0273	130	+ .0368
15	− .0860	75	− .0220	135	+ .0420
20	− .0806	80	− .0166	140	+ .0472
25	− .0752	85	− .0112	145	+ .0524
30	− .0698	90	− .0058	150	+ .0575
35	− .0645	95	− .0004	155	+ .0626
40	− .0592	100	+ .0049	160	+ .0677
45	− .0539	105	+ .0102	165	+ .0728
50	− .0486	110	+ .0156	170	+ .0779
55	− .0433	115	+ .0209	175	+ .0829
60	− .0380	120	+ .0262	180	+ .0879

* This table compiled from tables I. and IV. of Appendix 10 of Report of the U. S. Coast and Geodetic Survey for 1881.

Example.

From observations made at Sacramento, Cal., and at Summit on the top of the Sierra Nevada Mountains, the annual means were:

$$h' = 23.288 \text{ in.} \qquad t' = 42.1\ F.$$
$$h_1 = 30.014 \text{ in.} \qquad t_1 = 59.9.$$

From table I. we have

$$A' = 6901.0 \text{ feet.}$$
$$A_1 = -\ 12.7 \quad \text{"}$$
$$A' - A_1 = 6913.7 \quad \text{"}$$

From table II. we find for $t' + t_1 = 102^\circ.0$, $C = +\ .0070$.
$\therefore\ z = 6913.7\ (1 + .0070) = 6962$ feet.

138. Use of the Aneroid.—The aneroid barometer should be carried in a leather case, and it should not be removed from it. It should be protected from sudden changes of temperature, and when observations are made it should have the temperature of the surrounding outer air It should not be carried so as to be affected by the heat of the body, and should be read out of doors, or at least away from all artificially warmed rooms. Always read it in a horizontal position. The index should be adjusted by means of a screw at its back, to agree with a standard mercurial barometer, and then this adjustment left untouched.

When but a single instrument is used it is advisable to *pass* between stations as rapidly as possible, but to *stop* at a number of stations during the day for a half-hour or so, reading the barometer on arrival and on leaving. The difference of these two readings shows the *rate of change* of barometric readings ...spheric conditions, and from these iso... ...*uous correction-curve* can be constructed on profile or cross-section paper from which the instrumental corrections can be taken for any hour of the day.* The observations should be repeated the same day in reverse order, the corrections applied as obtained from this correction curve, and the means taken. Observations should be made when the humidity of the air is as nearly constant as possible, and never in times of changeable or snowy weather.

Let the student measure the heights of buildings, hills, etc., and then test his results by level or transit.

THE PEDOMETER.

139. **The Pedometer** is a pocket-instrument for registering the number of paces taken when walking. It is generally

FIG. 30.—FRONT VIEW.

FIG. 31.—BACK VIEW.

made in the form of a watch, the front and back views being shown in Figs. 30 and 31.

* Mr. Chas. A. Ashburner, Geologist of the Penn. Geol. Survey, has used this method with good results.

When the instrument is attached to the belt in an upright position, as here shown, the jar given it at each step causes the weighted lever shown in Fig. 31 to drop upon the adjustable screw *S*. The lever recovers its position by the aid of a spring, and in so doing turns a ratchet-wheel by an amount proportional to the amplitude of the lever's motion. This may be adjusted to any length of pace by means of the screw *S*, which is turned by a key. The face is graduated like that of a watch, and gives the distance travelled in miles. This instrument may also be used on a horse, and when adjusted to the length of a horse's step will give equally good results. The accuracy of the result is in proportion to the *uniformity* of the steps, after having been adjusted properly for a given individual. The instrument is only used on explorations, preliminary surveys, and reconnoissance-work.

The Length of Men's Steps has been investigated by Prof. Jordan,* of the Hanover Polytechnic School. From 256 step-measurements by as many different individuals, of lines from 650 to 1000 feet in length, carefully measured by rods and steel tapes, he concludes that the average length of step is 2.648 feet, ranging from 2.066 to 3.182 feet. The mean deviation from this amount for a single measurement was $\pm$ 0.147 feet, or $5\frac{1}{2}$ per cent. The average age of the persons making these step-measurements was 20 years. The length of step decreases with the age of the individual after the age of 25 to 30 years. It is also proportional to the height of the person. The results for 18 different-sized persons gave the following averages:

Height of person	5'.08	5'.25	5'.41	5'.58	5'.74	5'.90	6'.07	6'.23	6'.40	6'.56
Length of step..	2.46	2.53	2.56	2.59	2.62	2.69	2.72	2.76	2.79	2.85

* See translation in *Engineering News* for July 25, 1885.

On slopes the step is always shorter than on level ground, whether one goes up or down. The following averages from the step-measurement of 136 lines on mountain-slopes along trails were found:

Slope	0°	5°	10°	15°	20°	25°	30°
Length of step in ascending	2'.53	2'.30	2'.03	1'.84	1'.64	1'.48	1'.25
Length of step in descending	2'.53	2'.43	2'.36	2'.30	2'.20	1'.97	1'.64

The length of the step is also found to increase with the length of the foot. One steps farther when fresh than when tired. The increase in the length of the step is also in nearly direct proportion to the increase of speed in walking.

When the proper personal constants are determined, and when walking at a constant rate, distances can be determined by pedometer, or by counting the paces, to within about two per cent of the truth. *One should always take his natural step*, and not an artificial one which is supposed to have a known value, as three feet, for instance. Let a base be measured off and each student determine the length of his natural step when walking at his usual rate, or, what is the same thing, find how many paces he makes in 100 feet. He then has always a ready means of determining distances to an approximation, which in many kinds of work is abundantly sufficient.

THE ODOMETER.

140. **The Odometer** is an instrument to be attached to the wheel of a vehicle to record the number of revolutions made by it. One form of such an instrument is shown in Fig. 32 attached to the spokes of a wheel.

Each revolution is recorded by means of the revolution of an axis with reference to the instrument, this axis really being held stationary by means of an attached pendulum which does not revolve. The instrument really revolves about this fixed axis at each revolution of the wheel, and the number of times

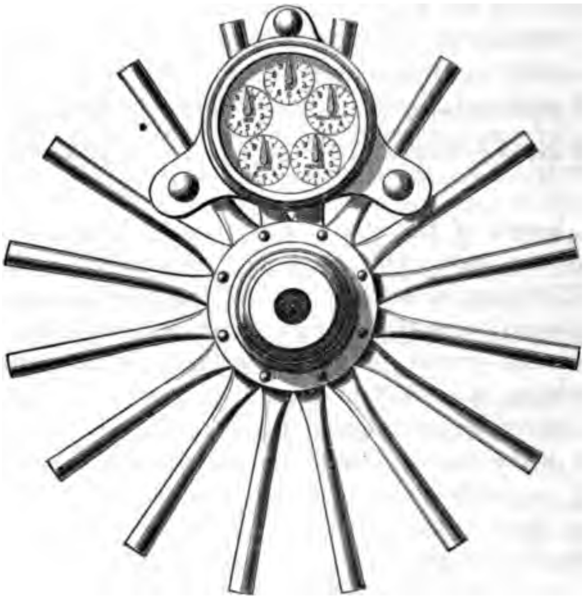

FIG. 32.

it does this is properly recorded and indicated by appropriate gearing and dials.

This method of measuring distances is more accurate than by pacing, as the length of the circumference of the wheel is a constant. This length multiplied by the number of revolutions is the distance travelled. It is mostly used by exploring parties and in military movements in new countries which have not been surveyed and mapped.

THE CLINOMETER.

141. The Clinometer is a hand-instrument for determining the slope of ground or the angle it makes with the horizon. It consists essentially of a level bubble, a graduated arc, and a line of sight, so joined that when the line of sight is at any angle to the horizon the bubble may be brought to a central position and the slope read off on the graduated arc. Such a combination is shown in Fig. 33. It is called the Abney level and

FIG. 33.

clinometer, being really a hand-level when the vernier is set to read zero. The position of the bubble is visible when looking through the telescope, the same as with the Locke hand-level, shown in Fig. 16, p. 82. The body of the tube is made square, so that it may be used to find vertical angles of any surface by placing the tube upon it and bringing the bubble to the centre. The graduations on the inner edge of the limb give the slope in terms of the relative horizontal and vertical components of any portion of the line; thus, a slope of 2 to 1 signifies that the horizontal component is twice the vertical. In reading this scale the edge of the vernier-arm is brought into coincidence with the graduation.

This instrument is very useful in giving approximate slopes in preliminary surveys, the instrument being pointed to a position as high above the ground as its own elevation when held to the eye.

THE OPTICAL SQUARE.

142. **The Optical Square** is a small hand-instrument used to set off a right angle. It is shown in Fig. 34, the method of

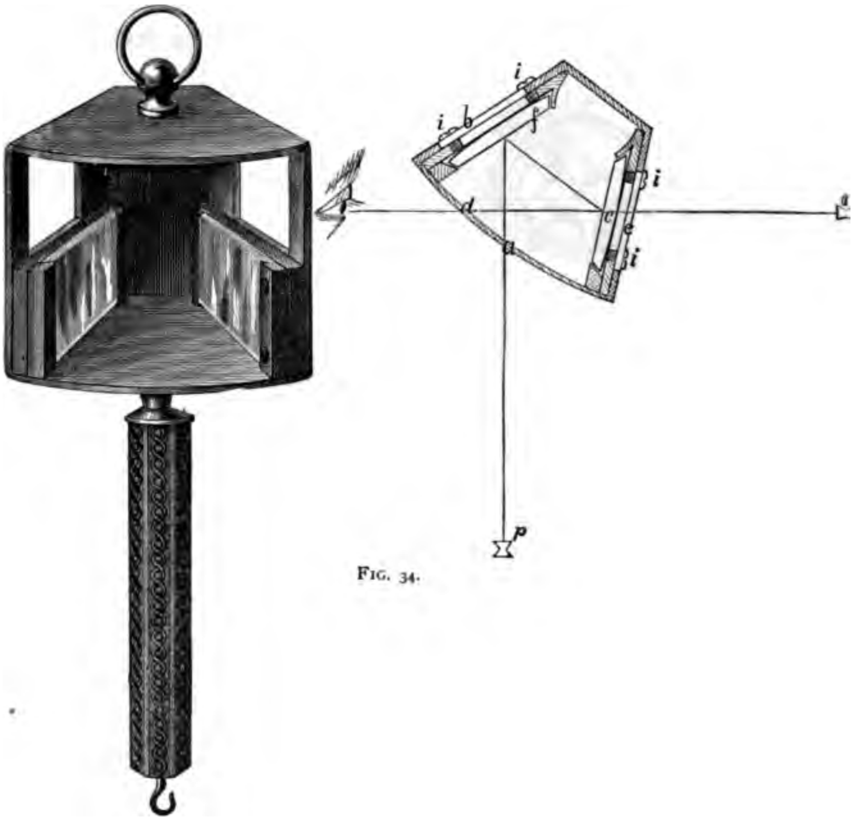

FIG. 34.

its use being evident from the figure. Thus, while the rod at *o* is seen directly through the opening, the rod at *p* is seen in the glass as the prolongation downwards of that of *o*, it being reflected from the mirrors *f* and *c* in succession, they having an angle of 45° with each other. By this means a line may be located at right angles to a given line at a given point, or a point in a given line may be found in the perpendicular to this line from a given point.

THE PLANIMETER.

143. **The Planimeter** is an instrument used for measuring areas that have been drawn to scale. It is a marked example of high mathematical analysis embodied in a very simple and useful mechanical appliance. Three of the best forms of the instrument will be described.

FIG. 35.

Fig. 35 is Amsler's Polar Planimeter. It consists of a metal arm, *ei*, carrying a needle point at *e* and pivoted at *i* to a frame through which slides a second metal arm, *h*, and to which is attached an axis, *ab*, carrying a rolling wheel, *c*, and a worm gear which turns a record disk, *l*. The arm *h* may be adjusted so that any required length, within the limits of the instrument, from pivot *i* to tracing point *d* may be used.

When in use the instrument rests on three points, *e*, *d*, and the circumference of the wheel *c*. To measure an area, the needle point *e* is fastened in the drawing in a convenient position, and the point *d* made to circumscribe the required area.

This causes the wheel c to rotate, and the number of revolutions made, as indicated by the record disk l and the vernier m, multiplied by a constant, is the required area. The determination of this constant, involving the theory of the instrument, will be given in such form as to be intelligible to students who have not studied the calculus.

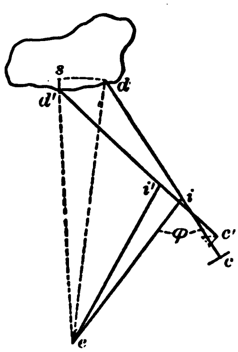

FIG. 36.

144. Theory of the Polar Planimeter.*—In Fig. 36 the essential parts of the instrument are lettered as in Fig. 35. The instrument is so constructed that the angle φ can never be less than 0° nor more than 180°; that is, neither d nor c can cross ci.

Any infinitesimal portion of d's path, in circumscribing an area, as dd', may be conceived to be the resultant of two infinitesimal component motions, ds and sd'; ds being described by a motion of d about e as a center, the angle φ remaining fixed, while sd' is motion of d directly toward e, or normal to ds, the angle φ changing in value. Each of these motions has its due effect in moving the wheel c and causing it to turn. For a given motion of either class the amount the wheel c will turn depends on the value of the angle φ, as will presently be shown. It is evident that, in circumscribing a closed area, the tracing point will move as much *toward e* as it does *from e*, and that for each element of motion toward e there will be a corresponding element from e with an equal value of φ. Hence the resulting turning of the wheel for *radial* motion of d is *nil* and need not be considered.

* The following demonstration has been given the author by Prof. Wm. G. Raymond, Rens. Poly. Inst., formerly of the University of California.

Corresponding right and left elements of circumferential motion of d are not made with equal values of φ, and hence the final record of the wheel is due to this class of motion. To show the effect of changes in φ (Fig. 36a):

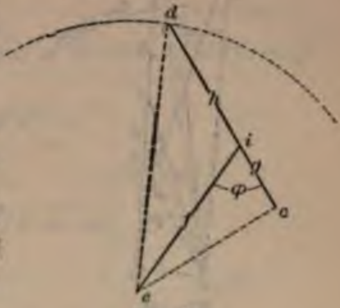

FIG. 36a.

When d is so situated that $e\,c\,d$ is a right angle, motion about e, φ remaining fixed, causes no rolling of the wheel c. For c rotates about e as a center with radius $e\,c$, which is normal to the axis of the wheel, and hence the axis of the wheel lies in the direction of motion and the wheel slips. The circumference that would thus be traced by d, φ remaining fixed, in a complete revolution about e, is called **the zero circumference**. Its radius $e\,d$ is easily shown to be

$$R = \sqrt{f^2 + h^2 + 2\,g\,h},$$

and

$$\varphi = \cos^{-1} \frac{g}{f}.$$

For φ less than this value, *i. e.*, d farther from e, right-handed, or clockwise, motion of d about e will cause c to partly slip and partly roll, and a moment's consideration will show that, looking from c to d, its roll will be clockwise. Rolling in this direction is called ***positive*** and the wheel is accordingly graduated.

If, on the other hand, d be brought nearer to e than the zero circumference, clockwise motion of d about e causes c to both slip and roll, and the roll is counter clockwise, or ***negative***.

Left-handed motion of d will cause c's roll to be ***negative*** for ***d outside*** the zero circumference and ***positive*** for ***d inside*** the zero circumference.

To determine the relation of the roll of the wheel to the area circumscribed (Fig. 36*b*):

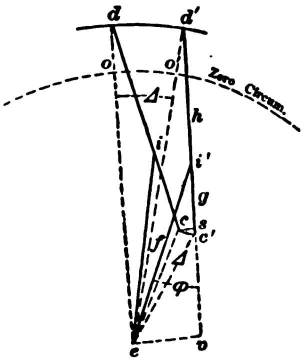

FIG. 36*b*.

Let $d\,d'$ be an infinitesimal circumferential component of d's motion due to the motion of the whole instrument about e through the infinitesimal angle Δ.

The wheel c moves through the arc $c\,c'$, partly rolling and partly slipping. The roll is that component of its motion $c\,c'$ normal to its axis. This component may be considered, in the infinitesimal triangle $c\,c's$, to be $c\,s$, normal to $c'i'$.

Let P be the arc corresponding to the small angle Δ when the radius is unity. Then

$$c\,c' = P \,.\, c'e,$$

and the *roll* of the wheel is

$$c\,s = P \,.\, c'e \,.\, \cos c'c\,s.$$

Since the angle Δ is very small, $c\,c'$ may be considered normal to $c'e$, hence

$$c'c\,s = e\,c'v,$$

$e\,v$ being a perpendicular on $i'c'$ produced.

Then $$c'v = e\,c' \,.\, \cos e\,c'v$$
$$= e\,c' \cos c'c\,s.$$

Whence, $$c s = P . c'v.$$

Also, $$c'v = f \cos \varphi - g.$$

Whence, $$c s = P (f \cos \varphi - g). \tag{1}$$

And this is the roll of the wheel due to motion of d through $d d'$.

To get an expression for the area $d d' o' o$:

By Trigonometry,

$$e d = \sqrt{f^2 + h^2 + 2 f h \cos \varphi},$$

$$d d' = e d . P.$$

The area of a sector is its arc multiplied by half its radius; whence,

$$\text{Area } e d d' = \tfrac{1}{2} P (f^2 + h^2 + 2 f h \cos \varphi). \tag{2}$$

Similarly, using the value of eo previously found,

$$\text{Area } e o o' = \tfrac{1}{2} P (f^2 + h^2 + 2 g h). \tag{3}$$

Subtracting (3) from (2), it is found that

$$\text{Area } d d' o' o = P h (f \cos \varphi - g), \tag{4}$$

which, it is observed, is equation (1) multiplied by h, whence the

Proposition.—*The distance rolled by the wheel c, for given* ***motion*** *of d, when multiplied by h, gives the area included be-*

tween d s path, the zero circumference, and the radial lines to d's initial and final position.

To show that in circumscribing a closed area with the tracing point d the roll of the wheel is correctly summed for motion of d right handed and left handed, and inside or outside the zero circumference :

In Fig. 36c, let the area $d\,d_1d_2d_3$ be traced, beginning at d, and moving clockwise. For motion from d to d_1 the wheel rolls positively an amount equal to the area $d\,d_1o'o$ divided by h; for motion from d_1 to d_2 the roll of the wheel will be neutralized by motion from d_3 to d; for motion from d_2 to d_3 the roll of the wheel is negative and equal to the area $d_2o'o\,d_3$ divided by h. Hence the resulting roll of the wheel due to circumscribing the area $d\,d_1d_2d_3$ is that area divided by h.*

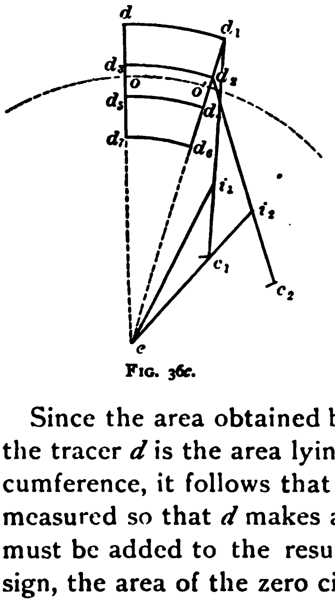

FIG. 36c.

Since the area obtained by the wheel for a given motion of the tracer d is the area lying between d's path and the zero circumference, it follows that if e is placed within the area to be measured so that d makes a complete revolution about e there must be added to the result obtained, paying attention to its sign, the area of the zero circumference.† It is

$$A = \pi\,(f^2 + h^2 + 2\,h\,g);$$

* Let the student reason similarly of the areas $d_3d_2d_4d_5$ and $d_5d_4d_6d_7$.

† Let the student discuss with a diagram the three cases: 1. Perimeter wholly without the zero circumference; 2. Perimeter wholly within the zero circumference; and, 3. Perimeter partly within and partly without the zero circumference.

h being known, if f and g are not furnished by the maker, the area may nevertheless be found by circumscribing a known area with the point e within it and comparing it with the result obtained by the instrument.

Since the roll of the wheel for each area circumscribed must be multiplied by h to obtain the area, and since the roll is equal to the circumference of the wheel multiplied by the number of revolutions (the quantity read from the scale), it becomes necessary to determine values for h for the various kinds of areas that may be measured, such that the work of multiplying may be a minimum.

145. To Find the Length of Arm for Given Conditions.—Let c be the circumference of the wheel and n the number of revolutions for a given area, A; then

$$A = h\,n\,c.$$

Since c is a fixed quantity, $h\,c$ may be placed equal to a convenient constant and a value for h determined. The only caution to be observed is that h must be made a convenient length for use. If it is desired to measure square inches,

let $$h\,c = 10;$$

whence, $$h = \frac{10}{c}$$

Since c is usually between two inches and three inches, this will give a convenient value for h, which value being set, any area circumscribed by d is given by

$$A = 10n.$$

The reading of the instrument is usually to one one-thousandth of a revolution, the disk recording whole revolutions and the scale and vernier fractional revolutions.

The value of c is usually furnished by the maker, but for an additional charge. If not so furnished, it may be found thus: Circumscribe any known area, with e outside of it, with a known length of arm h, and note the record n, of the wheel.

Then $$c = \frac{A}{h\,n}.$$

If the arm h is not graduated, it may be set by trial so that $A = 10n$ when the value of c is not required. If a diagram be drawn to a scale of f feet per inch, each square inch of paper represents f^2 square feet of actual area; and if it is desired to avoid multiplication, the arm h may be set so that a single revolution of the wheel shall correspond to a convenient number of units of actual area. Let A be the actual area represented by a on the diagram; then

$$f^2 a = A,$$

and $$a = h\,n\,c;$$

whence, $$A = f^2 h\,n\,c.$$

$f^2 h\,c$ may be assumed a convenient quantity and h determined.

Example.—The scale of a diagram is 40 feet per inch; then

$$A = 1600\,h\,n\,c.$$

Assume $$1600\,hc = 20000;$$

then $$h = \frac{20000}{1600\,c};$$

which will give a practical working value for h, and the figures representing the record of the wheel for any area circumscribed, without decimal point, multiplied by 20, will give the area in square feet. The instrument is supposed to read to thousandths of a revolution. If the horizontal scale is f feet per inch and the vertical scale is f' feet per inch, the equation for determining h would stand

$$A = ff'hnc.$$

Similarly, if a number of areas are to be determined and multiplied by a constant length, d, as in railroad earth-work, the expression for a single volume v_1 of area a_1 would be, dividing by 27 to reduce to cubic yards,

$$v_1 = \frac{f^2 a_1 d}{27};$$

but $$a_1 = hnc;$$

whence, $$v_1 = \frac{f^2 d}{27} hnc.$$

A convenient quantity may be chosen for $\frac{f^2 d}{27} hc$ and h determined.

The sum of a series of volumes, each of length d, is

$$V = v_1 + v_2 + v_3 + \text{\&c.} = \Sigma v$$

$$= \Sigma \frac{f'd}{27} hnc$$

$$= \frac{f'd}{27} hc\,\Sigma n;$$

whence it is only necessary to circumscribe each area in succession, add the records of the wheel, and apply the coefficient. If all the areas are platted from the same base-line the tracer may be moved continuously around the areas, traversing each area as many times as it is used in the summing, and only the final single record read, which is of course the sum of all the records. A great variety of other problems may be solved with the instrument.

146. The Suspended Planimeter.—This is shown in Fig. 37. It is essentially a polar planimeter, the pole being at *C*.

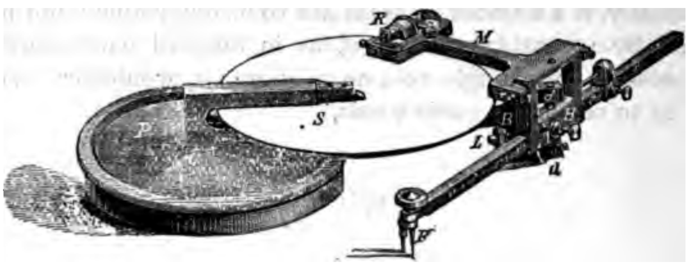

FIG. 37.

It has the advantage of allowing the wheel to move over the smooth surface of the plate *S*, instead of over the paper, thus giving an error about one sixth as great as that of the ordinary polar instrument. The theory of its action is essentially the same as the other.

147. The Rolling Planimeter is the most accurate instrument of its kind yet devised. Its compass is also indefinitely increased, since it may be rolled bodily over the sheet for any distance, on a right line, and an area determined within certain limits on either side. It is therefore especially adapted to the measuring of cross-sections, profiles, or any long and narrow surface. Fig. 38 shows one form of this instrument as designed by Herr Corradi of Zurich. It is a suspended planimeter, inasmuch as the wheel rolls on a flat disk which is a part of the instrument, but it could not be called a polar planimeter, the theory of its action being very different from that instrument. The frame B is supported by the shaft carrying

FIG. 38.

the two rollers R_1. To this frame are fitted the disk A and the axis of the tracing-arm F. The whole apparatus may thus move to and fro indefinitely in a straight line on the two rollers while the tracing-point traverses the perimeter of the area to be measured. The shaft carries a bevel-gear wheel, R_2, which moves the pinion R_3. This pinion is fixed to the axis of the disk, and turns with it, so that any motion of the rollers R_1 causes the disk to revolve a proportional amount, and the component of this motion at right angles to the axis of the wheel E is recorded on that wheel. If the instrument remains stationary on the paper (the rollers R not turning) and the tracing-point moved laterally, it will cause no motion of the wheel, since its axis is parallel to the arm F, and turns about the same axis with F, but 90° from it; the wheel E, therefore, moves parallel with its axis and does not turn.

148. Theory of the Rolling Planimeter.—This will be developed by a system of rectangular coördinates, the path of the fulcrum of the tracing-arm being taken as the axis of

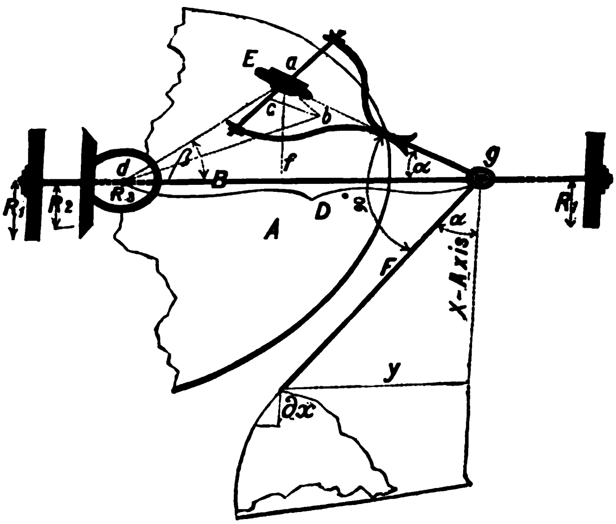

FIG. 39.

abscissæ. The path of the tracing-point will be considered as made up of two motions, one parallel to the axis of abscissæ and the other at right angles to it. The elementary area considered will be that included between the axis of abscissæ and two ordinates drawn to the extremities of an elementary portion of the path. But since this element of the perimeter is supposed to be made up of two right lines, one perpendicular to the axis of abscissæ and the other parallel to it, our elementary area must also be divided in a similar manner. It will at once be seen that one part of this area is zero, since the two ordinates bounding it form one and the same line. This is the part generated by the motion at right angles to the axis of abscissæ. Now, we have just shown in the previous article that the wheel-record for this part of the path is also zero.* We are brought therefore to this important conclusion: *that all components of motion of the tracing-point at right angles to the axis of abscissæ have no influence upon the result.* We will therefore only discuss a differential motion of the tracing-point in the direction of the axis of abscissæ.

In Fig. 39, which is a linear sketch of the instrument shown in Fig. 38, with the corresponding parts similarly lettered, it is to be shown that the motion of the wheel E caused by the movement of the tracing-point over the path dx is equal to the corresponding area ydx multiplied by some constant which is a function of the dimensions of the instrument.

It is evident that a motion of the tracing-point in the direction of the axis of abscissæ can only be obtained by moving the entire instrument on the rollers by the same amount, and therefore when the point moves over the path dx the circumferences of the rollers R_1 have moved the same amount. This causes a movement of the pitch circle of R_2 of $dx \frac{R_2}{R_1}$. This motion is conveyed to the disk through R_3, so that any point on this disk, as a, distant ad from the axis, moves through a

* This is not strictly correct, although it leads to correct results. The components of motion are really one parallel to the axis of x, and one about the pivot g as a center. This latter movement makes no record on the wheel, but it does generate an elementary area included between this element of the path, the ordinates to its extremities, and the axis of x. These areas are either added to or cut from the areas made by the onward element of motion in such a way as to exactly balance when the tracing point starts from and returns to the axis of x, or when it closes on the initial point, wherever it may be.

distance equal to $dx \dfrac{R_2}{R_1} \cdot \dfrac{ad}{R_3}$. Let *ab*, Fig. 39, be this distance. Then we have

$$ab = dx \frac{R_2}{R_1} \cdot \frac{ad}{R_3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The motion of that portion of the disk on which the roller rests, equal to *ab*, causes the circumference of the wheel E to revolve by an amount equal to the component of the distance *ab* perpendicular to the axis of the wheel. This component part of the disk's motion is *bc*, and this is the measure of the wheel's motion. It therefore remains to show that $bc = ydx$ multiplied by an instrumental constant.

Now, $$bc = ab \sin bac. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But $bac = \alpha + \beta$, since *gac* and *bad* are both right angles. Also, $bac =$ supplement of $dag = a + \beta$.

Also, from the triangle *dag*, we have

$$\sin dag : \sin agd :: D : ad,$$

or
$$\sin (\alpha + \beta) = \frac{D \sin \alpha}{ad}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since *Fga* is also a right angle, we have the angle formed by *Fg* and the axis of abscissæ equal to α, whence $\sin \alpha = \dfrac{y}{F}$.

We may now write:

$$bc = ab \sin (\alpha + \beta) = ab \frac{D \cdot \sin \alpha}{ad} = ab \frac{Dy}{F \cdot ad}. \quad . \quad (4)$$

Now, substituting the value of *ab* from (1), we have

$$cb = ydx \frac{D \cdot R_2}{F \cdot R_1 \cdot R_3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since D, R_2, F, R_1, and R_3 are all constants for any one instrument, we see that the wheel-record is a function of the area generated by the tracing-point and the instrumental constants, which was to be shown. It now follows that the summation of all these elementary areas included between the path of the tracing-point and the axis of abscissæ, is represented by the total *wheel-movement*, or the difference between its initial and final readings, provided the tracing-point starts from and closes on the axis of x, or closes on the starting-point, wherever that may be.

The following comparison between the rolling and the polar planimeters may be made: The axis of x corresponds to the zero circle; the unrecorded motion about the pivot g, Fig. 39, corresponds to the balanced record of the motion about c, Fig. 36; and the significant forward motion of the former to the motion about P as a center, in the latter.

As in the case of the polar instrument, the proper length of arm F, to be used with the rolling-planimeter to give results in any desired unit, depends on the other instrumental constants. These being known, the value of F may be computed in the same manner as with the polar planimeter.

149. To test the Accuracy of the Planimeter, there is usually provided a brass scale perforated with small holes. A needle-point is inserted in one of these and made fast to the paper or board, while the tracing-point rests in another. The latter may now be moved over a fixed path with accuracy. Make a certain number of even revolutions forward, or in the direction of the hands of a watch, noting the initial and final readings. Reverse the motion the same number of revolutions, and see if it comes back to the first reading. If not, the discrepancy is the combined instrumental error from two measurements due to slip, lost motion, unevenness of paper, etc.

If this test be repeated with the areas on opposite sides of the zero-circle in the case of the polar-planimeter, or on opposite sides of the axis of abscissæ in case of the rolling-planimeter, with the same score in both cases, it proves that the pivot-points *a*, *b*, *k*, and the tracing-point *d* (Fig. 35), are in the same straight line, in case of the polar instrument, and that the corresponding points in the suspended and rolling planimeters form parallel lines; in other words, that the axis of the measuring-wheel is parallel to the tracing-arm. If the results differ when the areas lie on opposite sides of the axis or zero-circle, these lines are not parallel and must be adjusted to a parallel position.

150. **Use of the Planimeter.**—The paper upon which the diagram is drawn should be stretched smooth on a level surface. It should be large enough to allow the rolling-wheel to remain on the sheet.

The instrument should be so adjusted and oiled that the parts move with the utmost freedom but without any lost motion. This requires that all the pivot-joints shall be adjustable to take up the wear. The rim of the measuring-wheel must be kept bright and free from rust. The instrument must be handled with the greatest care. Having set the length of the tracing-arm for the given scale and unit, it is well to test it upon an area of known dimensions before using. If it be found to give a result in error by $\frac{1}{n}$ of the total area, the length of the tracing-arm must be changed by an amount equal to this same ratio of its former length. If the record made on the wheel was too small then the length of the tracing-arm must be diminished, and *vice versa*. If the paper has shrunk or stretched, find the proportional change, and change the length of the tracing-arm *from its true length as just found*, by this same ratio, making the arm longer for stretch and shorter for shrinkage. Or the true length of arm may be used, and the results corrected for change in paper.

To measure an area, first determine whether the fixed point, or pole, shall be inside or outside the figure. It is preferable to have it outside when practicable, since then the area is obtained without correction. If, however, the diagram is too large for this (in case of the polar planimeter) the pole may be set inside. In either case inspection, and perhaps trial, is necessary to fix upon the most favorable position of the pole, so that the tracing-point may most readily reach all parts of the perimeter. If the area is too large for a single measurement, divide it by right lines and measure the parts separately. Having fixed the pole, set the tracing-point on a well-defined portion of the perimeter, and read and record the score on the rolling-wheel and disk. This is generally read to four places. Move the tracing-point carefully and slowly over the outline of the figure, in the direction of the hands of a watch, around to the initial point. Read the score again. If the pole is outside the figure, this result is always positive when the motion has been in the direction here indicated. If the pole is inside the figure, the result will be negative when the area is less than that of the zero-circle, positive if greater. With the pole inside the figure, however, the area of the zero-circle must always be added to the result as given by the score, and when this is done the sum is always positive, the motion being in the direction indicated. The area of this zero-circle is found in art. 144. to be $\pi\ (m^2 + l^2 + 2nl)$. The value of l, which is the length of the tracing-arm, is known. The values of m and n should be furnished by the maker. If these are unknown, the area of the zero-circle can be found for any length of arm l, by measuring a given area with pole outside and inside, the difference in the two scores being the area of this circle. By doing this with two very different values of l we may obtain two equations with two unknown quantities, m and n, from which the absolute values of these quantities may be found. Thus we would have:

$$A = \pi\,(m^2 + l^2 + 2nl);$$
$$A' = \pi\,(m^2 + l'^2 + 2nl');$$

whence
$$m^2 + 2nl = \frac{A}{\pi} - l^2;$$

$$m^2 + 2nl' = \frac{A'}{\pi} - l'^2;$$

wherein l, l', A, and A' are known. The values of m and n are then readily found.

In using the rolling-planimeter, it is advisable to take the initial point in the perimeter on the axis of abscissæ, as in this position any small motion of the tracing-point has no effect on the wheel, and so there is no error due to the initial and final positions not being exactly identical.

The planimeter may be used to great advantage in the solution of many problems not pertaining to surveying. In all cases where the result can be represented as a function of the product of two variables and one or more constants, the corresponding values of the variables may be plotted on cross-section paper by rectangular coördinates, thus forming with the axis and end-ordinates an area which can be evaluated for any scale and for any value of the constant-functions by setting off the proper length of tracing-arm. Thus, from a steam-indicator card the horse-power of the engine may be read off, and from a properly constructed profile the amount of earth-work in cubic yards in a railway cut or fill. Some of these special applications are further explained in Part II. of this work.

151. Accuracy of Planimeter-measurements.—Professor Lorber, of Loeben, Austria, has thoroughly investigated the relative accuracy of different kinds of planimeters, and the results of his investigations are given in the following table. It will be seen that the relative error is less as the area measured is larger. The absolute error is nearly constant for all areas, in the polar planimeter. The remarkable accuracy of the rolling-planimeter is such as to cause it to be ranked as an instrument of precision.

TABLE OF RELATIVE ERRORS IN PLANIMETER-MEASUREMENTS

AREA IN—		The error in one passage of the tracer amounts on an average to the following fraction of the area measured by—		
Square cm.	Square inches.	The ordinary polar planimeter-Unit of vernier: 10 sq. mm. = .015 sq. in.	Suspended planimeter-Unit of vernier: 1 sq. mm. = .001 sq. in.	Rolling planimeter-Unit of vernier: 1 sq. mm. = .001 sq. in.
10	1.55	$\frac{1}{75}$	$\frac{1}{625}$	$\frac{1}{1000}$
20	3.10	$\frac{1}{145}$	$\frac{1}{1111}$	$\frac{1}{2000}$
50	7.75	$\frac{1}{355}$	$\frac{1}{2500}$	$\frac{1}{3000}$
100	15.50	$\frac{1}{642}$	$\frac{1}{3147}$	$\frac{1}{5000}$
200	31.00	$\frac{1}{1274}$	$\frac{1}{7143}$	$\frac{1}{7693}$
300	46.50	....	$\frac{1}{9375}$	$\frac{1}{10000}$

THE PANTOGRAPH.

152. The Pantograph is a kind of parallel link-motion apparatus whereby, with one point fixed, two other points are made to move in a plane on parallel lines in any direction. The device is used for copying drawings, or other diagrams to the same, a larger, or a smaller scale. The theory of the instrument rests on the following:

PROPOSITION: ***If the sides of a parallelogram, jointed at the corners A, B, C, and D, and indefinitely extended, be cut by a right line in four points, as E, F, G, and H, then these latter points will lie in a straight line for all values of each of the parallelogram angles from zero to 180°, and the ratio of the distances EF, FG, and GH will remain unchanged.***

In Fig. 40, let A, B, C, D be the parallelogram, whose sides (extended) are cut by a right line in F, G, E, and H. It is evident that one point in the figure may remain fixed while

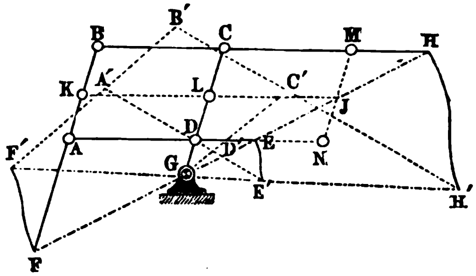

FIG. 40.

the angles of the parallelogram change. Let this point be G.

Since GC and GH, radiating from G, cut the parallel lines DE and CH, we have

$$GD : DE :: GC : CH.$$

Also, for similar reasons,

$$ED : DG :: EA : AF.$$

Now since the sides of the parallelogram, as well as all the intercepts, AF, GD, DE, and CH, remain constant as the angles of the figure change, when the figure has taken the position shown by the dotted lines, we still have

$$GD' : D'E' :: GC' : C'H';$$

also,

$$E'D' : D'G :: E'A' : A'F'.$$

From the first of these proportions we know that G, E', and H' are in the same straight line, and the same for G, E', and F; therefore, they are all four in the same straight line.

To show that they are the same relative distance apart as before we have,

$$FG : GE : EH :: BC : DE : CH - DE;$$

also,

$$FG : GE' : E'H' :: B'C' : D'E' : C'H' - D'E'.$$

But

$$BC = B'C',\ DE = D'E',\ \text{and}\ CH - DE = C'H' - D'E';$$

therefore we may write,

$$FG : GE : EH :: FG : GE' : E'H'.$$

Q. E. D.

It is evident that two of the points E, F, G, and H may become one by the transversal passing through the point of intersection of two of the sides of the parallelogram. The above proposition would then hold for the three remaining points.

In the Pantograph only three of the four points E, F, G, and H (Fig. 40) are used. One of these *may* therefore be taken at the intersection of two sides of the parallelogram, but it is not necessarily so taken. These three points are: the fixed point, the tracing-point, and the copying-point.

In Fig. 41, F is the fixed point, held by the weight P; B is the tracing-point, and D is the copying-point, or *vice versa* as to B and D. The parallelogram is E, G, B, H. The points

F, *B*, and *D* must lie in a straight line, *B* being at the intersection of two of the sides of the parallelogram. The points *A*, *E*, and *C* are supported on rollers. In Fig. 42, the fixed-

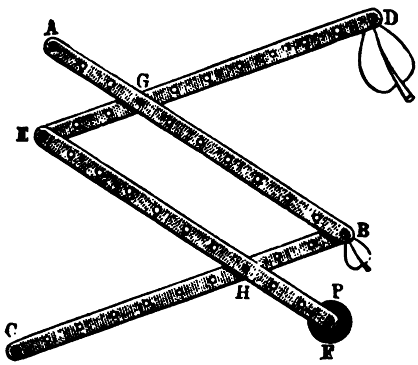

FIG. 41.

point is the point of intersection of two of the sides of the parallelogram. The upper left-hand member of the frame is not essential to its construction, serving simply to stiffen the

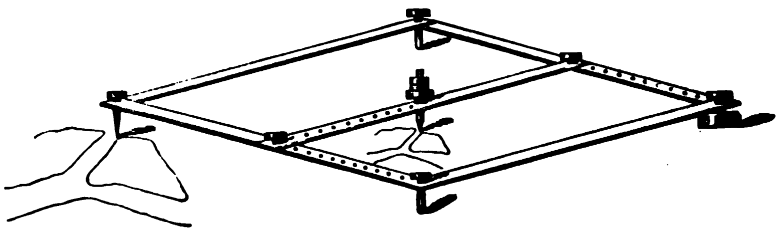

FIG. 42.

copying-arm, the fourth side of the parallelogram being the side holding the tracing-point.

In Fig. 43, neither of the three points is at the intersection of two of the sides of the parallelogram, and hence there is a fourth point unused, having the same properties as the fixed, tracing, and copying points, it being at the intersection of the line joining these three points with the fourth side of the parallelogram.

From the theoretical discussion, and from the figures shown, it becomes evident that there may be an indefinite variety of

FIG. 43.

combinations which will do the work of a pantograph. *The only essential conditions are that the fixed, the tracing, and the copying points shall lie in a straight line on at least three sides of a jointed parallelogram, either point serving any one of the three purposes.*

153. **Use of the Pantograph.**—The use of the instrument is easily acquired. Since both the tracing and copying points should touch the paper at all times, such a combination as that shown in Fig. 41 is preferable to those shown in Figs. 42 and 43, since in these latter the tracing point is surrounded by supported points, and so would not touch the paper at all times unless the paper rested on a true plane. In most instruments where the scale is adjustable, the two corresponding changes in position of tracing and copying points for different scales are indicated. To test these marks, see that the adjustable points are in a straight line with the fixed point, and to test the scale see that the ratio $\frac{FD}{FB}$ (Fig. 41) is that of the reduction

desired. Thus, if the diagram is to be enlarged to twice the original size, make $FD = 2FB$;

or make $$\frac{DE}{GE} = \frac{FE}{FH} = \text{scale of enlargement.}$$

If the drawing is to be reduced in size, make B the copying-point and D the tracing-point.

If the drawing is to be copied to the same scale, make $BF = BD$ and make B the fixed point. The figure is then copied to same scale, but in an inverted position.

In the best instruments the arms are made of brass, but very good work may be done with wooden arms.

PROTRACTORS.

154. A Protractor is a graduated circle or arc, with its centre fixed, to be used in plotting angles. They are of various designs and materials.

Semicircular Protractors, such as shown in Fig. 44, are usually made of horn, brass, or german-silver. They are graduated to degrees or half-degrees, and the angle is laid off by holding the centre at the vertex of the angle, with the plain edge, or the 0 and 180 degree line on the given line from which the angle is to be laid off.*

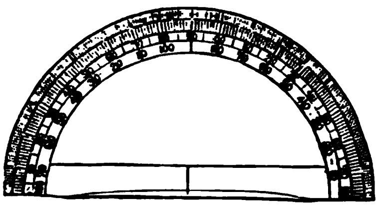

FIG. 44.

In the *full circle protractor*, shown in Fig. 45, there is a movable arm with a vernier reading to from 1 to 3 minutes. The horn centre is set over the given point, the protractor oriented with the zero of the circle on the given line, and the arm set to the given reading when the other line may be drawn.

* For an elegant style of protractor to be used in topographical work, see Fig 66, p. 273.

The *three-arm protractor*, Fig. 46, has one fixed and two movable arms by which two angles may be set off simultaneously. It is used in plotting observations by sextant of two

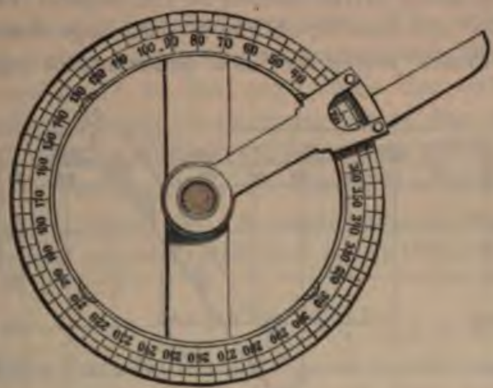

FIG. 45.

angles to three known points for the location of the point of observation. This is known as the three-point problem and is discussed in Chap. X.

FIG. 46.

Paper protractors are usually full circled, from 8 to 14 inches in diameter, graduated to half or quarter degrees. They are printed from engraved plates on drawing- or tracing-paper or bristol-board, and are very convenient for plotting topographical surveys. The map is drawn directly on the protractor sheet, the bearing of any line being taken at once from the graduated circle printed on the paper. These "protractor sheets" can now be obtained of all large dealers.

*The coördinate protractor** is a quadrant, or square, with

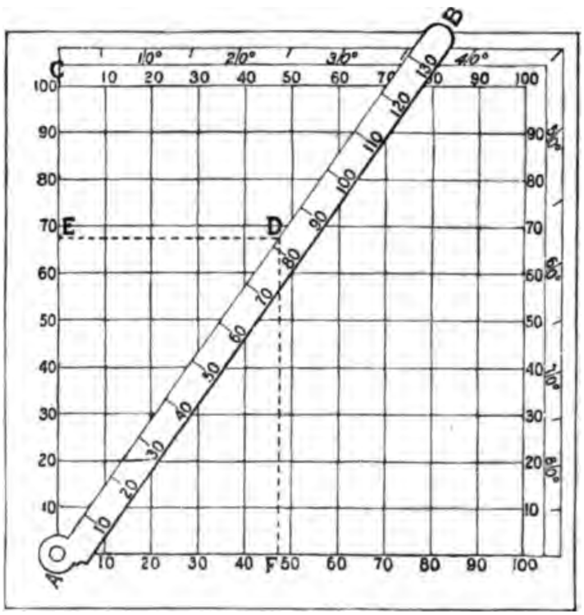

FIG. 47.

angular graduations on its circumference, or sides, and divided over its face by horizontal and vertical lines, like cross-section paper. A movable arm can be set by means of a vernier to read minutes of arc, this arm being also graduated to read distances from the centre outward. Having set this arm to read the proper angle, the latitude is at once read off on the

* Called a Trigonometer by Keuffel & Esser, the makers.

vertical scale and the departure on the horizontal scale for the given distance as taken on the graduated arm. A quadrant protractor giving latitudes and departures for all distances under 2500 feet to the nearest foot, or under 250 feet to the nearest tenth of a foot, has been used. The radius of the circle is $18\frac{3}{4}$ inches. Both the protractor and the arm are on heavy bristol-board, so that any change due to moisture will affect both alike and so eliminate errors due to this cause. The instrument was designed to facilitate the plotting of the U. S. survey of the Missouri River.* It has proved very efficient and satisfactory. A similar one on metal, shown in Fig. 47, is now manufactured, and serves the same purpose.

PARALLEL RULERS.

155. **The Parallel Ruler** of greatest efficiency in plotting is that on rollers, as shown in Fig. 48. The rollers are made of exactly the same circumference, both being rigidly attached to the same axis. It should be made of metal so as to add to its

FIG. 48.

weight and prevent slipping. It is of especial value in connection with the paper protractors, for the parallel ruler is set on any given bearing and then this transferred to any part of the sheet by simply running the ruler to place. Two triangles may be made to serve the same purpose, but they are not so rapid or convenient, and are more liable to slip. The parallel ruler is also very valuable in the solution of problems in graphical statics.

SCALES.

156. **Scales** are used for obtaining the distance on the drawing or plot which corresponds to given distances on the

* For sale by A. S. Aloe & Co., St. Louis, Mo.

object or in the field. There is such a variety of units for both field and office work, and a corresponding variety of scales, that the choice of the particular kind of scale for any given kind of work needs to be carefully made. Architects usually make the scale of their drawings so many feet to the inch, giving rise to a duodecimal scale, or some multiple of $\frac{1}{12}$.

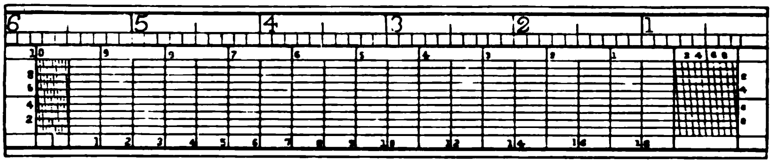

FIG. 49.

A surveyor who uses a Gunter's chain 66 feet in length plots his work to so many chains to the inch, making a scale of some multiple of $\frac{1}{792}$. An engineer usually uses a 100-foot chain and a level rod divided to decimal parts of a foot; so he finds it convenient to use a decimal scale for his maps and drawings, reduced to the inch-unit however. Here the field-unit is feet and the office-unit is inches, both divided decimally. This gives rise to a sort of decimal-duodecimal system, the scale being some multiple of $\frac{1}{120}$. Various combinations of all these systems are found.

Figure 49 shows one form of an ivory scale of equal parts for the general draughtsman. The lower half of the scale is designed to give distances on the drawing for 4, 40, or 400 units to the inch when the left oblique lines and bottom figures are used, and for 2, 20, or 200 units to the inch when the right oblique lines and top figures are used. Thus, if we are plotting to a scale of 400 feet to the inch, and the distance is 564 feet, set one point of the dividers on the vertical line marked 5, and on the fourth horizontal line from the bottom. Set the other leg at the intersection of the sixth inclined line with this same horizontal line, and the space subtended by the points of the dividers is 564 feet to a scale of $\frac{1}{4800}$.

Figure 50 is a cut of an engineer's triangular boxwood scale, 12 inches long, being divided into decimal inches. There are six scales on this rule, a tenth of an inch being subdivided into 1, 2, 3, 4, 5, and 6 parts, making the smallest

FIG. 50.

graduations $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{30}$, $\frac{1}{40}$, $\frac{1}{50}$, $\frac{1}{60}$ of an inch respectively. This is called an engineer's or decimal-inch scale The architect's triangular scale is divided to give $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{2}$, 2, 3, and 4 inches to the foot. Such a scale is of less service to the civil engineer.

BOOK II.

SURVEYING METHODS.

CHAPTER VII.

LAND-SURVEYING.*

157. Purposes.—All surveys of land, properly so called, are made

(*a*) For the purpose of establishing certain monuments, corners, lines, and boundaries, as in laying out and dividing land, or,

(*b*) For the purpose of identifying and locating such monuments, lines, and boundaries, after they have been established, as in all resurveys for location and area.

In all cases the boundary and dividing lines are the traces of vertical planes on the surface of the ground, and the area is the area of the horizontal plane included between the bounding vertical planes. In other words, the area sought is the area of the horizontal projection of the real surface.

158. In laying out Land the work consists in running the bounding and dividing lines over all the irregularities of the surface, leaving such temporary and permanent marks as the work may demand. These lines to lie in vertical planes, and their bearings and horizontal distances to be found. The bearing of a line is the horizontal angle it makes with a meridian plane through one extremity, and its length is the length of its horizontal projection. This reduces the plot of the work to what it would be if the ground were perfectly level. If all

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* See *Appendix G* for the essential requirements of a survey and map.

the straight lines of a land-survey lie in vertical planes, and if their bearings and horizontal lengths are accurately determined, then as a land survey it is theoretically perfect, whatever the purpose of the survey may be.

The needle compass and Gunter's chain have been universally and almost exclusively used in land surveying. Except in those localities where there is local attraction and very erratic changes in the declination of the needle, the compass is the best instrument that can be used for the purpose. Most of the trouble which has resulted from its use has arisen from a failure to make frequent determinations of the declination. An observation on Polaris by the method given in Art. 33, or by the use of Table XII., and its description in Art. 381*a*, should be made in each township, and a true meridian marked on the ground at every county-seat. The compass should be set up on this meridian as often as once a year, at about 10 o'clock A.M., and the declination noted. The annual change in declination found at the county-seat could then be attributed to each of the declinations found in the several townships of the county, and so a continuous corrected record of the true declination kept for all parts of the county.

159. Monuments.—All marks of whatever description left on or near the surface of the ground, such as stones, stakes, mounds, holes, or trees specially marked and described, for the purpose of designating a particular point on the surface as a landmark, are called monuments.

All land monuments set by surveyors should be stones, suitably cut and marked, and planted in the ground. Surveyors cannot insist too strongly on the necessity of setting permanent monuments to mark land boundaries at the time these boundaries are first established. The surveyor who first lays out the ground should always set permanent monuments before the survey passes beyond his own control. It will not do to trust that some one interested will replace his temporary marks by those of a more permanent character afterwards, both because this is likely to be entirely neglected, and also because of the cloud thus thrown over the authority of the monument itself from the fact that it is not the original mark.

Monuments are used not only to mark corners of tracts of land, but also to mark points in straight lines, as in State boundaries, and points fixed by triangulation in geodetic, geological, State, and municipal surveys.

160. Significance and Authority of Monuments.—Whenever monuments are placed in any scheme of land subdivision, and these monuments are described in the conveyance of such lands when sold, they thereby acquire a perpetual and controlling significance. It matters not how frail and temporary a monument may have been—a mere peg stuck in the ground—if it did at the time designate a particular point in the boundary of the tract, and if such monument is recognized in the deed, its position controls the location absolutely. In any subsequent survey for the location of the boundary it becomes supremely important to identify with certainty the true position of such monument. The field notes of the original survey, or any description of the boundaries in the deed, or the area called for, have no weight in determining the position of the lines and corners as against the *certain* identification of the monuments also recognized in the conveyance. What the conveyer sold and the purchaser bought was a certain fixed tract of land which should have been marked at one time by visible monuments. In this case the field notes are material evidence of the original position of the monuments, but since errors in surveying are not uncommon, and since the supposed area of the tract is computed from these field notes, neither the area nor the description by course and distance, called for in the deed, are allowed to hold as against the proved location of the original monuments, also called for in the deed.

Surveys are always subject to revision and correction. A monument once set and used in a conveyance cannot be changed, even though its position is not what it was intended to be, or not what it is said to be,* in the written description, without the free consent of all parties concerned. There is therefore an inviolableness and absoluteness of control in recognized monuments which does not pertain to any surveys or to any descriptions or areas dependent on surveys.†

161. Lost Monuments.†—When monuments have once been established and used in conveyances and afterward disappear or are lost, they cannot be re-established as an absolute authoritative control by any survey or agreement of surveys. Nothing but consent or acquiescence of all the parties in interest, or a judgment of the court can replace a lost monument. Surveys and the judgment of surveyors are valuable evidence in determining where the original monument was placed, but the surveyor has no authority or right to replace or re-establish a lost monument, or to certify to its position, unless he can find such trace of the original monument itself, or of a witness point, as may serve to identify its position with certainty. He may then replace it by a more permanent mark, and by recording a full description of his work the new monument may be recognized as having all the authority of the original. But any location of a monument based on the field notes of the original survey, even in conjunction with other well-authenticated monuments a considerable distance off, cannot serve to "establish" such monument. It serves only as so much evidence, to be taken in connection with all other evidence, material and personal, such as fence lines, acknowledged boundaries, testimony of witnesses, etc., which evidence may, and often does, outweigh the evidence furnished by the survey. In such a case the surveyor is an expert witness, engaged to interpret the original field notes and to find where they would place the lost monument; but inasmuch as the original field notes may not have agreed with the actual position of the monument, any number of resurveys or agreement

* See particular case 3, p. 231.

† See Arts. 302, 303, and 304 in Chap. XII., on City Surveying.

[illegible] into **Townships.**—From each "town- [illegible] parallel auxiliary meridians are [illegible] parallel. Since these meridians [illegible] the principal meridian, they will [illegible] when they reach the next standard [illegible] distances have been marked off [illegible] principal meridian, and it is from [illegible] that the next auxiliary meridians will [illegible] the next standard parallel, etc. Thus [illegible] becomes a "correction-line" for the [illegible] into "ranges" [illegible] numbered east and [illegible] These ranges are then cut [illegible] corresponding township [illegible] territory into [illegible] the narrowing [illegible] townships [illegible] base-line." [illegible] the third [illegible] ated as [illegible] contains

[illegible]

[illegible] ded [illegible] [illegible] is run [illegible] township line. [illegible] the west, and [illegible] When this [illegible] correction-line, [illegible] to it, and thus [illegible] section-lines as well by the national Government. The details of the system have been modified from time to time, but it remains substantially unchanged. The following is a synopsis of the method now used, which is given in detail in the *Manual of Surveying Instructions*, issued by the Commissioner of the General Land Office, at Washington, D. C., and obtained on application: *

163. **The Reference Lines** are first a Principal Meridian and an accompanying Base Line. There have been twenty-four sets of these meridian and base lines used in laying out the public lands in different parts of the United States, for a detailed description of which see Appendix E.

From the principal meridian and its accompanying base line, guide meridians and standard parallels are run north and south from the base line and east and west from the principal meridian, twenty-four miles apart in each direction. These lines are run with great care, using the solar compass or solar attachment. The magnetic needle cannot be relied on for this work, for two reasons: there may be local attraction from magnetic deposits, and the declination changes rapidly (about a minute to the mile) on east and west lines. The transit alone might be used to run out the meridians, as this consists simply of extending a line in a given direction. If the transit is used in running the parallels offsets must be taken as described in Art. 169, p. 185. The solar compass is the only surveying instrument that can be used for running a true east and west line an indefinite distance. The needle-compass would serve if there were no local attraction and if the true declination were known and allowed for at all points. The solar compass (or solar attachment) is the instrument for this work.

In running these reference-lines, every eighty chains (every mile) is marked by a stone, tree, mound, or other device, and is called a "section corner." Every sixth mile has a different mark, and is called a "township corner."

* The surveyor should obtain and follow the instructions in force at the time the original surveys were made in his locality.

164. The Division into Townships.—From each "township corner" on any standard parallel auxiliary meridians are run *north* to the next standard parallel. Since these meridians converge somewhat toward the principal meridian, they will not be quite six miles apart when they reach the next standard parallel. But the full six-mile distances have been marked off on this parallel from the principal meridian, and it is from these township corners that the next auxiliary meridians will start and run north to the next standard parallel, etc. Thus each standard parallel becomes a "correction-line" for the meridians. The territory has now been divided into "ranges" which are six miles wide, each range being numbered east and west from the principal meridian. These ranges are then cut by east and west lines joining the corresponding township corners on the meridians, thus dividing the territory into "townships," each six miles square, neglecting the narrowing effect of the convergence of the meridians. The townships are numbered north and south from the "principal base-line." The fifth township north of this base-line, lying in the third range west of the principal meridian, would be designated as "town five north, range three west." Each township contains thirty-six square miles, or 23,040 acres.

165. The Division into Sections.—The township is divided into thirty-six sections, each one mile square and containing 640 acres. This is done by beginning on the south side of each township and running meridian lines north from the "section corners" already set, marking every mile or "section corner," and also every half-mile or "quarter-section corner." When the fifth section corner is reached, a straight line is run to the corresponding section corner on the next township line. This will cause this bearing to be west of north on the west, and east of north on the east, of the principal meridian. When this northern township boundary is a standard or correction-line, then the sectional meridians are run straight out to it, and thus this line becomes a correction-line for the section-lines as well as for the township-lines. The east and west division-lines are run, connecting the corresponding section corners on the meridian section lines, always marking the middle, or quarter-section points. Evidently, to run a straight line between two points not visible from each other, it is necessary first to run a random or trial line, and to note the discrepancy at the second point. From this the true bearing can be computed and the course rerun, or the points on the first course can be set over the proper distance. The sections are numbered as shown in Figs. 51 and 52.

When account is taken of the convergence of meridians, the sections in the northern tiers of each township will not be quite one mile wide, east and west; but as the section corners are set at the full mile distance on the township-lines, the southern sections in the next town north begin again a full mile in width. In setting the section and quarter-section corners on the east and west town lines the full distances are given from the east toward the west across each township, leaving the deficiency on the last quarter-section, or 40-chain distance, until the next correction-line is reached, when the town meridians are again adjusted to the full six-mile distances.

166. The Convergence of the Meridians is, in angular amount,*

$$c = m \sin \tfrac{1}{2} (L + L');$$

where m = meridian distance in degrees, or difference of longitude, and L and L' are the latitudes of the two positions. In other words, the angular convergence of the meridians is the difference in longitude into the sine of the mean latitude.

The convergence in chains of two township-lines six miles apart, from one correction-line to another twenty-four miles apart, in lat. 40°, is

$$C = 24 \times 80 \times \sin c;$$

where c, in degrees, $= \frac{6}{53} \sin 40°$, since one degree of longitude in lat. $40° = 53$ miles. Thus $c = 4'.37$ for each six-mile dis-

* From Eq. (G), Appendix D, when $\cos \frac{1}{2} \Delta h$ is taken as unity.

tance, east or west, in lat. 40°. Whence $C = 2.42$ chains, which is what the northern tier of sections in the north range between correction-lines lacks of being six miles east and west.

In a similar manner, we may find that the north sections in

FIG. 51.

79.40 6 79.92	80 5 79.92	80 4 79.92	80 3 79.92	80 2 79.92	80 1 79.92
7 79.94	8	9	10	11	12 79.94
18 79.95	17	16	15	14	13 79.95
19 79.97	20	21	22	23	24 79.97
30 79.98	29	28	27	26	25 79.98
31 80	32 80	33 80	34 80	35 80	36 80

PRINCIPAL MERIDIAN.

CORRECTION-LINE.

a town are about six feet narrower, east and west, than the corresponding southern sections in the same town.

Figures 51 and 52 show the resulting dimensions of sections in chains when no errors are made in the field-work. The north and south distances are all full miles.*

In Fig. 51 it will be observed that in the northern tier of sections the meridians must bear westerly somewhat so as to meet the full-mile distance, laid off on the township-line.

* Of course all measurements in surveying are more or less inexact, and hence the d lengths on section lines deviate more or less from these theoretical amounts.

In Fig. 52 they continue straight north to the town line, which is in this case a correction-line. If the distances on this correction-line be summed they will be found to be 2.42 chains short of six miles as above computed.

CORRECTION-LINE.

78.08 6 78.10	79.90 5	79.90 4	79.90 3	79.90 2	79.90 1 79.92
7 78.12	8	9	10	11	12 79.94
18 78.13	17	16	15	14	13 79.75
19 78.14	20	21	22	23	24 79.97
30 78.16	29	28	27	26	25 79.98
31 78.18	32 80	33 80	34 80	35 80	36 80

PRINCIPAL MERIDIAN.

FIG. 52.

The law provides that all excesses or deficiencies, either from erroneous measurements or bearings or from the convergence of meridians, shall, so far as possible, be thrown into the northern and western quarter-sections of the township.

167. Corner Monuments have been established on all United States land surveys at all the corners of townships, sections, quarter-sections, and meandered lines, except at the point common to four-quarter sections, at the center of each section. These monuments are of various composition, as:

(*a*) A stone with pits and earthen mound.
(*b*) " " " a mound of stone.
(*c*) " " " bearing trees.
(*d*) A post in a mound of stone.
(*e*) " " " " " " earth.
(*f*) " " with bearing trees.
(*g*) A mound without post or stone.
(*h*) A tree without bearing trees.
(*i*) " " with " "

Whenever possible, certain descriptive marks and letters are cut on the stones, posts, or trees, such as described in the following sample field notes, taken from the *Manual of Instructions*, issued by the United States Land Commissioner, Washington, D. C., in which full illustrations and descriptions are found on all matters pertaining to the original surveys of public lands. It should also be stated, that any of the styles of marking above-named may be used for any kind of corner, and that the styles described below are not limited to the purposes there named.

STANDARD TOWNSHIP CORNER.

Stone, with Pits and Mound.

Set a — stone — × — × — × ins. — ins. in the ground for Standard Cor. to Tps. 5 N., R's 2 and 3 W., marked S. C., with 6 notches on N., E., and W. edges, dug pits 24 × 18 × 12 ins. crosswise on each line; N., E., and W. of stone 6 ft. dist., and raised a mound of earth 2½ ft. high, 5 ft. base, alongside.

STANDARD SECTION CORNER.

Post, with Bearing Trees.

Set a post 4 ft. long, 4 ins. square, 24 ins. in ground for Standard Cor., to secs. 35 and 36, marked
S. C. T. 5 N., R. 3 W., on N.;
S. 36 on E., and

S. 35 on W. faces, with 1 notch on E. and 5 notches on W. faces, from which

A — , — ins. diam. bears N. — °, E. — lks. dist., marked T. 5 N., R. 3 W. S. 36 B. T.

A — , — ins. diam. bears N. — °, W. — lks. dist. marked T. 5 N., R. 3 W. S. 35 B. T.

A — , — ins. diam. bears S. — °, E. — lks. dist., marked T. 5 N., R. 3 W. S. C. S. 35 and 36 B. T.

CORNER COMMON TO FOUR SECTIONS.

Mound, without Post or Stone.

Deposited a marked stone (charred stake or quart of charcoal) 12 ins. in the ground, for Cor. to Secs. 25, 26, 35, and 36; dug pits, 18 × 18 × 12 ins. in each Sec., 5½ ft. dist., and raised a mound of earth 2 ft. high, 4½ ft. base over it.

In S. C. pit drove a stake 2 ins. square, 2 ft. long, 12 ins. in ground, marked

T. 2 N., S. 25, on N. E.
R. 2 W., S. 36, on S. E.
S. 35, on S. W., and
S. 26, on N. W. faces with 1 notch on S. and E. edges.

QUARTER-SECTION CORNER.

Post in Mound.

Set a post 3 ft. long, 3 ins. square, with marked stone (charred stake or quart of charcoal), 12 ins. in the ground, for ¼ Sec. Cor., marked ¼ S. on N. (or W.) face; dug pits, 18 × 18 × 12 ins., N. and S. (or E. and W.) ft. base of post 5½ ft. dist., and raised a mound of earth 1½ ft. high, 3½ around post.

168. The Subdivision of Sections.- No interior section lines were run by the United States Deputy Surveyors, but quarter-section corners were set at the middle points of all the four sides of all sections, except those in the north and west tiers in each township. In order to satisfy the law, that so far as possible all excesses and deficiencies should be thrown into the north and west tiers of quarter-sections, the monuments on these section lines are placed just 40 chains from the next interior section lines. That is say, the east and west quarter-section line in the north tier of sections lies 40 chains from and parallel to the south side of such section, and the north and south quarter-section line in the west tier of sections lies 40 chains from and parallel to the east side of such sections. In all other cases the quarter-section lines are intended to be medial lines. The location of the quarter-section corners, when set, will control the position of these lines, however, so that nothing remains to be done in making a resurvey but to run trial or random lines through the section from one quarter-section corner to the opposite one, and by noting the errors, correct one-half of them at the centre of the section, and so obtain the point where the lines joining the opposite quarter-section corners of a section intersect. This is the interior quarter-section corner.

In case no quarter-section corner has been or can be set on one side of a section, the quarter-section line is to be extended from the opposite corner by a true north and south or east and west line.

The north and west tiers of quarter-sections in every township are called fractional quarters, and are divided again into one full half-quarter and a fractional half-quarter. The northern tier are so divided by an east and west line running just 20 chains north of the quarter-section line, and the western tier of quarter-sections are divided by a north and south line lying just 20 chains west of the quarter-section line, the N. W. quarter of Section 6 being classed with the northern tier of quarters. All other subdivided quarter-sections are divided into half quarter-sections by medial lines run north and south, and into quarter quarter-sections by medial lines run east and west between the corresponding sides of the quarter-section.

169. To Run Out a Parallel of Latitude or a True East and West Line.—A true east and west line is one which is at every point at right angles to a meridian passing through that point. It is therefore a constantly curving line, being always deflected toward the north in the northern hemisphere. Any line run on the earth's surface by prolongation by means of any surveying instrument will be a great circle. If the magnetic needle always pointed directly north, a line run at right angles to its direction would be a parallel of latitude. Since the solar compass, or attachment, always orients itself in the true meridian, any line run by it at right angles to the constantly observed meridian will be a true east and west line. This is the only instrument capable of running such a line directly.

This method is not so accurate, however, as to use a transit, and make frequent observations for azimuth. Then, starting out on a true east and west line, run out a straight line by prolongation (Art. 100, p. 95) for some twelve miles distance, and make corrections northward for the points on the true parallel. Then offset the proper distance, set the transit again on the parallel, and either make a new observation for azimuth or carry the old azimuth forward, correcting it to agree with the new meridian. To do this two tables are required: one to give the proper offsets from the great circle to the parallel of latitude tangent to it at the initial meridian, and the other to give the change in azimuth necessary to prolong the line from a new meridian when no new observation for azimuth can be obtained. These two tables* are combined in one on the following page. The angles there given are measured from the north point toward the point of tangency of the straight line

* Condensed from tables given in the "Manual of Instructions," issued by the Commissioner of the General Land Office, Washington, D. C., 1890.

with the parallel, which is the initial point from which the distances given in the table are measured. The convergence of the meridians for the corresponding distances is 90°, minus the angles given in this table.

The offsets are to be always measured to the north of the great circle or tangent straight line, in the northern hemisphere, and south from it in the southern hemisphere.

Having started from a given point due east, in latitude 40°, for instance, and run out a straight line for six miles, we find from the accompanying table that the true meridian is obtained by turning off from east to south or from west to north the angle 89° 55′ 38″, and the true position of the parallel at this point is 20.1 ft. north of the line. When twelve miles have been run out in one continuous tangent line the angle with the meridian is 89° 51′ 17″, and the parallel now lies 80.5 ft. north of the line.

ACUTE ANGLES WITH THE MERIDIAN, AND OFFSETS TO PARALLELS, AT POINTS ONE MILE APART ON A GREAT CIRCLE OR STRAIGHT LINE TANGENT TO THE PARALLEL AT THE INITIAL POINT.

Latitude	1 Mile from Tangent Point.		2 Miles.		3 Miles.		4 Miles.	
	Angle.	Offset.	Angle.	Offset.	Angle.	Offset.	Angle.	Offset.
°	° ′ ″	ft.	° ′ ″	ft.	° ′ ″	ft.	° ′ ″	ft
30	89 59 30	0.39	89 59 00	1.54	89 58 30	3.47	89 58 00	6.17
32	89 59 28	.42	89 58 55	1.67	89 58 23	3.76	89 57 50	6.67
34	89 59 25	.45	89 58 50	1.80	89 58 15	4.05	89 57 40	7.20
36	89 59 22	.48	89 58 44	1.94	89 58 07	4.36	89 57 29	7.75
38	89 59 19	.52	89 58 39	2.08	89 57 58	4.69	89 57 18	8.33
40	89 59 16	.56	89 58 33	2.24	89 57 49	5.03	89 57 06	8.95
42	89 59 13	.60	89 58 26	2.40	89 57 40	5.40	89 56 53	9.59
44	89 59 10	.64	89 58 20	2.57	89 57 30	5.79	89 56 40	10.29
46	89 59 06	.69	89 58 12	2.76	89 57 19	6.20	89 56 25	11.04
48	89 59 02	.74	89 58 05	2.95	89 57 07	6.65	89 56 09	11.82
50	89 58 58	0.79	89 57 56	3.17	89 56 54	7.12	89 55 53	12.68

Latitude	5 Miles.		6 Miles.		7 Miles.		8 Miles.	
	Angle.	Offset.	Angle.	Offset.	Angle.	Offset.	Angle.	Offset.
°	° ′ ″	ft.	° ′ ″	ft.	° ′ ″	ft.	° ′ ″	ft.
30	89 57 30	9.64	89 57 00	13.88	89 56 30	18.89	89 56 00	24.67
32	89 57 18	10.42	89 56 45	15.02	89 56 12	20.44	89 55 40	26.69
34	89 57 05	11.25	89 56 30	16.20	89 55 54	22.05	89 55 19	28.80
36	89 56 51	12.11	89 56 13	17.41	89 55 36	23.74	89 54 59	31.01
38	89 56 37	13.02	89 55 56	18.75	89 55 16	25.52	89 54 35	33.33
40	89 56 22	13.98	89 55 38	20.11	89 54 56	27.40	89 54 11	35.78
42	89 56 06	14.99	89 55 19	21.59	89 54 32	29.38	89 53 46	38.38
44	89 55 49	16.07	89 54 59	23.14	89 54 09	31.50	89 53 18	41.14
46	89 55 31	17.21	89 54 37	24.80	89 53 43	33.76	89 52 50	44.10
48	89 55 12	18.47	89 54 14	26.59	89 53 16	36.19	89 52 18	47.27
50	89 54 50	19.80	89 53 49	28.52	89 52 47	38.82	89 51 45	50.70

Latitude	9 Miles.		10 Miles.		11 Miles.		12 Miles.	
	Angle.	Offset.	Angle.	Offset.	Angle.	Offset.	Angle.	Offset.
°	° ′ ″	ft.	° ′ ″	ft.	° ′ ″	ft.	° ′ ″	ft.
30	89 55 30	31.23	89 55 00	38.55	89 54 30	46.65	89 54 00	55.52
32	89 55 08	33.78	89 54 35	41.71	89 54 03	50.47	89 53 30	60.06
34	89 54 44	36.45	89 54 09	45.00	89 53 34	54.45	89 52 59	64.80
36	89 54 20	39.25	89 53 42	48.45	89 53 04	58.63	89 52 26	69.77
38	89 53 55	42.19	89 53 14	52.08	89 52 33	63.02	89 51 53	75.00
40	89 53 28	45.29	89 52 44	55.91	89 52 00	67.65	89 51 17	80.51
42	89 52 59	48.57	89 52 12	59.97	89 51 25	72.56	89 50 38	86.35
44	89 52 28	52.07	89 51 38	64.28	89 50 48	77.78	89 49 58	92.57
46	89 51 56	55.81	89 51 02	68.90	89 50 08	83.37	89 49 14	99.22
48	89 51 20	59.83	89 50 23	73.86	89 49 25	89.37	89 48 28	106.36
50	89 50 43	64.17	89 49 41	79.22	89 48 39	95.86	89 47 37	114.08

FINDING THE AREA OR SUPERFICIAL CONTENTS OF LAND WHEN THE LIMITING BOUNDARIES ARE GIVEN.

170. The Area of a Piece of Land is the area of the level surface included within the vertical planes through the boundary-lines. This area is found in acres, roods, and perches, or, better, in acres only, the fractional part being expressed decimally. Evidently the finding of such an area involves two distinct operations, viz.: the Field-work, to determine the positions, directions, and lengths of the boundary-lines; and the Computation, to find the area from the field-notes. There are several methods of making the field observations, giving rise to corresponding methods of computation. Thus, the area may be divided into triangles, and the lengths of the sides, or the angles and one side, or the bases and altitudes measured, and the several partial areas computed. Or the bearings and distances of the outside boundary-lines may be determined and the included area computed directly. This is the common method employed. Again, the rectangular coördinates of each of the corners of the tract may be found in any manner with reference to a chosen point which may or may not be a point in the boundary, and the area computed from these coördinates. These three methods will be described in detail.

I. *Area by Triangular Subdivision.*

171. By the Use of the Chain Alone.—In Fig. 53 let *ABCDEF* be the corner boundaries of a tract of land, the sides being straight lines. Measure all the sides and also the diagonals *AC*, *AD*, *AE*, and *FB*. The area required is then the sum of the areas of the four triangles *ABC*, *ACD*, *ADE*, and *AEF*. These partial areas are computed by the formula

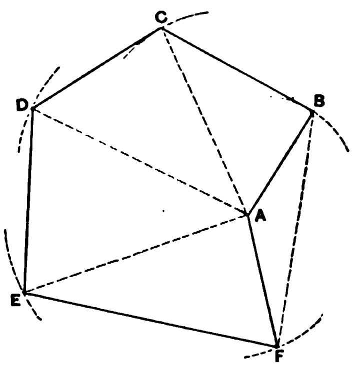


FIG. 53.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the half sum of the three sides a, b, c in each case.

For a Check, plot the work from the field-notes. Thus, take any point as A and draw arcs of circles, with A as the common centre, with the radii AB, AC, AD, AE, and AF taken to the scale of the plot. From any point on the first arc, as B, and with a radius equal to BC to scale, cut the next arc, whose radius was AC, giving the point C. From C find D with the measured distance CD, etc., until F is reached. Measure FB on the plot, and if this is equal to the measured length of this line, taken to the scale of the drawing, the field-work and plot are correct. It is evident the point A might have been taken anywhere inside the boundary-lines without changing the method.

172. By the Use of the Compass, or Transit, and Chain.—If the compass had been set up at A the outer boundaries could have been dispensed with, except the lines AB and AF. All that would be necessary in this case would be the bearings and distances to the several corners. We then have two sides and the included angle of each triangle given when the area of each triangle is found by the formula:

$$\text{Area} = \tfrac{1}{2}ab \sin C.$$

In this case there is no check on the chaining or bearings. The taking-out of the angles from the given bearings could be checked by summing them. This sum should be 360° when A is inside the boundary-line, and 360° minus the exterior angle FAB when A is on the boundary. If the boundary-lines be measured also, then the area of each triangle can be computed by both the above methods and a check obtained.

173. By the Use of the Transit and Stadia.*—Set up at A, or at any interior or boundary point from which all the corners can be seen, and read the distances to these corners and the horizontal angles subtended by them. The area is then computed by the formula given in the previous article. The distances may be checked by several independent readings, and the angles by closing the horizon (sum = 360°).

The above methods do not establish boundary-lines, which is usually an essential requirement of every survey.

II. *Area from Bearing and Length of the Boundary-lines.*

174. The Common Method of finding land areas is by means of a compass and chain. The bearings and lengths of the boundary-lines are found by following around the tract to

* The stadia methods are described in Chapter VIII.

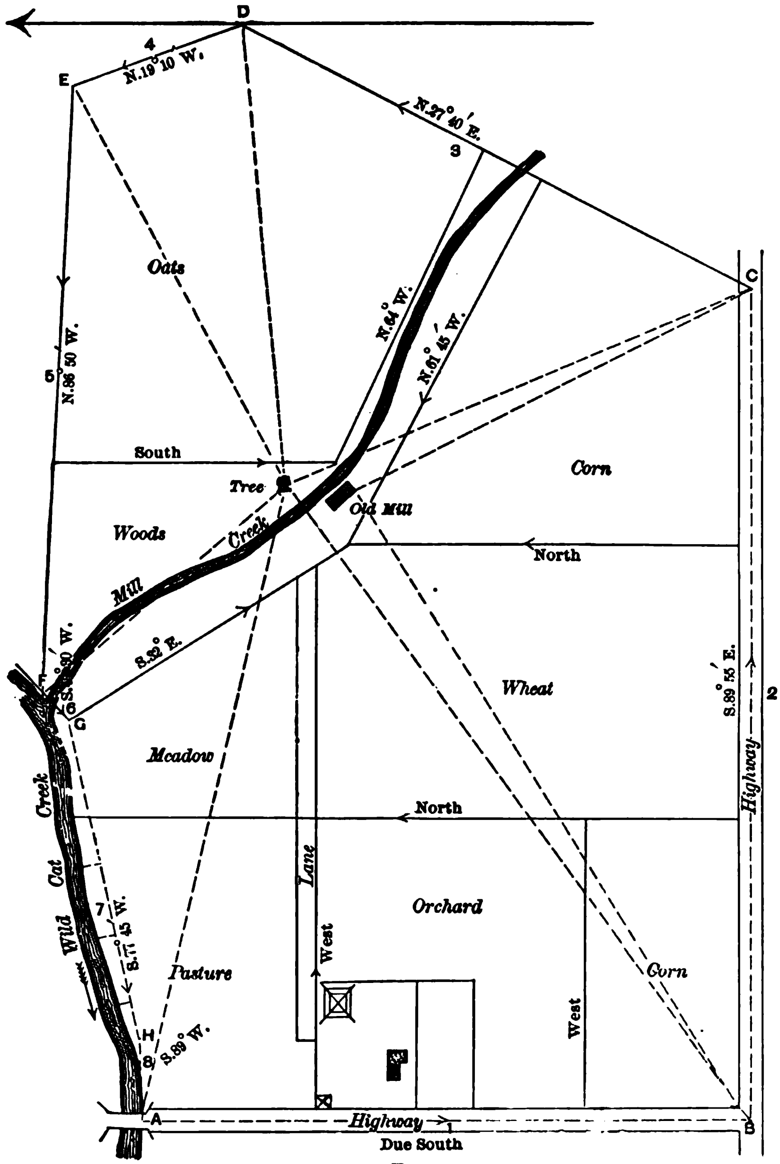
the point of beginning. If the boundary-lines are unobstructed by fences, hedges, or the like, then the compass is set at the corners, and the chaining done on line. If these lines are obstructed, then equal rectangular offsets are measured and the bearings and lengths of *parallel* lines are determined. In this case the compass positions at any corner for the two courses meeting at that corner are not coincident, neither are the final point of one course and the initial point of the next course, the perpendicular offsets from the true corner overlapping on angles less than 180° and separating on angles over 180°.

The chaining is to be done as described in Art. 4, p. 8, the 66-foot or Gunter's chain being used. Both the direct and the reverse bearing of each course should be obtained for a check as well as to determine the existence of any local attraction. For the methods of handling and using the compass see Chapter II.

175. **The Field-notes** should be put on the left-hand page and a sketch of the line and objects crossing it on the right-hand page of the note-book. The following is a convenient form for keeping the notes. They are the field-notes of the survey which is plotted on p. 192. It will be seen that the "tree" was sighted from each corner of the survey and its bearing recorded. If these lines were plotted on the map they would be found to intersect at one point. If the plot had not closed, then these bearings would have been plotted and they would not have intersected at one point, the first line which deviated from the common point indicating that the preceding course had been erroneously measured, either in bearing or distance, or else plotted wrongly. In general such bearings, taken to a common point, enable us to locate an error either in the field-notes or in the plot. The bearings of all division-fences were taken, as well as their point of intersection with the course, so that these interior lines could be plotted and a map of the farm obtained. The "old mill" is located by bearings taken from corners *B* and *G*. The reverse-bearings are given in parenthesis.

FIELD NOTES—COMPASS SURVEY. Oct. 23, 1885.

No. of Course.	Point.	Bearing.	Distance along the Course.	Remarks.
	Bearing-tree.......	S. 76° 50′ E....	Ch.	True bearings given.
	Pasture Fence.....	West..........	7.20	Variation of needle 5°
	Yard "	"	9.75	50′ east.
1	" "	"	11.54	Henry Flagg,
	Orchard "	"	13.90	*Compassman.*
	Corner B.....	South.........	25.42	Peter Long, } *Chainmen.*
Wt. = 1		(North)..... ..		John Short, }
	B. T..............	N. 54° 15′ E...		Courses 1 and 2 are
	Old Mill...........	N. 58° E...		along the centres of
2	Fence.............	North.........	12.50	the highway.
	"	"	24.10	
	Corner C..........	S. 89° 55′ E....	34.68	
Wt. = 1		(West).... ...		
	B. T..............	N. 22° 20′ W..		
	Old Mill...........	N. 26° 45′ W...		
	Fence.............	N. 61° 45′ W...	9.90	
3	Mill Creek........	....	10.70	
	Fence.............	N. 64° W......	12.45	
	Corner D..........	N. 27° 40′ E...	24.00	
Wt. = 3		(S. 27° 45′ W.)		
	B. T	S. 85° W......		
4	Corner E....... ..	N. 19° 10′ W..	7.40	
Wt. = 2		(S. 19° 15′ E.)..		
	B. T..............	S. 62° 30′ W...		
5	Fence......... ...	South	15.80	
	Corner F.....	N. 86° 50′ W..	25.58	
Wt. = 2		(S. 86° 45′ E.).		
	B. T..............	S. 40° 15′ E....		
	N. bank Mill Creek.	....	0.30	
6	S. " "	....	0.80	
	Corner G.....	S. 47° 30′ W...	1.50	
Wt. = 5		(N. 47° 30′ E.).		
	Fence.............	S. 32° E.......	0.00	
	Offset, 0.40.......	....	0.00	
	" .60.......	....	3.00	
	" .80.......	....	6.00	
7	" .70.......	....	9.00	
	" .30.......	....	12.00	
	" .20.......	....	13.60	
	Corner H..........	S. 77° 45′ W...	13.60	
Wt. = 3		(N. 77° 45′ E.).		
8	Corner A..........	S. 89° W......	3.53	
Wt. = 1		(N. 89° E.) ...		



D
4
N.19° 10' W.
E
N.27° 40' E.
3
C
Oats
N.64° W.
N.61° 45' W.
5
N.36° 50' W.
South
Tree
Corn
Old Mill
Woods
Creek
North
Mill
S.30° W.
S.32° E.
F
6
G
Wheat
S.89° 55' E.
2
Meadow
Highway
North
Creek
Cat
Lane
Orchard
7
S.77° 45' W.
Wild
West
Pasture
Corn
West
H
8
S.89° W.
A
Highway
1
B
Due South

FIG. 54.

COMPUTING THE AREA.

Fig. 55.

176. The Method stated. — In Fig. 55,* let $ABCDE$ be the tract whose area is desired. Let us suppose the bearings and lengths of the several courses have been observed. Pass a meridian through the most westerly corner, which in this case is the corner A. Let fall perpendiculars upon this meridian from the several corners, and to those lines drop other perpendiculars from the adjacent corners, as shown in the figure. Then we have:

$$\begin{aligned} \text{Area } ABCDE &= bBCDfb - bBAEDfb \\ &= bBCe + eCDf - (bBA + AEa + aEDf). \quad (1) \end{aligned}$$

Hence twice the area $ABCDE$ is

$$\begin{aligned} 2A &= (bB + eC)Bc + (eC + fD)Dd \\ &- (bB)Ab - (aE)Aa - (aE + fD)Eg. \quad . \quad . \quad . \quad (2) \end{aligned}$$

We will now proceed to show that these distances are all readily obtained from the lengths and bearings of the courses.

177. Latitudes, Departures, and Meridian Distances.— *The latitude* of a course is the length of the orthographic projection of that course on the meridian, or it is the length of the course into the cosine of its bearing. If the forward bearing of the course is northward its latitude is called its *northing*, and is reckoned positively; while if the course bears southward its latitude is called its *southing*, and is reckoned negatively.

* The lines OD and OX in this figure are used in art. 191.

The departure of a course is the length of its orthographic projection on an east and west line, or it is the length of the course into the sine of its bearing. If the forward bearing of the course is eastward its departure is called its *easting*, and is reckoned positively; while if its forward bearing is westward its departure is called its *westing*, and is reckoned negatively.

The meridian distance of a point is its perpendicular distance from the reference meridian, which is here taken through the most westerly point of the survey.

The meridian distance of a course is the meridian distance of the middle point of that course; therefore

The double meridian distance of a course is equal to the sum of the meridian distances to the extremities of that course. The D. M. D.'s of the two courses adjacent to the reference meridian are evidently equal to their respective departures. The D. M. D. of any other course is equal to the D. M. D. of the preceding course plus the departure of that course plus the departure of the course itself, easterly departures being counted positively and westerly departures negatively. This is evident from Fig. 55.

Thus in Fig. 55 Dd is the latitude and dC is the departure of the course DC. If the survey was made with the tract on the left hand, then the latitude of this course is positive and the departure negative; while the reverse holds true if the survey was made with the tract on the right hand. In this discussion it will be assumed that the survey is made by going around to the left, or by keeping the tract on the left hand, although this is not essential. The D. M. D. of this course CD is $fD + eC$; or it is the D. M. D. of $BC + cC + (-dC)$.

In equation (2), art. 176, the quantities enclosed in parentheses are the double meridian distances of the several courses, all of which are positive, while the distances into which these are multiplied are the latitudes of the corresponding courses. If we go around towards the left the latitudes of the courses

AB, *DE*, and *EA* are negative, and therefore the corresponding products are negative, while the latitudes of the courses *BC* and *CD* being positive, their products are positive.

We may therefore say that *twice the area of the figure is equal to the algebraic sum of the products of the double meridian distances of the several courses into the corresponding latitudes,* north latitudes being reckoned positively and south latitudes negatively, and the tract being kept on the left in making the survey. If the tract be kept on the right in the survey, then the numerical value of the result is the same, but it comes out with a negative sign.

178. Computing the Latitudes and Departures of the Courses.—Since the departure of a course is its length into the sine, and its latitude its length into the cosine, of its bearing, these may be computed at once from a table of natural or logarithmic sines and cosines. When bearings were (formerly) read only to the nearest 15 minutes of arc, tables were used giving the latitude and departure for all bearings expressed in degrees and quarters for all distances from 1 to 100. Such tables are called *traverse tables.* It is customary now, however, to read even the needle-compass closer than the nearest 15 minutes; and if forward and back readings are taken on all courses, and the mean used, these means will seldom be given in even quarters of a degree. If the transit or solar compass is used, the bearing is read to the nearest minute. The old style of traverse table is therefore of little use in modern surveying. The ordinary five- or six-place logarithmic tables of sines and cosines are computed for each minute of arc, and these may be used, but they are unnecessarily accurate for ordinary land-surveying. For this purpose a four-place table is sufficient. If the average error of the field-work is as much as 1 in 1000 (and it is usually more than this), then an accuracy of 1 in 5000 in the reduction is evidently all-sufficient, and this is about the average maximum error in a four-place table; that is, the average of the maximum errors that can be made in the different parts of the table.

Table III. is a four-place table of logarithms of numbers from 1 to 10,000, and Table IV. is a similar table of logarithms of sines and cosines, from 0 to 360 degrees. If a transit is used in making the survey, and if it is graduated continuously from 0 to 360 degrees, then the azimuths of the several sides are found, all referred to the true meridian or to the first side. If it is desired now to take out the latitudes and departures, the same as for a compass-survey, where the *bearings*

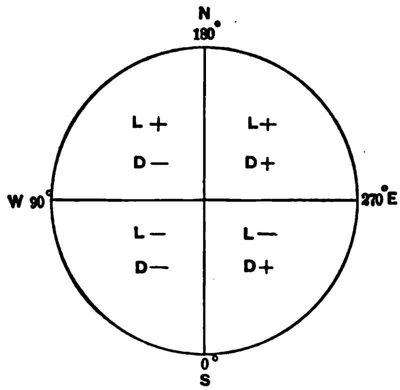


FIG. 56.

of the sides are given directly referred to the north and south points, it may be done by Table IV.

Since the log sine changes very fast near zero and the log cosine very fast near 90°, the table is made out for every minute for the first three degrees from these points; for the rest of the quadrant it gives values 10 minutes apart, but with a tabular difference for each minute. It is very desirable to make the table cover as few pages as possible for convenience and rapidity in computation. In this table the zero-point is south and angles increase in the direction SWNE, so that in the first quadrant both latitudes and departures are negative. In the second quadrant latitude is positive and departure negative, in the third both are positive, and in the fourth latitude is negative and departure positive. These relations are shown in Fig. 56. For any angle, falling in any quadrant, if reckoned from the south point in the direction here shown, the log sin (for departure) and log cosine (for latitude) may be at once found from Table IV. If these logarithms are both taken out at the same time and then the logarithms of the distance from Table III., this can be applied to both log sin and log cos, thus giving the log departure and log latitude, when from Table III. again we may obtain the lat. and dep. of this course, giving these their signs according to the quadrant in which the azimuth of the line falls.

If Table IV. is to be used for *bearings* of lines as given by a needle-compass, then enter the table for the given bearing, in the first set of angles, beginning at 0 and ending at 90°.

Example: Compute the latitudes and departures of the survey plotted in Fig. 55, p. 193, by Tables III. and IV. The following are the field-notes as they would appear, first, as read by a transit and referred to the true meridian; and, second, as read by a needle-compass:

Station.	Azimuth referred to the South Point.	Compass bearing.	Distance.
A	290° 45′	S. 69° 15′ E.	7.06
B	217° 15′	N. 37° 15′ E.	5.93
C	140° 30′	N. 39° 30′ W.	6.00
D	57° 45′	S. 57° 45′ W.	4.65
E	30° 00′	S. 30° 00′ W.	4.98

The following is a convenient form for computing the latitudes and departures:

	Course AB 4th Q.	Course BC 3d Q.	Course CD 2d Q.	Course DE 1st Q.	Course EA 1st Q.
log sin (dep.) =	9.9708	9.7820	9.8035	9.9272	9.6990
log dist. =	.8488	.7731	.7782	.6675	.6972
log dep. =	.8196	.5551	.5817	.5947	.3962
Departure =	+ 6.60	+ 3.59	− 3.82	− 3.93	− 2.49
log cos (lat.) =	9.5494	9.9009	9.8874	9.7272	9.9375
log dist. =	.8488	.7731	.7782	.6675	.6972
log lat. =	.3982	.6740	.6656	.3947	.6347
Latitude =	− 2.50	+ 4.72	+ 4.63	− 2.48	− 4.31

It is seen that Table IV. answers equally well for either set of bearings, and also that Table III. would have given the latitudes and departures to the fourth significant figure as well as to the third. If the proper quadrant is given for each course in the heading as shown above, then the signs may be at once given to the corresponding latitudes and departures.

179. Balancing the Survey.—If the bearings and lengths of all the courses had been accurately* determined, the survey would "close;" that is, when the courses are plotted successively to any scale the end of the last course would coincide on the plot with the beginning of the first one. Furthermore, the sum of the northings (plus latitudes) would exactly equal the sum of the southings (minus latitudes), and the sum of the

* The error of closure simply shows a *want of uniformity* of measurement, for if all the sides were in error by the same relative amount, the survey would close just the same. For instance, if an erroneous length of chain were used the survey might close but the area be considerably in error. See Arts. 18[illegible] and 182.

eastings (plus departures) would exactly equal the sum of the westings (minus departures). It is evident that such exactness is not attainable in practice, and that neither the north and south latitudes nor the east and west departures will exactly balance, there always being a small residual in each case. These residuals are called the errors of latitude and departure respectively. The distribution of these errors is called ***balancing the survey.***

In the form for reduction of the field-notes given below, wherein this example is solved, it is seen that the error of latitude is 6 links and the error of departure is 5 links. The distribution of these errors is made by one of the following:

FORM FOR COMPUTING AREAS FROM BEARINGS AND DISTANCES OF THE SIDES.

Stations.	Courses.		Dif. Lat.		Departure.		Balanced.		D. M. D.	+ Area.	− Area.
	Bearings.	Dist.	N. +	S. −	E. +	W. −	Lat.	Dep.			
		Ch.									
A	S. 69° 15′ E.	7.06	....	2.50	6.60	....	− 2.52	+ 6.61	6.61	....	16.66
B	N. 37° 15′ E.	5.93	4.72	....	3.59	....	+ 4.71	+ 3.60	16.82	79.22	....
C	N. 39° 30′ W.	6.00	4.63	....	....	3.82	+ 4.62	− 3.81	16.61	76.74	....
D	S. 57° 45′ W.	4.65	....	2.48	....	3.93	− 2.49	− 3.92	8.88	....	22.11
E	S. 30° 00′ W.	4.98	....	4.31	....	2.49	− 4.32	− 2.48	2.48	....	10.71
		28.62	9.35	9.29	10.19	10.24				155.96	49.48
			9.29			10.19				49.48	
			Error in lat. = .06		Error in dep. = .05					2) 106.48	

Area = 53.24 sq. ch.
= 5.324 *Acres*

$$\text{Error of closure} = \frac{\sqrt{5^2 + 6^2}}{2862} = 0.0027$$
$$= 1 \text{ in } 366.$$

RULES FOR BALANCING A SURVEY.

RULE 1. *As the sum of all the distances is to each particular distance, so is the whole error in latitude (or departure) to the correction of the corresponding latitude (or departure)*, each correction being so applied as to diminish the whole error in each case.

RULE 2. Determine the relative difficulties to accurate measurement and alignment of the several courses, selecting one course as the standard of reference. Thus, if the standard course would probably give rise to an error of 1, determine what the errors *for an equal distance* on the other courses would probably be, as $1\frac{1}{2}$, 2, 1, 0.5 etc. Multiply the length of each course by its number, or weight, as thus obtained. Then we would have:

As the sum of all the multiplied lengths is to each multiplied length, so is the whole error in latitude (or departure) to the correction of the corresponding latitude (or departure), each correction being so applied as to diminish the whole error in each case.

These two rules are based on the assumption that the error of closure is as much due to erroneous bearings as to erroneous chaining, which experience shows to be true in needle-compass work.

If, however, the bearings are all taken from a solar compass (or attachment) in good adjustment, or if the exterior lines are run as a traverse with a transit, so that the *angles* of the perimeter are accurately measured, then the above assumption does not hold, as it is highly probable that the error of closure is almost wholly due to erroneous chaining. Especially would this be highly probable if the azimuth is checked by occupying the first station on closing and redetermining the azimuth of the first course, as found from the traverse, and comparing it with the initial (true or assumed) azimuth of this course. If it thus appears that the traverse is practically correct as to angular measurements, it may be fairly assumed that the error of closure is almost wholly due to erroneous chaining. In this case use

RULE 3. *As the arithmetical sum of all the latitudes is to any one latitude, so is the whole error in latitude to the correction to the corresponding latitude*, each correction being so applied as to diminish the whole error in each case. Proceed similarly with the departures.*

In the solution given on p. 199 the first rule is applied. In ordinary farm-surveying it is not common to give the lengths of the courses nearer than the nearest even link or hundredth of a chain. In balancing, therefore, the same rule may be observed.

180. The Error of Closure is the ratio to the whole perimeter of the length of the line joining the initial and final points, as found from the field-notes. The length of this line is the hypotenuse of a right triangle of which the errors in latitude and departure are the two sides. Its length is therefore equal to the square root of the sum of the squares of these two errors. This divided by the whole perimeter gives the error of closure, which ratio is usually expressed by a vulgar fraction whose numerator is one, being $\frac{1}{366}$ in the above example.

The error of closure for ordinary rolling country should not

* It is evident that the courses could here be weighted for different degrees of difficulty in the chaining; but instead of multiplying the lengths of the courses by their weights, multiply the latitudes and departures by the weights of the corresponding courses, and then distribute the errors in latitude and departure by these multiplied latitudes and departures.

be more than 1 in 300. In city work it should be less than 1 in 1000, and should average less than 1 in 5000.

181. The Form of Reduction.—On p. 199, the ordinary form of reduction is shown. Here the courses are not weighted for different degrees of difficulty in chaining; and since it was a compass-survey the effect of erroneous bearings is supposed to equal that from erroneous chaining, and so the first rule for balancing is used. The balanced latitudes and departures having been found, the double meridian distances are next taken out. In taking out these it is preferable to begin with the *most westerly corner*, whether this be the first course recorded or not. In the example solved on p. 199, it is the first corner occupied, but in that given on p. 206 it is not the first course. By beginning with the most westerly corner (which is equivalent to passing the reference meridian through that corner), all the double meridian distances will be positive; otherwise some of them may be negative. If attention be paid to signs we may begin at any corner to compute the double meridian distances.

A check on the computation of the D. M. D.'s is that, when computed continuously in either direction and from any corner, the numerical value of the D. M. D. of the last course must equal its departure. This is a very important check and must not be neglected, as it proves the accuracy of all the D. M. D.'s.

We are now able to compute the double-areas according to equation (2), art. 176, since the terms entering in that equation have their numerical values determined. The several products, being the partial double-areas, are written in the last two columns, careful attention being paid to the signs of these products. Thus, when the reference meridian is taken through the most westerly corner, then all the D. M. D.'s are positive and the results take the sign of the corresponding latitude. If some of the D. M. D.'s are negative, then the signs of these partial areas are opposite to those of the corresponding latitudes. The algebraic sum of the partial double-areas is twice the area of the figure, as shown in eq. (2), Art. 176. If the distances are given in chains, then the area is given in sq. chains, and dividing by ten gives the area in acres. If the distances were given in feet, as it often is, being measured by a 100-foot chain or tape, then the area is in sq. feet, and this must be divided by 43560, the number of sq. feet in one acre, to give the area in acres. This is best done by logarithms, as shown in the example solved on p. 206. It is preferable to express areas in acres and decimals rather than in roods and perches, as was formerly the custom.

On the following page is the reduction of the field-notes given on p. 191. Here the several courses have been weighted for various degrees of difficulty in the chaining. Thus, the first and second courses were along the public highway and on even ground. These are taken as the standard and given the weight unity. The third course is on very uneven ground and is judged to give rise to about three times the error of courses one and two per unit's distance. It is therefore weighted three. The proper weight to give to the several courses is thus seen to depend on the character of the obstructions to accurate work, and represents simply the judgment of the surveyor as to the probable relation of these sources of error. The short course *FG* was very difficult to measure, as there were precipitous bluffs, and the course *GH* was also on very uneven ground.

Following the column of weights in the tabular reduction are the multiplied distances; the errors of latitude and departure are distributed according to the results in this column by Rule Two, p. 200. This survey was also made with a needle-compass.

In the following example the transit was used, and the

Station.	Courses. Bearings.	Courses. Distances.	Relative Weighting.	Multiplied Distances.	Dif. Lat. N. +	Dif. Lat. S. −	Departure. E. +	Departure. W. −	Balanced. Lat.	Balanced. Dep.	D.M.D.	+ Area.	− Area.
A	South.	25.42	1	25.42	......	25.42	......	......	−25.44	0.000	0.000	....	......
B	S. 89° 55′ E.	34.68	1	34.68	......	.05	34.68	......	− 0.09	+34.69	34.69	....	3.122
C	N. 27 40 E.	24.00	3	72.00	21.26	......	11.15	......	+21.19	+11.16	80.54	1706.643	......
D	N. 19 10 W.	7.40	2	14.80	6.99	......	......	2.43	+ 6.98	− 2.42	89.28	623.174	......
E	N. 86 50 W.	25.58	2	51.61	1.41	......	......	25.55	+ 1.36	−25.54	61.32	83.400	......
F	S. 47 30 W.	1.50	5	7.50	......	1.01	.....	1.10	− 1.02	− 1.09	34.69	....	35.380
G	S. 77 45 W.	13.60	3	40.80	......	2.88	......	13.29	− 2.92	−13.28	20.32	....	59.334
H	S. 89 W.	3.53	1	3.53	......	.06	.....	3.53	− 0.06	− 3.52	3.52	....	0.211
		135.71		249.89	29.66	29.42	45.83	45.90				2413.217	98.047
					29.42			45.83				98.047	

Error in lat. = 0.24 Error in dep. = 0.07

2) 2315.170

$$\text{Error of closure} = \frac{\sqrt{24^2 + 7^2}}{13571} = 1 \text{ in } 543.$$

Area = 1157.585 sq.chs.
= 115.76 acres.

survey began at A. The azimuth of the line AB (Fig. 57) was found by a solar attachment, and then the other courses ran as a traverse, the horizontal limb of the transit being oriented by the back azimuth of the last course. The azimuths of the courses are all referred to the south point as zero, and increase in the direction *SWNE*. After the last course FA was run, the instrument was carried to A and oriented by a back sight on F and the azimuth of AB again determined. This agreed so well with the original azimuth of this course that the azimuths of all the courses were proved to be correct.†

FIG. 57.*

The error of closure is therefore due to the chaining alone. A hundred-foot chain was used so that the distances are all given in feet. The obstructions to chaining were about uniform, so the courses are all given equal weight. In balancing, Rule Three must be used, since the errors are supposed to come only from the chaining.

If the errors in latitude and departure had been distributed by Rule One, or in proportion to the lengths of the courses, the resulting area would have been 56.41 acres, a difference of 0.07 acres, or about one eight-hundredth of the total area.

182. Area Correction due to Erroneous Length of

* The lines MB and OO' in this figure are used in art. 192.

† From the azimuth check here obtained, as compared to the errors in latitude and departure, decide whether the latter are due mostly to the chaining or whether the errors in azimuth have had an equal influence, and so determine whether to use rule 1 or rule 3 in balancing.

Station.	Azimuth.	Dist.	Dif. Lat. N. +	Dif. Lat. S. −	Departure. E. +	Departure. W. −	Balanced. Lat.	Balanced. Dep.	D. M. D.	+ Areas.	− Areas.
A	164° 03′	838 ft.	805	....	......	230	+ 806	− 231	2539	2,046.434	....
B	205 39	1004	905	....	434	......	+ 906	+ 432	2740	2,482,440	....
C	112 12	896	338	....	......	830	+ 339	− 834	2338	792,582	....
D	55 00	912	......	523	......	747	− 522	− 750	754	....	393,588
E	0 04	1542	......	1542	......	2	−1540	− 2	2	....	3,080
F	269 32	1392	11	....	1392	......	+ 11	+1385	1385	15,235	....
A (check)	164° 05′	6484 ft.	2059	2065	1826	1809				5,336,691	396,668
				2059	1809					396,668	
									2)	4,940,023	

Error in lat. = 6 17 = error in dep.

$$\text{Error of closure} = \frac{\sqrt{6^2 + 17^2}}{6484} = 1 \text{ in } 360. \qquad (43560 \text{ sq. ft.} = 1\text{A.})$$

$$\text{Area} = \begin{cases} 2,470,012 \text{ sq. ft.} \\ \text{or } \ 56.70 \text{ acres.} \end{cases}$$

Chain.—If the measuring unit has not the length assigned to it in the computation, then the computed area will be erroneous. Such an error will not show in the balancing of the work or elsewhere, and hence an independent correction must be applied for this error. If the chain was too long by one one-thousandth part of its length, for instance, then all the courses are too short in the same ratio. And since similar plane figures are to each other as the squares of their like parts, we would have

$$\text{true area} : \text{computed area} :: (1001)^2 : (1000)^2,$$

or $$\text{true area} = \tfrac{1002}{1000} \text{ computed area (nearly)};^*$$

or, in general, if l = length of chain and $\varDelta l$ = error in length, being positive for chain long and negative for chain short, and if $\varDelta l$ is small as compared with l, as it always is in this case, then if we let

A = true area, A' = computed area

C_A = correction to computed area,

and $\varDelta$ = relative error of chain,

we have $$A = \frac{l + 2\varDelta l}{l} A' = (1 + 2\varDelta)A';$$

whence, $$A - A' = C_A = 2\varDelta A'.$$

That is to say, *the relative area correction due to erroneous length of chain is twice the relative error of the chain, being positive for chain long, and negative for chain short.*

* The error in this approximation is one one-millionth in this case, and would always be inconsiderable in this class of problems.

FINDING THE AREA OR SUPERFICIAL CONTENTS OF LAND WHEN THE RECTANGULAR COÖRDINATES OF THE CORNERS ARE GIVEN WITH RESPECT TO ANY POINT AS AN ORIGIN.

183. Conditions of Application of this Method.—Where many tracts of land, all bounded by straight lines, are somewhat confusedly intermingled, as is the case in many of the older States, and where the area of each tract over an extended territory is to be found, this method is greatly to be preferred to that by means of the boundary-lines. In this case it is only necessary to make a general coördinate survey of the whole territory, as described in Chapter VIII., on Topographical Surveying, using the stadia for obtaining distances, and being careful to locate every corner of each tract. If areas alone are required, no attention need be paid to the obtaining of elevations for contour lines, and so the work is greatly facilitated. A transit and two or three stadia rods would be the instruments used. The survey would then be carefully plotted and the coördinates measured on the sheet, or they could be computed from the field-notes. If the plotting is carefully done the former method is preferable. It is best to choose the origin of coördinates entirely outside the tract and so that the whole area falls in one quadrant, thus making all the coördinates of one sign.

Large tracts of mineral land are sometimes acquired by large companies, including perhaps hundreds of individual estates. In such cases a topographical map of the region is necessary; and when this survey is made, a little extra care to obtain all the "corners" of private claims will enable the areas of all such lots to be determined with great accuracy and at small additional cost. The method probably has no advantages when the area of but a single tract is desired.

184. The Method of Finding the Area from the Rectangular Coördinates of the Corners is as follows:

Let Fig. 58 be the same tract as that given in Fig. 55, and

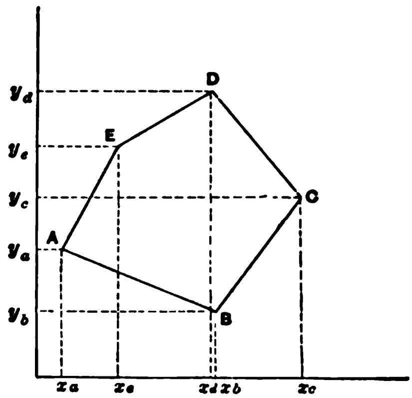


FIG. 58.

let the origin be one chain west of A and three chains south of B. Then, from the balanced latitudes and departures for this case, given on p. 199, we find the following coördinates of the corners y_a, y_b, etc., denoting the latitudes of the corners A, B, etc., and similarly with x_a, x_b, etc., for departures:

$$y_a = 5.52, \quad y_b = 3.00, \quad y_c = 7.71, \quad y_d = 12.33, \quad y_e = 9.84.$$

$$x_a = 1.00, \quad x_b = 7.61, \quad x_c = 11.21, \quad x_d = 7.40, \quad x_e = 3.48.$$

The area of the figure $ABCDE$ is equal to the areas

$$y_bBCy_c + y_cCDy_d - \{y_eEDy_d + y_aAEy_e + y_bBAy_a\};^*$$

or

$$A = \tfrac{1}{2}[(y_c - y_b)(x_b + x_c) + (y_d - y_c)(x_c + x_d) - (y_d - y_e)(x_d + x_e) - (y_e - y_a)(x_e + x_a) - (y_a - y_b)(x_a + x_b)]. \quad (1)$$

* Here y_b, y_c, etc., are used to designate points and not ordinates. In the following equations they are ordinates.

By developing equation (1) we obtain

$$A = \tfrac{1}{2}\left[y_a x_e - y_a x_b + y_b x_a - y_b x_c + y_c x_b - y_c x_d \right.$$
$$\left. + y_d x_c - y_d x_e + y_e x_d - y_e x_a\right].^* \quad (2)$$

From this we may obtain either of the following:

$$\left.\begin{aligned} A &= \tfrac{1}{2}\left[y_a(x_e - x_b) + y_b(x_a - x_c) + y_c(x_b - x_d) \right. \\ &\qquad \left. + y_d(x_c - x_e) + y_e(x_d - x_a)\right]; \\ \text{or} \\ A &= -\tfrac{1}{2}\left[x_a(y_e - y_b) + x_b(y_a - y_c) + x_c(y_b - y_d) \right. \\ &\qquad \left. + x_d(y_c - y_e) + x_e(y_d - y_a)\right]. \end{aligned}\right\} \quad (3)$$

From these equations we may obtain the following

RULE FOR FINDING THE AREA OF A CLOSED FIGURE BOUNDED BY STRAIGHT LINES FROM THE RECTANGULAR COÖRDINATES OF THE CORNERS.

Multiply the $\left\{\begin{matrix}\textit{abscissa} \\ \textit{ordinate}\end{matrix}\right\}$ *to each corner by the difference between the* $\left\{\begin{matrix}\textit{ordinates} \\ \textit{abscissæ}\end{matrix}\right\}$ *of the two adjacent corners, always making the subtraction in the same direction around the figure, and take half the sum of the products.*

The student will observe that this is simply a more general case of the former method of computing the area from the latitudes and double-meridian distances.

* If these coördinates be arranged thus :

$$\begin{array}{ccccccccccc} \dfrac{x_a}{y_a} & \times & \dfrac{x_b}{y_b} & \times & \dfrac{x_c}{y_c} & \times & \dfrac{x_d}{y_d} & \times & \cdots & \times & \dfrac{x_a}{y_a} \end{array}$$

then in accordance with formula (2), the area is equal to the sum of the products of the quantities joined by the broken lines minus the sum of the products of the quantities joined by the full lines.

185. The Form of Reduction for this case is given below.

Corner.	Ordinates (y).	Abscissæ (x).	Difference between Alternate Abscissæ.	Double Areas.
A	5.52	1.00	− 4.13	− 22.80
B	3.00	7.61	− 10.21	− 30.63
C	7.71	11.21	+ .21	+ 1.62
D	12.33	7.40	+ 7.73	+ 95.31
E	9.84	3.48	+ 6.40	+ 62.98

Plus areas = 159.91
Minus areas = 53.43
2) 106.48
Area = 53.24 sq. chns.
= 5.324 acres.

This is the same result as found on p. 199 by the other method, as it should be, since the same balanced latitudes and departures were used in each case.

It is also evident that after the balanced latitudes and departures are obtained for the ordinary perimeter-survey, the area may be computed by this form—from equations (3), p. 210, if preferred. Or, if the coördinates of the corners are taken at once from a map, or computed from traverse lines, the bearings and lengths of the courses joining such corners could readily be computed. Thus, the length of any course, as BC, is $BC = \sqrt{(x_c - x_b)^2 + (y_c - y_b)^2}$, while its bearing is the arc whose tan is $\dfrac{x_c - x_b}{y_c - y_b}$.

186. Supplying Missing or Erroneous Data.—In any closed survey there are two geometric conditions that must be fulfilled, viz.:

1. The sum of all the latitudes must be zero.
2. The sum of all the departures must be zero.

These two conditions give rise to two corresponding equations.

If l_1, l_2, l_3, etc., be the lengths of the several courses, and if θ_1, θ_2, θ_3, etc., be their compass-bearings, then our two geometric conditions give

$$l_1 \sin \theta_1 + l_2 \sin \theta_2 + l_3 \sin \theta_3 + \text{etc.}, = 0. \quad . \quad . \quad (1)$$

$$l_1 \cos \theta_1 + l_2 \cos \theta_2 + l_3 \cos \theta_3 + \text{etc.}, = 0. \quad . \quad . \quad (2)$$

Since we have *two* independent equations, we can solve for *two* unknown quantities. These two unknowns may be any two of the functions entering in the above equations. Thus, if any two distances, any two bearings, or any one distance and any one bearing are missing, they may be found from these equations. Or, if but one bearing or distance is missing, it may be found from one of these equations and the other equation used for balancing either the latitudes or departures. When all bearings and distances are given, these equations are really used in balancing; but if they are both used to determine missing quantities, there can be no balancing of errors, for when the missing quantities are computed by these equations, both latitudes and departures will exactly balance. In other words, all the errors of the survey are thus thrown into these two quantities.

This artifice should therefore never be resorted to except where it is impracticable to actually measure the quantities themselves in the field.

There are four cases to be solved:

I. Where the bearing and length of one course are unknown.

II. Where the bearing of one course and length of another are unknown.

III. Where two bearings are unknown.

IV. Where two lengths are unknown.

The bearings will be reckoned from both north and south points around to the east and west points, as is common in compass surveying. Then the length of a course into the sin of its bearing gives its departure, and into the cos of its bearing gives its latitude. North latitude is plus and south latitude minus; east departure plus and west departure minus.

In every case let the sum of the departures of all known courses, taken with the opposite sign, be D, and the sum of their latitudes, taken with the opposite sign, be L. Then D and L are the departure and latitude necessary to close the survey.

CASE I.—*Bearing and length of one course unknown.*

The two condition equations here become

$$\left.\begin{array}{l} l_m \sin \theta_m = D; \\ l_m \cos \theta_m = L. \end{array}\right\} \quad \dots \dots \quad (3)$$

Whence

$$\tan \theta_m = \frac{D}{L}. \quad \dots \dots \quad (4)$$

Having found the bearing, find l_m from either of equations (3). Particular attention must always be paid to the signs of D and L. Evidently $\sin \theta_m$ (dep.) and $\cos \theta_m$ (lat.) have the same signs as D and L respectively, whence the quadrant which includes the bearing may be determined and the proper letters applied. For this purpose Fig. 56 may be consulted.

CASE II.—*The bearing of one course and the length of another unknown.*

In this case let a be the known bearing of the course whose length is unknown, and let l be the known length of the course whose bearing is unknown. Then we have

$$\left.\begin{array}{l} l_m \sin a + l \sin \theta_n = D; \\ l_m \cos a + l \cos \theta_n = L. \end{array}\right\} \quad \dots \dots \quad (5)$$

[redacted]

If we let $\sin a = s$, and $\cos a = c$, we have

$$l_m = sD + cL \pm \sqrt{l^2 - (D^2 + L^2) + (sD + cL)^2}. \quad . \quad (6)$$

Here there are two values of l_m which will satisfy the equation, and so there are two solutions to the problem. If the surveyor has no knowledge whatever of either the unknown length or bearing, the problem is indeterminate. If he has seen the tract he could usually tell which length or which resulting bearing was the correct one, when the problem would become determinate. When l_m is found, substitute in one of equations (5) and find θ_n. Pay careful attention to the signs of the trigonometrical functions of all bearings. When the two unknown courses are nearly at right angles with each other the problem is impracticable.

CASE III.—*When two bearings are unknown.*

Let l' and l'' be the known lengths of the courses whose bearings are unknown. Then the equations become

$$\left.\begin{aligned} l' \sin \theta_m + l'' \sin \theta_n &= D; \\ l' \cos \theta_m + l'' \cos \theta_n &= L. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Whence $$\cos \theta_n = \frac{KL \pm D\sqrt{L^2 - K^2 + D^2}}{D^2 + L^2}; \quad . \quad . \quad . \quad (8)$$

Where $$K = \frac{l''^2 - l'^2 + D^2 + L^2}{2l''}.$$

This case is also indeterminate unless one is able to tell which of the two sets of bearings is the correct one.

CASE IV.—*When the lengths of two courses are unknown.*

Let a and b be the known bearings of the courses whose lengths are unknown.

Our equations here become

$$\left.\begin{aligned} l_m \sin a + l_n \sin b &= D; \\ l_m \cos a + l_n \cos b &= L. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whence $$l_n = \frac{L \sin a - D \cos a}{\sin a \cos b - \cos a \sin b}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This case is determinate.

In case there is but one unknown, then either one of equations (3) will solve. In taking out either the sine or the cosine from the tables, however, two angles will always be found equidistant from the east or west point if the sine, and equidistant from either the north or south point if the cosine, either of which may be chosen. In such case *both sine and cosine* must be found, when the signs alone of these two functions will determine the quadrant in which the bearing is found. Hence, if the single unknown is a bearing; *both* of the equations (3) must be used in order to determine which of the two bearings given by the table is the correct one, but *one alone* is sufficient to obtain the numerical value of the bearing. Thus, if the sine equation is used to compute the bearing, then the latitude may be taken out for the given length and bearing; and these will then not balance, but will have to be balanced in the usual way, while the departures will, of course, balance, since the residual departure D necessary to close the survey as to departures was used to compute the corresponding bearing. The reverse of this would be true if the cosine equation were used to compute the bearing.

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IRREGULAR

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areas are then computed as trapezoids,

the distance along the base-line being the altitude and the half-sum of the adjacent offsets being the mean width. The offsets should therefore be run at such intervals as to make this method of computation sufficiently accurate. Such offsets were taken from the course *GH* in Fig. 54, the notes for which are given on p. 191.

The work of computation may be shortened by using a modified form of the method of areas from the rectangular coördinates of the corners, which, in this case, are the ends of the offset lines. Let Fig. 59 be an area to be determined from the offsets from the line *AK*. The position and lengh of the offsets are given. Take the origin at *A* and let the distances along *AK* be the abscissæ, and the lengths of the offsets be the ordinates. Using the second of equations (3), p. 210, we have

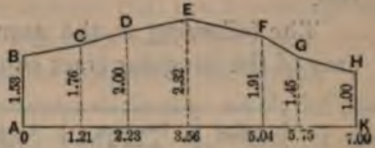

FIG. 59.

$$A = {}^{*}\tfrac{1}{2}[x_a(y_k - y_b) + x_b(y_a - y_c) + x_c(y_b - y_d) \\ + x_d(y_c - y_e) + x_e(y_d - y_f) \\ + x_f(y_e - y_g) + x_g(y_f - y_h) \\ + x_h(y_g - y_k) + x_k(y_h - y_a)]. \quad (1)$$

But here x_a, x_b, y_a, and y_k are all zero; also $x_h = x_k$, hence this equation becomes

$$A = \tfrac{1}{2}[x_c(y_b - y_d) + x_d(y_c - y_e) + x_e(y_d - y_f) \\ + x_f(y_e - y_g) + x_g(y_f - y_h) + x_h(y_g + y_h)]. \quad (2)$$

From eq. (2) we have the

* The plus sign is here used, since we have gone around the figure in a direction opposite to that followed in the general case

RULE FOR FINDING AREAS FROM RECTANGULAR OFFSETS AT IRREGULAR INTERVALS.

Multiply the distance along the course of each intermediate offset from the first by the difference between the two adjacent offsets, always subtracting the following from the preceding. Also multiply the distance of the last offset from the first by the sum of the last two offsets. Divide the sum of these products by two.

The following is the numerical reduction for finding the area of the irregular tract shown in Fig. 59.

Offset.	Distance from A.	Length of Offset.	Differences.	Products.
	ch.	ch.	ch.	sq. ch.
B	0.00	1.53		
C	1.21	1.76	− 0.47	− 0.57
D	2.23	2.00	− 0.56	− 1.25
E	3.56	2.32	+ .09	+ .32
F	5.04	1.91	+ .87	+ 4.38
G	5.75	1.45	+ .91	+ 5.23
H	7.00	1.00	+ 2.45	+17.15

2) 25.26

Area = 12.63 sq. chs.
= 1.263 acres.

It is evident that an area bounded on all sides by irregular or curved lines could have a base-line run through it, and offsets taken from this line to both boundaries and the area computed by this method. Example 10, p. 235, should be so computed.

189. **The Method by Offsets at Regular Intervals.**—If the intervals between the offsets, or ordinates, are all equal the computation is much simplified. On the assumption that the area is a series of trapezoids, we have the

RULE FOR FINDING THE AREA FROM RECTANGULAR OFFSETS AT REGULAR INTERVALS.

Add together all the intermediate offsets and one half the end offsets, and multiply the sum by the constant interval between them.

The following rules for finding areas are found from the successive orders of differences in each case and may all be derived by a rigid development.* They assume that the bounding-line is curved and that rectangular ordinates have been measured at uniform intervals from a base-line traversing the figure.

Let the common interval between ordinates be d; let the lengths of the ordinates be $h_0, h_1, h_2 \ldots . h_n$; and let the number of intervals be N.

I. $N = 1, \quad A = \frac{d}{2}(h_0 + h_1)$, Trapezoidal Rule.

II. $N = 2, \quad A = \frac{d}{3}(h_0 + 4h_1 + h_2)$, Simpson's $\frac{1}{3}$ Rule.

III. $N = 3, \quad A = \frac{3d}{8}(h_0 + 3h_1 + 3h_2 + h_3)$, Simpson's $\frac{3}{8}$ Rule.

IV. $N = 4, \quad A = \frac{2d}{45}[7(h_0 + h_4) + 32(h_1 + h_3) + 12h_2]$.

V. $N = 6, \quad A = \frac{3d}{10}[h_0 + h_2 + h_4 + h_6 + 5(h_1 + h_3 + h_5) + h_3]$.

This is called Weddel's Rule. If a quadrant be computed by this rule, the result is $0.779r^2$ instead of $0.785r^2$, the true value.

When an area, bounded by a base line and two end ordi-

* See appendix C.

[redacted]

nates, be divided by imaginary lines parallel to the end ordinates and equally spaced, as in Fig. 60, and if the middle ordi-

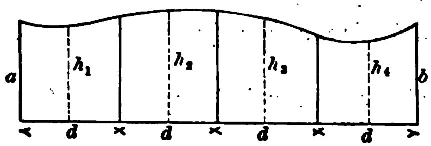

Fig. 60.

nates of these partial areas be measured, then if d = common width of the partial areas and h_1, h_2, h_3, etc., their middle ordinates, a the first end ordinate and b the last one, we have, approximately,

$$\text{I. } A = d\Sigma h,$$

where Σh signifies the summation of all the h's.

The following rules are, however, more accurate:

$$\text{II. } A = d\Sigma h + \frac{d}{12}(a - h_1 + b - h_n), \text{ Poncelet's Rule;}$$

or, [Rule.

$$\text{III. } A = d\Sigma h + \frac{d}{72}(8a + h_2 - 9h_1 + 8b + h_{n-1} - 9h_n), \text{ Francke's}$$

The various rules above given are often used to determine areas of irregular figures such as steam diagrams, cross-sections of structural forms, streams, excavations, etc. The most ready and accurate means of determining all such areas, however, is by means of the planimeter.

THE SUBDIVISION OF LAND.

190. The Problems arising in the subdivision of land are of almost infinite variety. All such problems are solved by the application of the fundamental principles and relations of geometry and trigonometry with which the student is supposed to be familiar. There are, however, two classes of problems of such frequent application that they will be given in detail.

191. To cut off from a Given Tract of Land a Given Area by a Right Line, starting from a Given Point in the Boundary.—In Fig. 55, p. 193, let O be the middle point on the line AB, from which a line is to be run in such a manner as to cut off three acres from the western portion of the tract. We may at once assume that the dividing-line will cut the side DC in some point X, whose distance from D is to be found. First compute the area $OAED$, using the balanced latitudes and departures given on p. 199, we have the following:

Course.	Lat.	Dep.	D. M. D.	Double Areas.	
				+	−
	ch.	ch.			
AO	− 1.26	+ 3.30	3.30	....	4.16
OD	(+ 8.07)	(+ 3.10)	9.70	78.28	....
DE	− 2.49	− 3.92	8.88	....	22.11
EA	− 4.32	− 2.48	2.48	....	10.71
	(− 8.07)	(− 3.10)	Sums	+ 78.28	36.98
				− 36.98	
				2) 41.30	
			Area	= 20.65 sq. chs.	
				= 2.065 acres.	

Here the latitude and departure of the course OD are such as to make the latitudes and departures balance. The area is found to be 2.065 acres, leaving 0.935 acres to be laid off from OD by the line OX. It remains now to find the point X.

First compute the length and bearing of the line OD from Case I., p. 213.

Thus we have

$$\tan\theta = \frac{D}{L} = \frac{+3.10}{+8.07} = 0.384.$$

Whence $\theta = 21°$ from the table of natural tangents. From the table of natural sines, we find $\sin 21° = 0.358$.

Hence from eq. (3), p. 213, we have

$$l \sin\theta = D, \qquad \text{or} \qquad 0.358l = 3.10.$$

Whence $\qquad l = 8.66$ chains.

The bearing is evidently N. 21° E.

We now have to find the distance DX such that the area ODX shall be 9.35 sq. chains. Since the area of any triangle is one half the product of two sides into the sine of the included angle (another way of saying it is equal to half the base into the altitude), we have

$$9.35 = \tfrac{1}{2}(8.66 \times DX)\sin ODX. \quad . \quad . \quad . \quad . \quad (1)$$

From the bearings of OD and DX we find the angle ODX to be 60° 30′, hence $\sin ODX = 0.870$, from which we find

$$DX = 2.48 \text{ chains.}$$

The length and bearing of the line OX may be computed from its latitude and departure, the same as was done for the line OD above, or we may compute the angle DOX and length

OX by solving the triangle DOX. The bearing of OX may then be found, and the line run from O. There will then be two checks on the work, viz.: the measured lengths of OX and DX must be equal to their computed values.

To find the angle DOX, let the three angles of the triangle be D, O, and X, and the sides opposite these angles be d, o, and x, respectively. Then we have

$$\tan \tfrac{1}{2}(X - O) = \frac{x - o}{x + o} \tan \tfrac{1}{2}(X + O).$$

This equation gives the angle $(X - O)$, whence

$O = \frac{1}{2}(X + O) - \frac{1}{2}(X - O)$, and $X = \frac{1}{2}(X + O) + \frac{1}{2}(X - O)$

Also, $$d = OX = OD \frac{\sin D}{\sin X},$$

and $$o = DX = OD \frac{\sin O}{\sin X}.$$

We therefore have the following

RULE FOR CUTTING OFF A GIVEN AREA BY A LINE STARTING FROM A GIVEN POINT IN THE BOUNDARY

Having first surveyed the tract and plotted the same, join the given point on the plot with the corner which will give the nearest approximation to the desired area. Compute the length and bearing of this line, and of the area thus cut off. Subtract this area from the desired area, and the remainder is the area to be cut off in the form of a triangle, one side of which has bearing and distance given, and another side has its bearing alone given. From these data compute the lengths and bearings of the other sides, one of which is the line sought. This line may then be run, and its length measured, as well as the length of the portion of the opposite boundary cut off, for a check on the accuracy of the work.

192. To cut off from a Given Tract of Land a Given Area by a Right Line running in a Given Direction.

[redacted]

—Let the problem be to cut off 30 acres from the northern portion of the tract shown in Fig. 57, p. 205, by a line whose bearing is N. 80° E., or whose azimuth is 260°.*

Pass a line parallel to the required line through the corner nearest to the probable position of the desired line. Let *MB*, Fig. 57, be such a line. Compute the lengths of the lines *EM* and *MB* by Case IV., p. 214.

From the computation, p. 206, we have the following:

Courses	Azimuth.	Lengths.	Balanced Latitudes.	Balanced Departures.	D. M. D.'s	Double Areas.
BC	205° 39′	1004 ft.	+ 906 ft.	+ 432 ft.	2738	+ 2,480,628
CD	112 12	896	+ 339	− 834	2336	+ 791,804
DE	55 00	912	− 522	− 750	752	− 392,544
EM	0 04	(926)	− 926	− 1	1	− 926
MB	260 00	(1171)	+ 203	+1153	1153	+ 234,059
			(+ 723)	(−1152)		2) 3,113,021

Therefore to close requires $L = -723$ and $D = +1152$. Area = 1,556,510 sq. ft. = 35.73 ac's.

From equation (10), p. 215, we have

$$EM = \frac{D \cos 260° - L \sin 260°}{\sin 259° 56'}$$

$$= \frac{(+1152)(+.1736) - (-723)(+.9848)}{+.9846}$$

$$= \frac{200 + 712}{.9846} = 926 \text{ ft.}$$

* In this problem it would have shortened the operation somewhat if the meridian of the survey had been taken parallel to the dividing-line. The bearings could have all been changed to give angles from this meridian, and original computation made from these new bearings.

Whence from eq (9), we have

$$MB = \frac{D - EM \sin 4'}{\sin 260^\circ}$$

$$= \frac{+1152 - (926)(-.0011)}{+.9848.} = 1171 \text{ ft.}$$

Inserting these values of the lengths of the courses *EM* and *MB*, we can compute the area *BCDEM*. This is found to be 35.73 acres, or 5.73 acres too much. The problem now is to pass a line north of *MB* and parallel to it, so that the area included between the parallel lines and the intercepted portions of *EF* and *BC* shall be 5.73 acres, or 249,710 sq. ft. Let *OO'* be such a line. This line can be run when either *MO* or *BO'* is known. It is best, however, to compute both these distances, using one for a check. To find these distances,

Let x = perpendicular distance between the parallel lines *MB* and *OO'*.

Let angle $\qquad EMB = EOO' = \theta,$

and angle $\qquad OO'B = \phi.$

Then we have

$$\text{Area } MOO'B = \overline{MB} \,.\, x - \tfrac{1}{2}x^2 \cot \theta + \tfrac{1}{2}x^2 \cot \phi$$

$$= \overline{MB} \,.\, x + \tfrac{1}{2}x^2 (\cot \phi - \cot \theta) \,.\,. \quad . \quad (1)$$

Since θ and ϕ are known angles, their cotangents are known quantities in any case. So, for simplicity, let

$$(\cot \phi - \cot \theta) = K;$$

also, let the distance $MB = D$,

and area $MOOB = A$.

Then the equation becomes

$$A = Dx - \tfrac{1}{2}Kx^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$x^2 - \frac{2D}{K}x = \frac{2A}{K} ;$$

$$x = -\frac{D}{K} \pm \sqrt{\frac{2A}{K} - \frac{D^2}{K^2}}$$

$$= -\frac{D}{K} \pm \frac{1}{K}\sqrt{2AK - D^2} ;$$

$$= \frac{1}{K}\left(\pm\sqrt{2AK - D^2} - D\right). \quad . \quad . \quad . \quad . \quad (3)$$

That sign of the radical is to be used which will give a positive value to x. The other sign would give the value of x to be used in laying off the given area on the opposite side of MB provided the sides CA and DB were continuous in that direction.

Using equation (3) for the problem in hand, we have

$x =$ [illegible]

$y =$ [illegible]

$A =$ [illegible]

$D =$ [illegible]

$K =$ [illegible]

whence $x = \frac{1}{0.5397}(\pm \sqrt{269,537 + 1,371,241} - 1171)$

$= 203.6$ feet.

We can now find MO and BO' from

$$MO = \frac{x}{\sin \theta} \quad \text{and} \quad BO' = \frac{x}{\sin \phi};$$

whence $MO = 206.8$ feet, and $BO' = 250.6$ feet.

The length of the line OO' is

$$OO' = MB + x(\cot \phi - \cot \theta).$$

We may therefore write the following

RULE FOR CUTTING OFF A GIVEN AREA BY A LINE PASSING IN A GIVEN DIRECTION.

Having first surveyed the tract and plotted the same, pass a line on the plot in the required direction through the corner which will give the nearest approximation to the desired area. Compute the lengths of the two unknown courses bounding this area, and then the area itself. Subtract this from the given area, and the remainder is the area which is to be cut off by a line parallel to the first trial line. This auxiliary area will always be a trapezoid, whose area, the length and bearing of one of the parallel sides, and the bearings of the remaining sides are known. The lengths of these sides may then be computed, one of the end lengths laid off, and the dividing-line run. Measure the length of this line and also of the other end line for checks.

PRINCIPLES AND LAWS BEARING ON THE RESURVEY OF PRIVATE LANDS.*

193. The Problem Stated.—In all resurveys of private lands, whether for running boundaries, computing areas, or for parting off or dividing land, it is first necessary to examine the description of the tract as given in the deed of conveyance, and then to identify such marks, corners, boundaries, and monuments on the ground as have been used in the description. The identification of these monuments often taxes the expertness and skill of the surveyor to the utmost, and it is here that the greatest experience and judgment are required. The original monuments, if there were any placed, may have been entirely lost, and they may or may not have been replaced by others. If they have been, it remains to be determined how reliable these secondary monuments are. In the absence of monuments specially set other natural or artificial features may have been used in the description or have by use acquired the force and authority of monuments. There may also be gross discrepancies between the position of the monuments and the description or area named in the deed. There may also be a controversy between the parties in interest as to the real boundaries which the surveyor may be wholly unable to decide. This much, however, is certain, that any location of a corner or line by course and distance is likely to be very uncertain and unsatisfactory, especially where the needle-compass was used in the original survey, and that every effort should be made to find some trace of the original monuments, if any were set, or to decide from all the evidence available, both material and personal, what the real boundaries are.

The surveyor has no judicial authority to fix or establish

* The rules laid down here are mostly derived from State Supreme Court decisions in cases which arose over boundaries established by the compass and chain, and hence do not apply so well to city and town surveys made with greater exactness. See also Appendix G.

anything.* He is simply an expert witness called in to assist by his knowledge and experience, and with the aid of his surveying instruments, to find the true lines and boundaries. His acts are at all times subject to review in the courts, and he should try and subject his decisions to the same rules and precedents which are likely to govern the court. He may thus save himself from much embarrassment and his client from unnecessary expense.

Where a surveyor makes gross blunders in his work, showing incompetency, he is often held for any damages which may result. In many cities licensed surveyors are under a heavy bond and are held liable for any erroneous locations they may make.

Surveyors' records and plats should always be complete and definite, otherwise they cannot be admitted as evidence.

The following propositions are based on judicial decisions which are thought to have all the force and authority of law, in the absence of special statutes governing the case:

194. The Interpretation of Descriptions in Deeds and the Identification of Boundaries.—*General Rules.*—1. If the description is inconsistent, insufficient, doubtful, or capable of two or more constructions, the purchaser is to be given the reasonable benefit of such defects. That is to say, the grantor is required to convey the land under the most favorable legitimate construction which may be put upon the description the grantor has used to describe it.

But if the intention is evident on the face of the instrument, or if the parties by their acts have shown a mutual agreement or acquiescence in a certain interpretation of the description, this meaning will hold and bind the parties.

2. Where any inconsistency in the description arises from a false or impossible statement, and by rejecting such evident error the remaining description becomes consistent and possible, then such part should be rejected and the deed allowed to stand.

Also, when parts of the description are certain and others uncertain, if the inconsistency can be removed by rejecting one or more of the uncertain portions, this may be done.

* See the valuable paper, by Chief-Justice Cooley, on "The Judicial Functions of Surveyors," read before the Michigan Association of Engineers and Surveyors, and printed in full in Appendix A. See also Appendix G.

Again, an entire side which has evidently been omitted from the description may be supplied if the monuments and area may thereby be made to agree with the description. If possible, however, every call in the description should be satisfied by the surveyor in locating the property.*

3. All descriptions in deeds must be construed in the light of what was known to and in the minds of the parties at the time the description was first written, and with reference to such plats, facts, and monuments as then existed.

4. The law presumes the deed was drawn with an honest intent to convey the property. The description must, therefore, be construed, if possible, in such a way as to make it effectual rather than void.

5. If the lines are fixed by monuments, and these can be clearly identified, and are used in the description, such lines are the true boundaries, and are to be determined by a resurvey, even though they differ from the plat, or from the description given in the deed. Any conclusive evidence of the original location of these lines and monuments will, therefore, overrule all surveys or other forms of evidence of where they *should* have been. If the boundaries were not marked at the time the plat was made, then the description is to govern, subject to the rules on excess and deficiency given below.

6. Where land is simultaneously subdivided into numerous tracts, as in the case of the United States land surveys and in the case of town plats, all the marks, lines, and monuments set in the original survey for subdivision serve as marks, lines, and monuments for every tract or lot in the original survey and are some evidence of the location of each tract or lot. In the absence of monuments marking the location of a particular tract or lot other monuments of the same original survey may be used, but monuments placed in preceding or subsequent surveys, or in surveys of adjoining territory not a part of the given subdivision, cannot be so used.

7. In the absence of monuments which can be identified, *conclusive* evidence of the original position of such monuments, or of the lines themselves, may set aside the courses and distances called for in the deed. In short, boundaries may be proved on such testimony and evidence as may be adduced to establish any other fact. The surveyor should, however, give great weight to the courses and distances called for, as a part of such material evidence.

8. Where streets have been opened and used for a long period and the lines marked by fences or other material boundary, and these lines have been acquiesced in without protest, such marks obtain the force and authority of monuments and should not be disturbed because of any disagreement with the original plat and description.

9. All monuments established in United States land surveys are presumed to be equally well placed and have equal authority or weight in determining boundaries. Thus a quarter-section corner has the same weight as a township or section corner, even in fixing a township line. Also, section corners set on lines closing on the north and west sides of townships, though not lying on the original township lines, should govern the location.

* See also Art. 304, Chap. XII., on City Surveying.

10. In the United States land surveys the several sized tracts are section, quarter section, half-quarter section, and quarter-quarter section. These are presumed to contain 640, 160, 80, and 40 acres respectively, and the Government sold them as containing these amounts.* The manner of subdividing a section is defined by law (see p. 183), and hence any actual subdivision of section into quarters and a quarter-section into halves and quarters again is always subject to revision and correction until the law is satisfied, except that the quarter-section corners, planted on the section lines by the United States Deputy Surveyors, cannot be changed. Other subdivisions than those here named are not subject to any law as to the methods to be pursued.

11. In order that monuments may control when inconsistent with the courses and distances used in the description, they must be mentioned in the deed. If not so mentioned, or if mentioned but not capable of identification, then the courses and distances govern.

Particular Cases.—1. Where land is described as being "owned and occupied," the actual line of occupation is a material call of the deed.

2. Where a boundary line is defined by distance and terminus at a known point or line, this known terminus fixes the length of the line. If the position of the terminus is uncertain the distance governs.

3. Acquiescence in a given boundary erroneously placed does not alone fix the boundary if the issue has not arisen, and if the fact of such error has not come to the attention of the parties.

4. A course described as running from point A to point B is presumably a straight line, but if not so stated it may be construed as a crooked or curved line if it is understood to follow some natural feature of the landscape.

5. The terms "southerly," "westerly," etc., are to be construed as meaning due south, due west, etc., if there is nothing to indicate the contrary. Also where terms of approximation are used, such as "about," "as near as may be," and the like, if the exact figures given fit the case and satisfy the description as well as any other, the interpretation is limited to the figures stated.

6. Where the described boundaries are complete and consistent, but inconsistent with the stated area, the boundaries hold as against the area. If the boundaries are doubtful the area may control.

7. Where the call is simply for a given area without dimensions it must be taken in the form of a square if such a rendering is not excluded by some other condition. If one side of the tract is given in line and distance it must be laid out as a rectangle upon such side.

8. In case of an accompanying plat showing monuments, courses, and distances, which plat is referred to in the deed, but the description not repeated in the text, the description will hold.

* Whenever these legal subdivisions are mentioned *as such* in deeds of conveyance, as "the S. E. 40 acres of the N. E. quarter of section 10," etc., nothing more is intended than simply "the S. E. quarter of the N. E. quarter of section 10," etc., and the conveyor cannot be held for the full area named.

9. Land bordering on a public highway usually takes to the center of the highway, unless expressly stated to the contrary. This does not apply to city and town lots where the streets have been reserved in the original plat.

10. All surveys and descriptions should close when platted, and the surveyor is usually at liberty to use his judgment in correcting either courses or distances or both, where no monuments are identified with sufficient certainty to give them authority. Where monuments remain they control the boundaries so far as they go, in which case the description is not obliged to close, or if it is made to close the points marked by monuments must not be disturbed.

In closing a survey or description, there being no other guide, that method will be used which will convey the greater quantity of land in accordance with the principle that the description is to be construed in favor of the purchaser.

11. Where the bearings of all the courses are given and one course can be identified with certainty, the declination (or "variation") of the needle used in the original survey should be found by setting up on this course and the declination thus found used for all the courses.

12. When a course is defined as starting at a given point on a navigable stream or traveled highway and running a certain distance to another point on said stream or highway, the distance is to be measured along the line of the bank of the stream or along the highway and not in a straight line, unless it is specifically so stated. If the stream is not navigable and the presumption is against it being a boundary, the distance is to be measured in a straight line.

But where a tract of land is described as bordering on or fronting a certain distance on a stream, in the absence of other controlling facts such distance must be measured in a straight line between the extremities of the opposite boundaries.

195. Water Boundaries and Meandered Lines.—1. Meandered lines on the United States land surveys were run for the purpose of outlining lakes and rivers and are in no sense boundary lines. They served for computing the areas of the fractional quarter-sections which were used in the first sales by the Government. But the real boundary is the center of the stream if not navigable and the line of ordinary high water or line of vegetation if navigable. In the case of lakes and ponds described as boundary lines the ownership is to the water's edge.

2. In extending side boundaries beyond the meandered lines to the river-bank or lake-shore, such extensions beyond the meandered lines should run at right angles to the shore. This rule also applies to city lots and to all lands fronting on bodies of water. An exception to the rule is the following:

3. When the waters of a lake recede from drainage or any other cause, or when a river or stream shifts its course, the accretion or "made land" belongs to the abutting property and should be divided in proportion to the original lengths of water-frontage. If the land thus acquired is the valuable consideration then the original side boundaries are to be extended so as to divide the new area among

the abutting tracts in proportion to their original frontages; but if the length of the new frontage is the desirable thing, then the new *line of frontage* is to be saved to the original tracts possessing such privileges, in due proportion. In either case the extension of the side boundaries will usually involve angles at the meandered or original water-line and these extensions will, in general, run nearly at right angles to the new shore line.

4. A "bank" of a stream is the continuous line where vegetation ceases. A "shore" is the exposed ground below the bank line.

5. The rights of ownership extend to the centre lines of non-navigable streams and lakes, but only riparian rights obtain in the beds of navigable streams or lakes.

6. Where land is specifically bounded by "the bank," or "along the bank" of a stream or lake, these words will exclude all ownership of the bed of such stream or lake. Whether they would exclude riparian rights also would depend on the circumstances of the case and on the understanding of the parties.

In computing the area of a survey the terms "from," "to," or "with" the bank of a stream mean to low water mark.

7. In the case of meandered river-banks or lake-fronts on the United States land surveys, the computed areas included only up to the meandered line, all outside of that belong to the tract by a natural right. Hence in any subsequent sales of the tract the area should only be computed to the meandered line unless the conveyance specifically calls for an extension "to" or "along" the shore or bank, in which case the area would be computed to low water mark, as above stated.

8. Similarly, when an area is to be laid off from a tract bounded in part by a meandered line, this area should be computed only up to the meandered line unless otherwise specifically stated.

9. Islands in streams unsurveyed by the United States and unappropriated belong to the abutting land on that side of the *filum aquæ* or the central thread of the low water channel on which the island itself lies.

196. Surplus and Deficiency.*—1. Surplus or deficiency, either of distances or of areas, does not invalidate a conveyance.

2. In the case of contiguous tracts where no monuments were established, or where they have been lost, the purchasers receive their full measure of ground, in the order of purchase from the original owner, the last purchaser receiving the surplus or losing the deficiency.

In the case of city lots, sold by number, any surplus or deficiency found on the ground should be divided proportionally among all the lots affected, but a suit for a proportionate part of the surplus would probably not hold, and in case of deficiency, if all but the last purchaser should take his full portion, the last man would probably have to content himself with the remainder, and pay for only so much as he gets.

* See also Art. 305, Chap. XII., on City Surveying.

EXAMPLES.

1. Compute the area, plot the survey, and determine error of closure from the following field-notes :

Station.	Bearing.	Distance.
A	S. $46\frac{1}{4}°$ E.	20.00 ch.
B	S. $74\frac{1}{2}$ E.	30.95
C	N. $33\frac{1}{4}$ E.	18.80
D	N. 56 W.	27.60
E	W.	21.25
F	S. $51\frac{3}{4}$ W.	13.80

Answer $\begin{cases} \text{Area} = 104.4 \pm \text{acres.} \\ \text{Error of closure} = 1 \text{ in } 201. \end{cases}$

This being a compass-survey, the errors in latitude and departure must be distributed in proportion to the lengths of the courses, regardless of their bearings, or according to Rule 1, p. 200. If the errors in the bearings (or deflection angles) had been very small as compared with the errors in measuring the distances, as is the case when the deflection angles are measured with a transit, then Rule 3, p. 201, should have been used.

2. Find the area and error of closure from the following field-notes :

Station.	Bearing.	Distance.
A	E.	130 rods.
B	N. 8° E.	137
C	N. 81 W.	186
D	S.	54
E	S. 36 W.	125
F	S. 45 E.	89
G	N. 40 E.	70

What would be the resulting difference in area from the use of Rules 1 and 3?

3. In Example 1, suppose the length and bearing of the first course were unknown. Let these be found as in Case I., Art. 186.

4. Suppose the length of course A and bearing of B are unknown in same example. Compute by Case II.

5. Let the first two bearings be unknown. Compute them by Case III.

6. Let the lengths of the first two courses be unknown. Find them by Case IV.

7. Let it be required to cut off twenty-five acres from the west end of the tract given in Example 1 by a line passing through a point on the course BC at a distance of ten chains from B. Find the length and bearing of the division-line and the other intersecting point on the boundary.

8. Let it be required to divide the tract given in Example 1 into three equal portions by north and south lines. Find the length and points of intersection of such lines with the boundary-lines.

9. Compute the coördinates of the corners of the tract given in Example 1, taken with reference to a point 35 chains directly south of A, and then compute the area of the tract from these coördinates by the formula given in Example 1. This area should, of course, be the same as that obtained by any other method where the same balanced latitudes and departures are used.

10. An irregular tract of land has a straight line run through it and rectangular offsets taken to the boundary. Find the area of the tract from the following notes:

Distance.	Width.
ch.	ch.
0	2.35
10	8.42
14	12.60
20	11.38
25	10.75
28	6.15
30.50	0.00

Is it significant whether or not this tract lies on both sides or wholly on one side of the base-line?

11. Compute the area of the tract of which the following are the field-notes.

The rectangular offsets are taken on both sides of a straight axial line, R signifying right and L left.

Distances.	Side.	Width or Length of Offset.	Distances.	Side.	Width or Length of Offset.
ch.		ch.	ch.		ch.
0	R	4.23	18	R	15.80
0	L	0.00	20	L	5.00
5	R	7.16	25	R	12.20
7.50	L	3.45	30	L	2.62
10	R	12.68	30	R	6.48
10	L	6.00	30	L	0.00
12	R	10.75			

CHAPTER VIII.

TOPOGRAPHICAL SURVEYING BY THE TRANSIT AND STADIA.*

197. A Topographical Survey is such a one as gives not only the geographical positions of points and objects on the surface of the ground, but also furnishes the data from which the character of the surface may be delineated with respect to the relative elevations or depressions.†

198. There are three general methods of making such a survey.

First, with a compass (or transit) and chain, to determine geographical position, and with a level for obtaining relative elevations.

Second, with a plane-table, either with or without stadia-rods.

Third, with a transit instrument and stadia rods.

The first method is very laborious, slow, and expensive. It is therefore not adapted to large areas. The second method has been more extensively used for this purpose than any other. The use of the plane-table is fully described in Chapter V. This method is giving place, however, to the third, which has been in use in America since about 1864, when it was officially adopted on the United States Lake Survey. The system was first used in Italy about 1820. In what follows, the third method will alone be described.

* The word "stadia" is Italian and was originally used to designate the rod used by the inventor of the method. It is now too firmly established to be changed. On the U. S. Coast and Geodetic Survey the word "telemeter" is used in place of "stadia," but this, which very properly means *distance-measurer*, has been appropriated for other appliances used for *measuring at a distance*, as temperature, for example. It would therefore seem that "stadia" is the better word to use.

† See also Appendix G.

199. The Principle of the location of points by the transit and stadia, both horizontally and vertically, is that of polar coördinates. That is, the location of the point geographically is by obtaining its angular direction from the meridian through the instrument, which is read on the limb of the transit, and its distance from the instrument, which is read through the telescope on the stadia-rod which is held at the point. This distance is found by observing what portion of the image of the graduated rod is included between certain cross-hairs in the telescope. The farther the rod is from the instrument, the greater is the portion of the rod's image which falls between the cross-wires.

For elevation, the vertical angle is read on the vertical circle of the transit, when the telescope is directed towards a point of the stadia-rod as far from the ground as the telescope is above the stake over which it is set. The tangent of this angle of elevation, or depression, into the given horizontal distance is the amount by which the point is above or below the instrument station.

In this way, both the chain and levelling-instrument are dispensed with, and the slow and laborious processes of chaining over bad ground, and levelling up and down hill, are avoided. The horizontal distances are obtained as well, in general, as by the chain: and the levelling may be done within a few tenths of a foot to the mile which is amply sufficient for topographical purposes.

THEORY OF STADIA MEASUREMENTS.*

200. Fundamental Relations.—In Fig. 61 let LS be any lens, or combination of lenses, used for the object-glass of a telescope.

* For a good description of A New Prismatic Stadia see *Jour. Assn. Eng. Soc's*, vol. xiii., p. 43 (Jan., 1894). One half of the telescope objective is covered with a prism which causes two portions of the rod, rays from which form a fixed angle at the objective, to coincide in the image, as in a sextant, thus dispensing with the use of stadia wires.

Let A_2B_2 be a portion of the object (in this case the stadia-rod), and let A_1B_1 be its image. The point of the object A_2 has its image formed at A_1, and so with B_2 and B_1.

Let F be the position of the image for parallel rays, or for an object an infinite distance away; and let C be the centre of

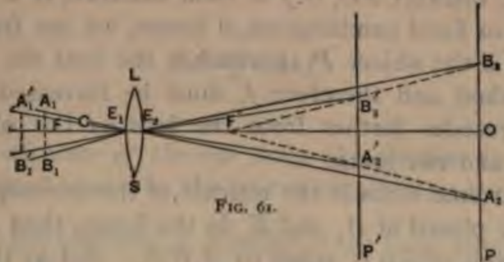

FIG. 61.

the instrument, or the intersection of the plumb-line, extended, with the axis of the telescope.

Let E_1 and E_2 be the "principal points,"* and let the distance $FE_1 = f$ (focal length),

$$\left.\begin{array}{l} IE_1 = f_1 \\ OE_2 = f_2 \end{array}\right\} \text{(conjugate foci),}$$

$A_1B_1 = i$ (for image, intercepted portion),

$A_2B_2 = s$ (for stadia, intercepted portion).

Then, since A_1E_1 is parallel to A_2E_2, and B_1E_1 is parallel to B_2E_2, we have

$$A_1B_1 : A_2B_2 :: IE_1 : OE_2,$$

or,

$$i : s :: f_1 : f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, from the law of lenses we have

* As optics is generally taught in the English text-books, E_1 and E_2 are made to coincide in a point at or near the centre of the lens; and this is called the "optical centre." The "principal points" of the ordinary objective fall inside the surfaces of the lens, but they never coincide. The ordinary theory is sufficiently approximate for the development of stadia formulæ but it saves confusion to make the conditions rigid, and it is equally simple.

[redacted]

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

On these two equations rests the whole theory of stadia measurements.

Since the distance $FE_1 = f =$ focal distance, is a constant for any lens or fixed combination of lenses, we see from equation (2) that if the object P approaches the lens the distance f_2 is diminished, and therefore f_1 must be increased; that is, the image recedes farther from the lens as the object approaches it, and *vice versa.*

If the extreme wires in the reticule of the telescope be supposed to be placed at A_1 and B_1 in the figure, then $A_1E_1B_1$ is the visual angle which is equal to $A_2E_2B_2$. But as the image changes its distance from the objective as the object is nearer to or farther from the instrument, so the reticule is moved back and forth,* for it must always be in the plane of the image. Therefore $IE_1 = f_1$ is a variable quantity, while A_1B_1 is constant for fixed wires. *Therefore the visual angles at E_1 and E_2 are variable.*

If these angles were constant, the space intercepted on the rod, and the distance of the rod from the objective, would be in constant ratio. Since this is not true, we must find the relation that does exist between the distance E_2O and the space intercepted on the rod, A_2B_2.

From equation (1) we have

$$\frac{1}{f_1}=\frac{s}{if_2};$$

but from equation (2) $\quad \frac{1}{f_1}=\frac{1}{f}-\frac{1}{f_2}.$

* If the objective is moved in focusing it does not appreciably affect these relations.

Equating these two values of $\frac{1}{f_1}$, we have

$$\frac{s}{if_1} = \frac{1}{f} - \frac{1}{f_1};$$

or

$$f_1 = \frac{f}{i}s + f; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is, ***the distance of the rod from the objective*** is equal to the intercepted space in the rod multiplied by the constant ratio $\frac{f}{i}$, plus the constant f, where f is the focal length of the objective, and i is the distance between extreme wires. If the distance between the extreme wires be made 0.01 of the focal length of the objective, then the distance of the stadia-rod *from the objective* (rigidly from E_1) is a hundred times the intercepted space on the rod, plus the focal length of the objective.

Again, if a base be measured in front of the instrument, ***with its initial point a distance* f *in front of the object-glass of the telescope,*** then the rod may be held at any point on this base-line, and ***its distance from the initial point, and the space intercepted by the extreme wires, will be in constant ratio.***

The lines A_1F' and B_1F' in Fig. 61 show this relation, for they are the lines defining the space on the rod which is intercepted by the extreme wires as the rod moves back and forth. Evidently the rod cannot approach so near as F', for then the image would be at an infinite distance behind the lens. Usually the extreme position of reticule does not correspond to a position of rod nearer than ten to fifteen feet.

It must be remembered that any motion of the eye-piece, with reference to the image and wires, is only made to accommodate different eyes, and has no effect in changing the relation of wire interval and image. The eye-piece is simply a magnifier with which to view the image and wires, but in all erecting-instruments it also reinverts the image so as to make it appear upright. The effect of the eye-piece has no place in the discussion of stadia formulæ.

If the distance of the stadia is to be reckoned from the centre of the instrument, which it usually is, and if this distance $= d$, and the distance from the centre of the instrument to the objective (CE_1 in Fig. 61) $= c$, then we have, from (3),

$$d = f_1 + c = \frac{f}{i}s + f + c. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since f, i, and c are constant for any instrument, we may measure f and c directly, and then find the value of i by a single observation. Proceed as follows:

1st. Measure the distance from the centre of the instrument (intersection of plumb-line with telescope) to the objective, and call this c.

2d. Focus the instrument on a distant point, preferably the moon or a star, and measure the distance from the plane of the cross-wire to the objective, and call this f.

3d. Set up the instrument, and measure the distance $f + c$ forward from the plumb-line, and set a mark. From this mark as an initial point, measure off any convenient base, as 400 feet.

4th. Hold the rod at the end of this base, and measure the space intercepted by the extreme wires. If we call the length of this base b, and the distance intercepted s, then we have, from equation (3),

$$b = \frac{f}{i}s,$$

or

$$i = \frac{s}{b}f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Here we have the value of i in terms of known quantities.

If it is desirable to set the wires at such a distance apart that $\frac{s}{b}$ will be a given ratio, as $\frac{1}{100}$, then i must equal $0.01f$. It is possible to set the wires by this means to any scale, so that a rod of given length may read any desired maximum distance.

If it is desired that $\frac{f}{i}$ should be determined with great accuracy for a given instrument, with wires already set, so as to have a coefficient of reduction for distance, for readings on a rod graduated to feet and tenths, for instance, proceed as follows:

Make two sets of observations for distance and intercepted interval. The distances should differ widely, as 50 feet and 500 feet, or 100 feet and 1000 feet, according to the length of rod used. The shorter distance should not be less than 50 feet, and the longer one not more than 1000 feet with the most favorable conditions of the atmosphere. The distances are to be measured from the centre of the instrument. Make several careful determinations of the wire interval at each position of the rod, and take the mean of all the results at each distance, and call that the wire interval, s, for that distance, d. We then have two equations and two unknown quantities, these latter being $\frac{f}{i}$ and $(f + c)$ in the formula, equation (4),

$$d = \frac{f}{i}s + (f + c).$$

Here the d and s are observed, and $\frac{f}{i}$ and $(f + c)$ are found. Knowing these, a table could be prepared giving values of d for any tabular value of s for that instrument.

This applies to the reading of distances from levelling-rods.

Some engineers prefer, in this case, to observe the wire interval for various measured distances, from the shortest to the longest, to be read in practice, and prepare a table by interpolation. If the observed positions are sufficiently numerous, this method should give identical results with those obtained by the use of the formula. The two methods may be used to check each other.

From equation (4) we see that the distance of the rod from the centre of the instrument is a constant ratio $\left(\frac{f}{i}\right)$ times the intercepted space on the rod, plus a constant $(f + c)$.

If diagrams or designs be drawn on the stadia-rod to the scale $\frac{i}{f}$, or so that $10 \times \frac{i}{f}$ yards on the rod would correspond to 10 yards in distance, and if the rod were decorated with symbols of this size, then the distance of the rod from the instrument could be read at once by noting how many symbols were intercepted between the wires. To this distance must then be added the small distance $(f + c)$, which is from 10 to 16 inches in ordinary field-transits. On all side-readings, taken only to locate points on a map, this correction need not be added, as one foot is far within the possibilites of plotting.

201. On the Government Surveys the base is usually measured *from the centre of the instrument*, and its length is taken as about a mean of those which the stadia is intended to measure, and the symbols scaled by this reading. Then, of course, the distance read is always in error by a small amount, except when it is the same as the base for which it was graduated. For all shorter distances the reading is too small, and for all greater distances the reading is too large. Sometimes several different lengths of base are taken, as 400, 600, and 800 feet, all from centre of instrument, and a mean value of wire interval used for giving the scale for the diagrams. This is practically the same as the other, for in either case the scale is correct for but a single distance.

The correction to any reading on a stadia so graduated, in order to give the distance from the centre of the instrument, is

$$K = (c + f)\left(1 - \frac{B}{B'}\right), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where K = correction, in feet;
B = distance read on stadia, in feet;
B' = length of base, in feet, for which the stadia was graduated.

If $B' = 1000$ feet, $B = 100$ feet, and $c + f = 1.5$ feet, then

$$K = 1.5\left(1 - \tfrac{100}{1000}\right) = +1.35 \text{ feet.}$$

If B had been 2000 feet, then

$$K = 1.5\left(1 - \tfrac{2000}{1000}\right) = -1.5 \text{ feet.}$$

These corrections are not usually applied.

Another method of determining the scale for graduating the rod is to measure the base from the plumb-line, as above, and then, from a fixed point on the lower part of the rod, find the intervals that correspond to various distances, as 100 feet, 200 feet, 300 feet, etc., and mark these on the board, always keeping the lower wire on the fixed, initial point of the rod. Then each 100-foot space is subdivided into ten equal parts, or symbols; so that, in reading the rod afterwards, if the lower wire is always set on the initial point, the reading always gives the correct distance from the centre of the instrument.

The objection to this method is that the initial point on the rod cannot always be seen, on account of obstructions. It is also desirable to avoid reading either wire near the ground, on account of excessive refraction there.

202. Graduation of Rod to Eliminate the Effects of Differential Refraction.—Mr. L. S. Smith, C. E., has

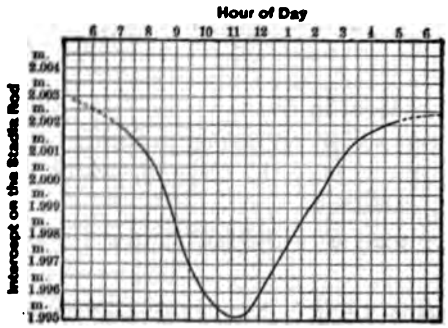

FIG. 61a.

shown* that the effect of refraction is much greater near the ground than it is a few feet up, and also that it is

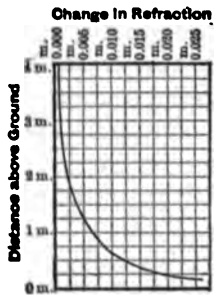

FIG. 61b.

greater in the middle of the day than it is mornings and evenings. If the stadia rod be graduated for average

* See *Bulletin of the University of Wisconsin*, Engineering Series, vol. i., No. 1895. Also *Engineering News*, vol. xxxiii., p. 364.

conditions, it will be found that the distances read on the rod are too large in the morning and evening hours, and too small in the hours from 9 A. M. to 2 P. M. This is shown in Fig. 61*a*. In Fig. 61*b* the rapid change in the refraction is indicated as the line of sight approaches the ground. In both of these figures the metre is the unit of measure.

As a result of Mr. Smith's elaborate experiments, the following rules for graduating (or standardizing) a stadia rod, become imperative. These rules apply to any or all the methods of graduating described in the previous article.

1. *Determine the wire interval for various distances over average ground and for all working hours of the average day.*

2. *For a radical change of field or season conditions, redetermine the wire interval, and let this redetermination cover all working hours, of the average clear-cloudy day.*

3. *It is best to avoid reading the lower cross-wire near the ground, either in the interval determination or on the field work, but the interval determination readings should agree in this respect with the average field practice.*

If stadia rods be graduated, or standardized, in this way, the accuracy attainable will be considerably increased. The importance of this matter had not been fully understood previous to Mr. Smith's experiments.

203. Adaptation of Formulæ to Inclined Sights.—The discussion given in Art. 200 is applicable to horizontal sights only.

If the rod be held on the top of a hill, and the telescope pointed towards it, the reading on the rod will give the linear

[redacted]

distance from instrument to rod, ***provided the rod be held perpendicular to the line of sight.*** As it would be inconvenient to do this, let the rod be held vertical in all cases. When the line of sight is inclined to the rod, the space intercepted is

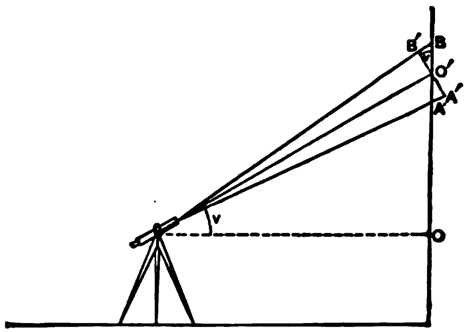

FIG. 62.

increased in the ratio of 1 to the cos of the angle with the horizon.

Thus, the space $A'B'$ (Fig. 62) for the rod perpendicular to the line of sight becomes AB for the rod vertical. But $A'B' = AB \cos v$.*

Let $A'B' = r'$, the reading on the stadia for perpendicular position; and

Let $AB = r$, the actual reading obtained for a vertical position.

Then $r' = r \cos v$.

But in equation (4) we have $\frac{f}{i} s = r'$, and therefore $r' + c$

---

* This assumes that $A'B'$ is perpendicular to CB and CA, which it is practically, since the angle ACO' is so very small, usually about 15′.

$+f$ is the distance CO'; whereas the distance on the horizontal, CO, is generally desired, and for this we have

$$CO = d = CO' \cos v = (r' + c + f) \cos v$$
$$= r \cos^2 v + (c + f) \cos v. \quad (7)$$

This is the equation for reducing all readings on the stadia to the corresponding horizontal distances.

The vertical distance of O' above O is equal to CO' sin V.

But $$CO' = r' + f + c = r \cos v + f + c,$$

hence

$$OO' = h = r \cos v \sin v + (f + c) \sin v$$
$$= \tfrac{1}{2} r \sin 2v + (f + c) \sin v. \quad (8)$$

Equation (8) is used for finding the elevation of the point on which the stadia is held above or below the instrument station.

204. Table V.*gives the values d and h computed from these formulæ for a stadia reading of 100 feet (or metres, or yards), with varying angles up to 30°.

It will be noted that the second term in the right member of equations (7) and (8) is always small, and its value depends on the instrument used. The values of this term are taken out separately in the table; and three sets of values are given of $(c+f)$,—viz., 0.75 feet, 1.00 feet, and 1.25 feet. If the work does not require great accuracy, these small corrections may be omitted.

The use of the table directly involves a multiplication for

* See also Colby's Slide Rule, p. 265.

every result obtained. Thus, if the stadia reads 460 feet, the angle of inclination 6° 20′, and we have $f+c=1$ foot, then

$$d = 4.60 \times 98.78 + 0.99 = 455.4 \text{ feet,}$$

and $$h = 4.60 \times 10.96 + 0.11 = 50.53 \text{ feet.}$$

The table is not generally used for reductions for d when the angle of elevation is less than 3 to 5 degrees. When $v = 5°\ 44'$, this reduction amounts to just *one per cent.* When an error of 1 in 100 can be allowed, then the reduction to the horizontal would not be used under 6°. If the second term in $c+f$ be also neglected, these two errors tend to compensate; and if $c+f$ for the instrument used is 1 foot, and both these corrections be omitted, they do exactly compensate when the

stadia	reading is	100 feet,	vertical	angle	5° 44′.
"	"	200	"	"	4° 04′.
"	"	300	"	"	3° 20′.
"	"	400	"	"	2° 52′.
"	"	500	"	"	2° 32′.
"	"	1000	"	"	1° 46′.
"	"	2000	"	"	1° 18′.

Therefore the reduction to the horizontal need *never* be made when v is less than 2°, and it generally may be neglected when v is less than 6°.

In obtaining the difference of elevation, h, the term in $c+f$ may be omitted for all angles under 6° if errors of 0.1 foot are not important. For elevations on the main line, however, this term should always be included.

In practice, therefore, the tables are mostly used to obtain the difference of elevation from the given stadia reading and angle of elevation.

PORRO'S TELESCOPE.

204a. The Reading Angle with Vertex at the Center of the Instrument.—In 1823 Mr. Porro, a Piedmontese officer, and afterwards a professor at Milan, invented a telescope which brings the vertex of the reading angle, $A_2F'B_2$, Fig. 61, to the center of the instrument, and so gives the true reading for all distances, without the $(c + f)$ correction, which must always be applied with the ordinary telescopes. Although this is not a very important matter in stadia work, yet because of this slight correction, or source of error when not applied, many engineers and surveyors have heretofore declined to use the stadia methods at all. The great advantages of these methods are coming to be better known, however, and soon the demand for Porro's telescope may warrant its manufacture for ordinary transits. It is not now (1890) made anywhere in America. This telescope would serve as well for all other purposes, and although really little better for stadia work, it removes an objection which has, more than anything else, caused the stadia methods to be generally ignored.

The construction of the telescope is shown in the accompanying figure.

The lens at O is the objective, having a longer focal length than the ordinary objective. At P is an

Fig. 61a.

auxiliary lens, by means of which all pencils of rays originating on the reading-angle lines, CA_2 or CB_2, are brought to a focus somewhere on the *parallel* lines mA_1 and nB_1, respectively. In the figure only one such pencil of rays is shown, which emanates from B_2. The cross-wires are at A_1 and B_1, and since all pencils of rays originating on the reading-angle lines, which now meet at the center of the instrument, will be brought to a focus on horizontal lines through the cross-wires, it follows that the intercepted space on the rod is always proportional to the distance of the rod from the center of the instrument.

The point F_p is the principal focus of the lens P, its focal length being much smaller than that of the lens O, since its principal focus is at a considerable distance back of P, as at F_0'. The point F_0 is the position of the principal focus of O in front, and is the point where the reading angle has its vertex with the ordinary telescope, as shown in Fig. 61. The points C and F_p are conjugate foci of the lens O. An image will always be formed to the right of P, even for an object nearer to the objective than F_0. The movements of the objective (or eyepiece) for focusing at different distances is less for this telescope than for the ordinary telescope. The relative position of the lenses O and P is fixed, and they must move together if the objective is moved in focusing.

The significance of the arrangement lies in the fact that the ray of light which traces the line B_2C, the limiting line for the reading angle, traverses the principal focus of the lens P, and hence emerges from this lens along the horizontal line, nB, on which the cross-wire is placed. The lenses O and P are so placed that the point F_p, which is the principal focus of P, is also the focus of O, which is conjugate with the point C, the center of the instrument. Any further discussion of the theory of this instrument is out of place here until it is manufactured and used in this country.*

* For the mathematical discussion of this telescope see an article by the author in *Engineering News*, November 8, 1890.

205. Reduction Diagram.—Since the use of these tables involves a multiplication each time, and since a table for varying distances and angles would be very voluminous, it is preferable to take out the elevations from a diagram. Such a diagram has been prepared, to be used in place of the table. It is arranged with both coördinates in feet, but can be used for both coördinates in metres, since the same unit is used for both. It will only be neccessary to re-number the divisions, to adapt it to the new scale.

This diagram has been prepared with great care, and is arranged to give distances to 500 yards or metres, or 1500 feet, with elevations to 50 feet. For longer distances or higher elevations for a single pointing, the results may be obtained from the table. Elevations are taken off from the diagram to the nearest tenth of a foot, with great readiness; as the smallest spaces are 2 millimetres square, and these correspond to two-tenths of a foot in elevation. It is of more convenient use than extended tables, and is just as accurate; the nearest tenth of a foot being quite as exact as one is warranted in writing elevations when obtained in this manner.

Corrections to the distances read are also obtained from this diagram for large vertical angles.*

THE INSTRUMENTS.

206. The Transit.—That the transit may be best adapted to this work, there are certain features it should possess, though all of them are by no means essential. They will be named in the order of their importance.

1st. The horizontal limb should be graduated from zero to 360°, preferably in the direction of the movement of the hands of a watch.

* The diagram is printed on heavy lithographic paper 20 by 24 inches, from an engraved plate, and can be had from the publishers of this volume. Price 50 cents, post paid. See also Colby's Slide Rule, p. 265.

[redacted]

2d. The instrument should have a vertical circle rigidly attached to the telescope axis, and not simply an arm that is fastened by a clamp-screw, and which reads on a fixed arc below. So much depends on the vertical circle holding its adjust ment that its arrangement should be the best possible. Since the telescope is not transited, the vertical circle need not be complete.

3d. The telescope should be inverting, for two reasons: *first*, in order to dispense with two of the lenses, and so obtain a better definition of image; and, *second*, that the objective may have a longer focal length, thus giving a flatter image and a less distorted field.

4th. The stadia wires should be fixed instead of adjustable, as in the latter case they are not stable enough to be reliable.

5th. The bubbles on the plate of the instrument should be rather delicate, so that a slight change in level may become apparent. They should also hold their adjustments well. This is very important, in order that the readings of the vertical angles may be reliable. It is also of great importance in carrying azimuth where the stations are not on the same level.

6th. The horizontal circle should read to thirty seconds; and there should be no eccentricity, so that one vernier-reading shall be practically as good as two.

7th. The instrument (or tripod) should have an adjustable centre, for convenience of setting over points.

8th. A solar attachment to the telescope will be found very convenient. In most regions the azimuth can be checked up by the reading of the needle, but in many places this is not reliable.

207. Setting the Cross-wires.—The engineer should always have at hand a spider's cocoon of good wires, and a small bottle of thick shellac varnish. If the dry shellac is carried it may be dissolved in alcohol. If no such cocoon is at hand a spider may be caught and made to spin a web. The small, black, out-door spider makes a good web for stadia purposes. A new wire should be allowed to dry for a few minutes, and an old one should be steamed to make it more elastic. The wires for stadia-work should be small, round, and opaque. Some wires are translucent, and some are flat and twisted like an auger-shank.

Scratches must be made across the face of the reticule where the wires are to lie. These must be made with great care, so as to have them equally spaced from the middle wire, parallel to each other, and perpendicular to the vertical wire. The distance apart of the extreme wires is to be computed by equation (5) for any desired scale on the rod.

Take a piece of web on the points of a pair of dividers, by wrapping the ends several times about the points, which should be separated by about an inch; stretch the wire, by spreading the dividers, as much as it will bear; and lay the dividers across the reticule in such a way that the web comes in place. The dividers must be supported underneath, so that the points will drop just a trifle below the top of the reticule; otherwise they would break the web. Move the dividers until the web is seen, by the aid of a magnifying-glass (the eye-piece will do), to be in exact position. Then take a little shellac on the end of a small stick or brush, and touch the reticule over the web, being careful to have no lateral motion in the movement. The shellac will harden in a few minutes, when the dividers may be removed. Shellac is not soluble in water.

208. Graduating the Stadia-rod.—The stadia-rod is usually a board one inch thick, four or five inches wide, and twelve to fourteen feet long. Sometimes this is stiffened by a piece on the back. To graduate the rod, it is necessary to know what space on the rod corresponds to a hundred feet (or yards, or metres) in distance. Either of the three methods cited on pp. 230–1 may be used for doing this, but the first is recommended. Thus, measure off $c + f$ in front of the plumb line, and set a point. From this point measure off any convenient base, as 200 yards, on level ground, and hold the blank rod (which has had at least two coats of white paint), at the end of this base-line. Have a fixed mark or target on the upper part of the rod, on which the upper wire is set. Have an assistant record the position of the lower wire as he is directed by the observer. Some sort of an open target is good for this purpose, but any scheme is sufficient that will enable the observer to fix the position of the extreme wires at the same moment with exactness. This work should be done when there is no wind, and when the atmosphere is very steady: a calm, cloudy day is best. Repeat the operation until the number of results, or their accordance, shows that the mean will give a good result. If the base was 200 yards long, divide this space into two equal parts, then each of these parts into ten smaller parts, and finally each small space into five equal parts; and one of these last divisions represents two yards in distance. Diagrams are then to be constructed on this scale, in such a way that the number of symbols can be readily estimated at the greatest distance at which the rod is to be read. The individual symbols should be at least three inches across; so that, if one of these is to represent ***ten units***, as yards or metres, then 100 units will cover $2\frac{1}{2}$ feet, and a rod 14 feet long will read a distance of 560 units (yards or metres). If it is desired to read distances of a quarter of a mile or more, the rod should be graduated to read to yards (or five-foot units, or metres); but if it is not to be used for distances over 500 to 1000 feet, it might be graduated to read to feet. This question must be decided before the wires are set, and then they must be spaced accordingly.

In measuring the base, care should be taken to test the chain or tape carefully by some standard.

If the rod is to be graduated to read to feet, of course the base should be some even hundreds of feet, as 600.

In Fig. 63 are shown four designs for stadia-rods which have been long in use, and are found to work well. They are intended to be all in black on a white ground.* It will be noticed that the shortest lines in these diagrams all cover a space of *two units* on the rod. In diagrams 2 and 3 the units are either yards or metres, while in 1 they are units of five feet each. In diagram 4 the units are of two feet each. The

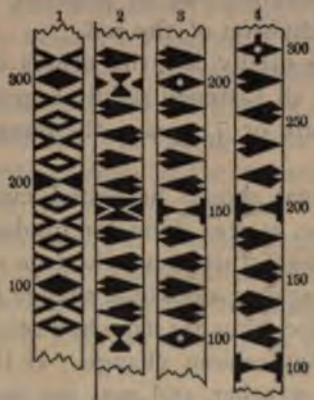

FIG. 63.

successive units are found at the middles and limits of these lines and spaces. Wherever the wire falls, there should be a *white ground* on some part of the cross-section ; and the more white ground the better, provided the figures are distinct. The black paint may be put on heavy, so that one coat will be sufficient.

The 50- and 100-unit marks should be distinguished by special designs. There should usually be at least two boards with each instrument, and sometimes three and four are needed. Of course, these are all duplicates. After the *unit scale* is obtained, or the space on the rod corresponding to a hundred

* Some engineers prefer red on the 100-unit figures.

units in distance, these 100-unit spaces should be so distributed as to be *symmetrical with reference to the ends of the rod.* The reason of this will appear later. Having determined how many 100-unit spaces there will be on the rod, fix the position of the two end 100-unit symbols with reference to this symmetry, and then the rod is subdivided from these points.

Special pains should be taken to have the angular points of the diagrams well defined and in position. These points are on the lines of subdivision of the rod.

After one rod is subdivided, the others of that set may be laid alongside, and all fastened rigidly together; and then, by means of a try-square or T-square, the remaining rods may be marked off.

The wire interval should be tested every few months by remeasuring a base, as was done for graduation, and reading the rod on it, to see if this shows the true measured distance. This is to provide against a possible change in the value of the wire interval. If the wires are stretched reasonably tight when they are put in, they seldom change, If they are too loose, they swell in wet weather, and may sag some. The reticule should be so firm that the variable strain on the adjusting-screws will not distort it appreciably.

If the wire interval is found to have changed, either the rods must be regraduated, or else a correction must be made to all readings of importance. What are called the "side shots," which make up a large proportion of the readings taken, would not need to be corrected.

If the wires are adjustable, any unit scale may be chosen at pleasure, and the wires adjusted to this scale. Then, if the intervals change, the matter is corrected by adjusting the wires. The adjustable wires are generally used to obtain distances from levelling-rods, where it is desirable that each foot on the rod shall correspond to a hundred feet in distance. For the ordinary stadia-rods, fixed wires are preferable.

GENERAL TOPOGRAPHICAL SURVEYING.*

209. The Topography of a region includes not only the character and geographical distribution of the surface-covering, but also the exact configuration of that surface with reference to its elevations and depressions. Thus any point is geographically located when its position with reference to any chosen point and a meridian through it is found, but to be topographically located its elevation above a chosen level surface must also be known. A topographical survey consists in locating by means of three coördinates a sufficient number of points to enable the intervening surface to be known or inferred from these. Evidently the points chosen should be such as would give the greatest amount of information. As for geographical outline, the corners, turns, or other critical points are chosen, so for configuration the points of changes in slope, as the tops of ridges and bottoms of ravines, or the brow and foot of a hill, are chosen as giving the greatest information.

210. Field-work.—Let it be required to make a topographical survey of either a small tract, a continuous shoreline, or of a large area, for the purpose of making a contour map.

In case of the small tract, any point may be taken as a point of reference, and the survey referred to it as an origin. In case of an extended region, a series of points should be determined with reference to each other, both in geographical position and in elevation. These determined points should not be more than about three miles apart. The points of elevation or bench-marks need not be identical with those fixed in geographical position. These last are best determined by a system of triangulation, and are called "triangulation stations." In the succeeding discussion, the symbol △ will be used for *triangulation station*, and B.M. for *bench-mark*.

* See Appendix F for field methods used on the Mississippi River Survey, and foot-note p. 662.

First, a system of triangulation points is established, the angles observed, azimuths and distances computed, and the stations plotted to scale on the sheet which is to contain the map. This plotting is best done, for small areas, by computing the rectangular coördinates (latitudes and departures), and plotting them from fixed lines which have been drawn upon the map, accurately dividing it into squares of 1000 or 5000 units on a side. They may, however, be plotted directly from the polar coördinates (azimuth and distance) as given by the triangulation reduction. For this purpose, the sheet on which the map is first drawn, called the ***field-sheet***, should have ***a protractor circle printed upon it, about twelve inches in diameter***. These ***protractor sheets*** of drawing-paper can be obtained of most dealers in drawing-materials, or the protractor circle may be printed to order on any given size or quality of paper.* These protractor circles are very accurate, and are graduated to 15′ of arc. Plotting can be done to about the nearest 5′.

Second, a line of levels is run, leaving B.M.'s at convenient points whose elevation are computed, all referred to a common datum. If the △'s are not also B.M.'s, then a B.M. should be left in the near vicinity of each △. This is not essential, however.

Third, the topographical survey is then made, and referred to, or hung upon, this skeleton system of △'s and B.M.'s.

The topographical party should consist of the observer, a recorder, two or three stadia-men, and as many axemen as may be necessary, generally not more than two.

The azimuth, preferably referred to the true meridian, is known for every line joining two △'s, as well as the length of such line.

Set up the transit over a △, and set the horizontal circle

* Messrs. Queen & Co., Philadelphia, or Blattner & Adam of St. Louis, can furnish such sheets.

(which should be graduated continuously from 0° to 360° in the direction of the hands of a watch) so that vernier A will read the same as the azimuth of the triangulation line by which the instrument is to be oriented. Clamp the plates in this position, and set the telescope to read on the distant △. Now clamp the instrument below, so as to fix the horizontal limb, and unclamp above. The azimuths of the triangulation lines are generally referred to the south point as the zero, and in small systems of this sort the forward and back azimuths are taken to be 180° apart. When the instrument has been set and clamped, all subsequent readings taken at that station are given in azimuth by the readings of vernier A on the horizontal limb. For any pointing, therefore, the reading of this vernier gives the azimuth of the point referred to the true meridian, and the rod reading gives the distance of the point from the instrument station. These enable the point to be plotted on the map. To draw the contour lines, elevations must also be known.

If the elevation of the △ is known, measure the height of instrument (centre of telescope) above the △ on the stadia,* as soon as the instrument is levelled up over that station. Suppose this comes to the 212-unit mark. Write in the note-book, as a part of the general heading for that station, "Ht. of Inst. = 212." Then, for all readings from that station for elevations, bring the middle horizontal wire to the 212-unit mark on the rod, and read the vertical angle. From this inclination and distance, the height of the point above or below the instrument station is found. If the rod be graduated symmetrically with reference to the two ends, one need not be careful always to keep the same end down, and so errors from this cause are avoided.

* Or, if preferred, a light staff, about five feet long, may be carried with the instrument for this purpose, it being graduated the same as the stadia rods for this instrument.

The record in the note-book consists of—

1st. *A Description of the Point*, as, "N.E. cor. of house," "intersec. of roads," "top of bank," "C.P." for "contour point," which is taken only to assist in drawing the contours, "⊡ 16" for "stadia station 16," etc.

2d. *Reading of Ver. A.*

3d. *Distance.*

4th. *Vert. Angle.*

These four columns are all that are used in the field. There should be two additional columns on the left-hand page, for reductions, viz.:

5th. *Difference of elevation*, corresponding to the given vertical angle and distance, and which is taken from a table or diagram.

6th. *Elevation*, which is the true elevation of each point referred to the common datum.

The right-hand page should be reserved for sketching.

It will be found most convenient to let the sketching proceed from the bottom to the top of the page; as in this case the recorder can have his book properly oriented as he holds it open before him, and looks forward along the line. The notes may advance from top to bottom, or *vice versa*, as desired. If there are many "side shots" from each instrument station, one page will not usually contain the notes for more than two stations, and sometimes not even for one.

The sketch is simply to aid the engineer when he comes to plot the work, and may often be omitted altogether. One soon becomes accustomed to impressing the characteristics of a landscape on his memory so as to be able to interpret his notes almost as well as though he had made elaborate sketches. For beginners the sketches should be made with care. The observer should usually make his own sketches and plot his own work.

After the instrument is oriented over a station, and its height taken on the stadia, the stadia-men go about holding the rods at all points which are to be plotted on the map, either in position or in elevation, or both. The choice of points depends altogether on the character of the survey; but since a single holding of the rod gives the three coördinates of any point within a radius of a quarter of a mile, it is evident the method is complete, and that all necessary information can thus be obtained. For very long sights, the partial wire intervals (intervals between an extreme and the middle wire) may be read separately on the stadia, and in this way twice as great a distance read as the rod was designed for. The limit of good reading is, however, usually determined by the state of the atmosphere, rather than by the length of the rod. When the air is very tremulous, good readings cannot be made over distances greater than 500 feet; while, when the atmosphere is very steady, a half-mile may be read with equal facility.

Before the instrument is removed from the first station, the forward stadia-man selects a suitable site for the next instrument station (generally called *stadia station*, and marked ⊡, to distinguish it from a triangulation station, ◬), and drives a peg or hub at this point. This peg is to be marked in red chalk, with its proper number, and should have a taller marking-stake driven by the side of it. The peg for the ⊡ should be large enough to be stable; for it must serve as a reference point, both in position and elevation, during the period of the survey. It is often desirable to start a branch line, or to duplicate some portion of the work, with one of these stations as the starting-point; and, since each ⊡ is determined, in position and elevation, with reference to all the others, one can start a branch line from one of these as readily as from a ◬. It is not usually necessary to put a tack in the top, but the centre may be taken as the point of reference. The stadia-man first holds his stadia carefully over the centre of this ⊡, *with its edge towards the instrument*, so as to enable the observer to get a more accurate setting for azimuth. The observer could just as well bisect the face of the rod; but, if held in this position, the centre of the rod may not be so nearly over the centre of the peg as when held edgewise. This holding of the rod edgewise for azimuth checks the carelessness of the stadia-man, and is done only for readings on instrument stations.

At a signal from the observer, the stadia is turned with its face to the instrument, and the observer reads the distance and vertical angle.

It is advisable, in good work, to re-orient and relevel the instrument just before reading to the forward ⊡. The transit is very apt to get out of level after being used for some time, with more or less stepping around it, and the limb may have shifted slightly on the axis, both of which might be so slight as to make no material difference for the side readings, but which would be important in the continued line itself. It is best, therefore, to level up again, and reset on the back station, before reading to the forward one. If it is inconvenient for the rear rodman to go back to this station to give a reading, a visible mark should be left there, to enable the observer to reset upon it for azimuth, as it is not necessary to read distance and vertical angle again.

When the instrument is moved, it is set up over the new station, and the new height of instrument determined and recorded. The rear stadia-man is now holding his rod, edgewise, on the station just left: and by this the observer orients his instrument, ***making vernier A read*** 180° ***different from its previous reading on this line.*** Clamping the plates at this reading, the telescope is turned upon the rod on the back station, and the lower plate clamped for this position. The circle is now oriented, so that, for a zero-reading of vernier A, the telescope points south.

It will be noted that the telescope is never reversed in this work.

The *distance* and *vertical angle* should both be reread, on this back reading, for a check. If the vertical circle is not in exact adjustment, this second reading of the vertical angle will show it, for the numerical value of the angle should be the same, with the opposite sign. If they are not the same, then the numerical mean of the two is the true angle of elevation, and the difference between this and the real readings is the index error of the vertical circle. This error may be corrected in the reduction, or the vernier on the vertical circle may be adjusted.

The second reading of the vertical angle on the stadia-stakes is thus seen to furnish a constant check on the adjustment of the vertical circle, and should therefore never be neglected. If the circle is out of adjustment by a small amount, as one minute or less, in ordinary work it would not be necessary either to adjust it or to correct the readings on side-shots, for the elevations of contour points are not required with such extreme accuracy. The mean of the two readings on stadia-stakes would still give the true difference of elevation between them, so that there would be no continued error in the work.

The work proceeds in this manner until the next ◬ is reached. In coming to this station, it is treated exactly as though it were a new ⊡; and the forward reading to it, and the back reading from it, are identical with those of any two consecutive ⊡'s. Having thus occupied the second ◬, and having oriented the instrument by the last ⊡, turn the telescope upon some other ◬ whose azimuth from this one is known. The reading of vernier A for this pointing should be this azimuth, and the difference between this reading and the known azimuth of the line is the accumulated error in azimuth due to carrying it over the stadia line. This error should not exceed five minutes in the course of two or three miles in good work.

The check in distance is to be found from plotting the line, or from computing the coördinates of the single triangulation line, and also of the meandered line, and comparing the results.

The elevations are checked by computing the elevation of the new △ from the stadia line, and comparing this with the known elevation from the line of levels.

In case the elevations of the △'s are not given, but only certain B.M.'s in their vicinity, then the check can be made on these just the same. Thus, in starting, read the stadia on the neighboring B.M., and from this vertical angle compute the elevation of the △ over which the instrument sets, and then proceed as before. In a similar manner, the check for elevation at the end of the line may be made on a B.M. as well as on the △.

A quick observer will keep two or three stadia-men busy giving him points; so that in flat, open country, with long sights, it may be advisable to have three or even four stadia-men for each instrument. In hilly country more time will be required in making the sketches, and hence fewer stadia-men are required.

After the instrument is oriented at each new station, the needle should be read as a check. To make this needle-reading agree with the readings of the verniers on the horizontal circle (the north end with vernier A, and the south end with vernier B, for instance), graduate an annular paper disk the size of the needle circle, and figure it continuously from 0° to 360°, *in the reverse direction to that on the horizontal limb of the instrument*, and paste it on the graduated needle-circle in such a position that the north end of the needle reads zero when the telescope is pointing south. If the variation is 6° east, this will bring the zero of the paper scale 6° east of south on the needle-circle. This position of the paper circle is then good within the region of this variation of the needle. When survey extends into a region where the variation is different the scale will have to be reset.

With these conditions, when the instrument is oriented for a zero-reading when the telescope is south, the reading of the north end of the needle will always agree with the reading of vernier A, and the south end with vernier B. It is so easy a matter to let the needle down, and examine at each ⊡ to see if this be so, that it well pays the trouble. No record need be made of this reading, as it is only used to check large errors.

211. Reducing the Notes.—The only reduction necessary on the notes is to find the elevation of all the points taken, with reference to the fixed datum, and sometimes to correct the distance read on the rod for inclined sights. The difference of elevation between the ⊡ and any point read to, as well as the correction to the horizontal distance, can be taken from Table V. or from the diagram. The methods of using these have been explained (see pp. 247–51). A very accurate and by far the most expeditious method of taking out these differences of elevation is by means of the Colby Slide Rule shown in Fig. 63*a*. This is a slide rule fifty inches long, graduated so as to

FIG. 63*a*.

give differences of elevation for any distance and for any angle up to nineteen degrees. In addition to giving the differences of elevation in the same unit (and it is immaterial *what* unit) of measure used in reading the distances, this slide rule will also give with equal facility the differences of elevation in *feet* when the distances are read in either meters or yards. Fig. 63*a* shows only about one-tenth of the slide rule, reduced one-half. It is for sale by leading dealers, and by B. H. Colby, St. Louis, Mo. After the differences of elevation are taken out, the final elevations of the points are to be computed by adding algebraically the difference of elevation to the elevation of ⊡.

The following is a sample page with these reductions:

GAZZAM, *Observer*.

APRIL 20, 1883. BAIER, *Recorder*.

At ⊡ 4. Ht. of Inst. = 87. Elevation = 24′.94.

Object.	Azimuth. Ver. A.	Distance.	Vert. Angle.	Difference of Elevation.	Elevation above Datum.
		yds.			
⊡ 3	328° 10′	199	− 0° 10′	− 1′.56	—
Bridge	127° 40′	70	+ 0° 32′	+ 1′.9	26′.8
S.E. cor. of house.........	142° 35′	90	+ 0° 15′	+ 1′.2	26′.1
On road	180° 25′	114	+ 0° 7′	+ 0′.7	25′.6
Water-level, foot of hill....	230° 15′	224	− 0° 57′	− 10′.9	14′.0
⊡ 5......................	128° 33′ 30″	216	+ 0° 55′	+ 10′.38	—
C.P	190° 48′	210	+ 1° 2′	+ 11′.4	36′.3
At ⊡ 5. Ht. of Inst. = 78. Mean = + 10′.26. 35′.20.					
⊡ 4......................	308° 33′ 30″	215	− 0° 54′	− 10′.13	—
S.W. cor. of house.........	43° 30′	104	+ 3° 3′	+ 16′.0	51′.2
Edge of bank..............	332° 10′	98	+ 1° 57′	+ 10′.1	45′.3
S.E. cor. of R.R. station...	85° 30′	158	+ 1° 2′	+ 8′.5	43′.7
Railroad track............	43° 55′	40	+ 2° 53′	+ 6′.0	41′.2
" "	79° 30′	270	+ 0° 9′	+ 2′.1	37′.3
⊡ 6	79° 30′	200	− 0° 2′	− 0′.36	—
At ⊡ 6. Ht. of Inst. = 79. Mean = − 0′.54. 34′.66.					
⊡ 5......................	259° 30′	200	+ 0° 4′	+ 0′.72	—
Cor. of house..............	277° 55′	112	+ 3° 26′	+ 19′.7	54′.4
Top of hill	87° 25′	198	+ 4° 48′	+ 49′.3	84′.0
Wagon road..........	58° 15′	186	+ 4° 25′	+ 42′.9	77′.6
⊡ 8......................	40° 37′	216—3 / 213	+ 6° 33′	+ 73′.53	—
C.P.......	41° 45′	111	+ 4° 41′	+ 27′.0	61′.7
⊡ 7......................	5° 25′	194	+ 0° 12′	+ 2′.04	—

It will be noted that the reading on ⊡ 5 from ⊡ 4 has a distance of 216 yards, and a vertical angle of $+0^\circ\ 55'$; while on the back reading, from ⊡ 5 to ⊡ 4 the distance is 215 yards, and the vertical angle $-0^\circ\ 54'$. The distance was probably between 215 and 216 yards, and the vertical circle was probably slightly out of adjustment. The difference of elevation is taken out for both cases, however, being respectively 10.38 feet and 10.13 feet. The mean of these is 10.26 feet, which stands as a part of the general heading at ⊡ 5. The true elevation of ⊡ 5 is then found by adding 10.26 to 24.94, giving 35.20 feet, which is also set down as part of the general heading.

The elevations on the side-readings from this station can now be taken out. These side-elevations are only used for obtaining the contours, and hence are only taken out to tenths of a foot. When the contours are ten feet apart or more, these side-elevations need only be taken out to the nearest foot. The elevations of the stadia stations should, however, always be taken out to hundredths, to prevent an accumulation of errors in the line.

The reduction for distance may also be taken from that portion of the diagram arranged for this purpose. This is used the same as the other portion; and the *correction* is found, which is to be always subtracted from the rod-reading. Thus, in the reading on ⊡ 8 from ⊡ 6, we have a reading of 216 yards, and a vertical angle of $6^\circ\ 33'$. The correction here is $2.16 \times 1.3 = 2.8$ yards, as found from the table. Calling this 3 yards it is subtracted from the 216, leaving 213 yards as the distance to be plotted. It is only the stadia-line distances that need ever be corrected in this way, the corrections being usually so small that it is not important on the side-shots.

It will be noted that two ⊡'s were set from ⊡ 6. This was done because a branch-line was run from ⊡ 6 over the bluffs. In order to make it unnecessary to occupy ⊡ 6 again when the branch-line came to be run, ⊡ 8 was set while ⊡ 6 was occupied in the main-line work. When the branch-line came to be run, the instrument was taken directly to ⊡ 8, and oriented on ⊡ 6 by the readings previously taken from ⊡ 6.

The right-hand page of the note-book, opposite the notes given above, is occupied with a sketch of the locality, with the ⊡'s marked on, the general direction of the contour lines, the railroad, stream, houses, etc.*

212. Plotting the Stadia Line.—It is customary to first plot the stadia stations alone, from one ⊡ to the next, to find whether or not it checks within reasonable limits. This part of the work should be done with extreme care, so that if it does not check it cannot be attributed to the plotting. In case it does not check within the desired limit, then the line of investigation will be about as follows until the error is found:

1st. Replot the stadia line.

2d. Recompute and replot the triangulation line.

3d. By examining the discrepancy on the plot, try and decide whether the error is in azimuth or distance, and, if possible, where such error occurred, and its amount.

4th. Examine the note-book carefully, and see if there is any evidence of error there.

5th. If there is a large probability that the error is of a certain character, and that it occurred at a certain place, take the instrument to that station, set it up, and redetermine the azimuths or distances which seem to be in error.

6th. If there is no high probability of any certain errors to be examined for in this way, then go back and run the line over, *taking readings on ⊡'s only.* If the elevations had been found to check, the vertical angles may be omitted on this duplicate line; and, on the other hand, if the plot came out all right, but the elevations could not be made to check, then a duplicate line must be run to determine this alone; and in this

* These notes were taken from a field-book of a topographical survey of Cœur Lake by the engineering students of Washington University.

case the vertical angles between ⊡'s are all that need be read. In cases of this kind, it will be found a great help to have the ⊡'s so well marked that they can be readily found.

With reasonable care in reading and in the handling of the instrument, it will never be necessary to duplicate a line entire, for all readings between ⊡'s are checked. The vertical angles and distances are checked by reading them forward and back over every stadia line; and the azimuth is checked by the needle readings, and also when the second △ is reached.

If, in the progress of the work, the readings on the back ⊡ for distance and vertical angle do not fairly agree with these quantities as read from the previous station, the recorder should note the fact: and the observer should then re-examine these readings; and, if found to be right, the first readings, taken from the other station, should be questioned, and the mean not taken in the reduction.

For plotting the stadia lines a parallel ruler (moving on rollers) is very desirable; otherwise, triangles must be used. The plotting is done by setting the parallel ruler or triangle on the proper azimuth as found from the protractor printed on the sheet, moving it parallel to itself to the station from which the point is to be plotted, and drawing a pencil line in the right direction. Then, with a triangular scale,—or, better, with a pair of dividers and a scale of equal parts,—lay off the correct distance on this line; and this gives the point.

If the instrument was oriented in the field for a zero reading for a south pointing, then the protractor on the sheet must have its south point marked zero, and increase around to 360° in the same direction in which the limb of the instrument increases, preferably in the direction of the movement of the hands of a watch.

213. Check Readings.—To enable the observer to locate large errors in azimuth or distance, or both, it is a good practice to take azimuth readings to a common object from a series of consecutive stations, if such be possible. If the plot does not close, go back and plot in these azimuths; and if there has been no error in azimuth or distance between ⊡'s, and no error in reading the azimuths for these pointings, then all these lines will meet in a common point on the plot. If all but one intermediate line meet at a point, then the error probably was in reading the azimuth of this pointing alone. If several of the first pointings intersect in a point, and the remaining pointings of the set taken to this object intersect in another point, then it is highly probable that the error was in reading the azimuth or distance of the line connecting these two sets of ⊡'s; and the relative position of the points of intersection will enable the observer to decide whether the error was in azimuth or distance, and about how much. If, in this way, the error be located, the instrument can be taken to this point, and the readings retaken.

214. Plotting the Side-readings.—Having plotted the stadia line and made it check, the next step is to go back and plot in the side-readings. For doing this, a much more rapid method may be used than that described above.

Divide the sheet into squares by horizontal and vertical lines spaced uniformly at from 1000 to 5000 units apart, according to scale. These lines are to be used for orienting the auxiliary protractor, and also to test the paper for stretch or shrinkage.

The side-readings are now plotted by the aid of a paper protractor, such as is shown in Fig. 64. This is made from a regular field-protractor sheet. The graduated circle printed on the sheet is used; and this is some 12 inches in diameter, and graduated to 15 minutes. The sheet is trimmed down to near the graduated circle, and the edges divided, as shown in the figure, to any convenient small scale.* This sheet is to be

* It is sometimes desirable to make the open space *DFE* rectangular and graduate the sides of the space *ABF* instead of the outer edges. The protractor can then be used nearer the edge of the sheet.

laid upon the plot, with its centre, *C*, coinciding with the ⊡. It is oriented by bringing the corresponding spaces on opposite edges to coincide with any one of the spaced lines on the plot. This circle then has its position parallel to that of the protractor circle printed on the sheet, and an azimuth taken from the one will agree with an azimuth taken from the other. When this auxiliary protractor has been so centred and oriented, let it be held in place by weights. Now the part *ADEB* folds back, on the line *AB*, into the position indicated by the dotted lines. The portion *DEF* is cut out entirely, so that when the flap is turned back the space *AFB* is left open. This space is to be large enough to include the longest side-readings when plotted to scale; that is, the radius, *CF*, of the circle to the scale of the drawing must exceed the longest readings. We now have a protractor circle about the ⊡, with this station for its centre.

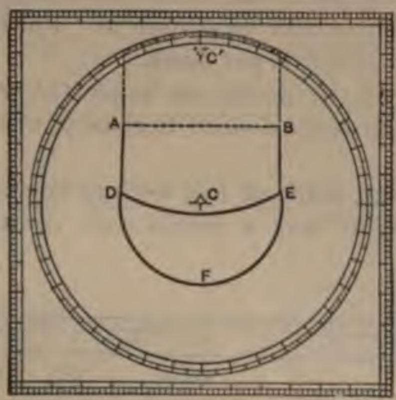

FIG. 64.

Take a triangular scale, select the side to be used in laying off the distances, and paste a piece of strong paper on the lower side at the zero point. Make a needle-hole through this paper close to the edge, at the zero of the scale. Fasten a needle through this hole into the point which marks the exact position of the ⊡. The scale can now swing freely around the needle, on the auxiliary protractor; and its zero remains at the centre of the station from which the points are to be plotted.

To plot any point, swing the scale around to the proper azimuth, and at the proper distance mark with the pencil the position of the point. If this marks a feature of the landscape, it should be drawn in at once, before going farther; and if the elevation of the point will be needed in sketching the contours, this should also be written in. For contour points, the elevation is all that is put down.

In this manner the points can be plotted very rapidly. A six-inch triangular scale, divided decimally, will be found best for this.

If there is very much of this work to be done, it might be found advisable to have a special scale constructed for the

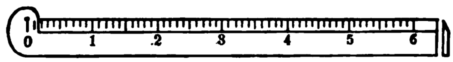

FIG. 65.

purpose. Fig. 65 is one form of such a scale drawn one-third size, which would be found very convenient and cheap. It should be graduated on a bevel edge, and to such a scale that the units of distance used on the rod may be plotted to the scale of the drawing. The small needle-hole, in line with the graduated edge, should be only large enough to fit the needle-point used, so that there would be no play. The rule then turns on an accurate centre, which will not wear. Such scales, six inches long, could be constructed very cheaply of German silver by any instrument-maker.

A special form of protractor, shown in Fig. 66, has also been used with great success in France and on the Mississippi River surveys.*

It is essentially a semicircular protractor, provided with

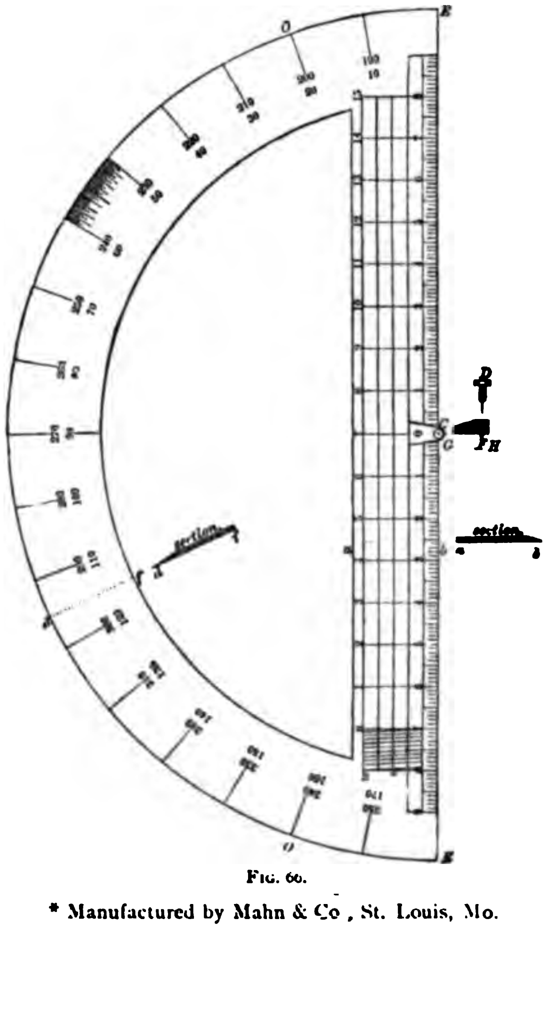

FIG. 66.

* Manufactured by Mahn & Co., St. Louis, Mo.

a needle-pointed pivot at its centre, and having the straight edge graduated so that distances can be measured off each way from the pivot; the angular deflection is given by the graduated circle, reading from a point marked on the paper. The bottom of the plate is flush with the bottom of the protractor, and the hole *F* is at the centre, and should be only large enough to admit a fine needle. The screw *D* has a hole drilled in its axis to admit the needle-point. It is also split, so that when it is screwed down it will clamp the needle firmly. If the latter is broken, it can readily be replaced by a new one. In addition to the scale on the beveled edge, a diagonal scale is also provided as shown. This instrument combines all the requisites for rapid and accurate plotting of points located by polar co-ordinates or by intersections.

In using this protractor the needle-point is placed at, say, the first station, and pressed firmly down. A meridian line is then decided upon, and a point is marked on it at the outer edge of the protractor circle. This will be the initial point from which the angles will be read. As azimuth is read from the south around by the west, it is plain that the circle, numbered as shown and revolved about the pivot till the proper reading coincides with the meridian line, will give the direction of the required point along the graduated diameter, while from the latter the distance can be pricked off. A point can be plotted in any direction without lifting the protractor from its position.

In going to the second station it is not necessary to draw a meridian line through it. The azimuth between the first and second stakes being known, if the pivot be set at the latter, and the protractor revolved so that the straight edge coincides with the line passing through the two stakes, then the point on the circle corresponding to the azimuth of the line will be a point on the meridian line. This point being marked the paper is the origin for the angles plotted from the second station, and it is evident that they will bear the proper relations to the points plotted from the first station.

Other methods are employed for plotting the side shots, such as solid half-circle protractors, of paper or horn, weighted in position, with their centres over the station. This is oriented on a meridian drawn through the point, and then all the points plotted whose azimuth falls between 0° and 180°, when the protractor is laid over on the other side, and the remaining points plotted. In this case the ruler is laid across the protractor, with some even division at the station. This method is more troublesome, less rapid, and defaces the drawing more, than the other methods given above. The plotter should have an assistant to read off to him from the note-book. When all the elevations have been plotted, the contour lines are sketched in.

The plotting should keep pace with the field-work as closely as possible, being done at night and at other times when the field-work is prevented or delayed. In difficult ground the map could be carried into the field and the contours sketched in on the ground. At least the stadia lines should be plotted up and checked before the observer leaves the immediate locality. Where the elevations are checked on B.M.'s, these checks should be immediately worked out. This much, at least, could be done each evening for that day's work.

215. Contour Lines.—In engineering drawings the configuration of the surface is represented by means of contour lines. A contour line is the projection upon the plane of the paper of the intersection of a horizontal, or rather level, plane with the surface of the ground. These cutting level planes are taken, five, ten, twenty, fifty, or one hundred feet apart vertically, beginning with the datum-plane, which is usually taken below any point in the surface of the region. Mean sea-level is the universal world's datum which should always be used when a reasonably accurate connection with the sea can be obtained.* Such contour lines are shown on Plate II. The proper drawing of these contours requires some accurate knowledge of the surface to be depicted, aside from the elevations of isolated points plotted on the map. This knowledge may consist of a vivid mental picture of the ground, derived from personal observation, or it may be gained from sketches made upon the ground. Even with this knowledge the draughtsman must keep vividly in mind the true geometrical significance of the contour line, in order to properly depict the surface by this means. The ability to draw the contour lines accurately on a field-sheet is the severest test of a good topographer. They are first sketched and adjusted in pencil and then may be drawn in ink.

A few fundamental principles may be stated that will assist the young engineer in mastering this art.

1. All points in one contour line have the same elevation above the datum-plane.

2. Where ground is uniformly sloping the contours must be equally spaced, and where it is a plane they are also straight and parallel.

3. Contour lines never intersect or cross each other.

4. Every contour line must either close upon itself or extend continuously across the sheet, disappearing at the limits of the drawing. It cannot have an end within these limits (an apparent exception, though not really one, is the following).

5. No contour should ever be drawn directly across a stream or ravine. The contour comes to the bank, turns up stream, and disappears in the outer stream line. If the bed of the stream, or ravine, ever rises above this plane, then the contour crosses it; but in the case of a stream the crossing is never actually shown. In the case of a ravine the crossing is shown, if points have been established in its bed.

6. Where a contour closes upon itself, the included area

* See in Chapter XIV., Precise Levelling, Art. 408.

is either a hill-top or a depression without outlet. If the latter, it would in general be a pond or lake. In other words, such contours enclose either maximum or minimum points of the surface.

7. If a higher elevation seems to be surrounded by lower ones on the plot, it is probably a summit; but if a lower elevation seems to be surrounded by higher ones, it is probably a a ravine, or else an error; otherwise it is a depression without outlet, in which case there would probably be a pool of water shown.

8. Contour lines cut all lines of steepest declivity, as well as all ridge and valley lines, at right angles.

9. Maximum and minimum ridge and valley contours must go in pairs; that is, no single lower contour line can intervene between two higher ones, and no single higher contour line can intervene between two lower ones.

10. Vertical sections, or profiles, corresponding to any line across the map, straight or curved, can be constructed from a contour map, and conversely a contour map may be drawn from the profiles of a sufficient number of lines.

11. Each contour is designated by its height above the datum-plane, as the fifty-foot contour, the sixty-foot contour, etc. In flat country, where the contour lines are few and wide apart, always put the number of the contour on the *higher* side, otherwise it sometimes may be impossible to tell on which side is the higher ground.

12. In taking surface-elevations for determining contour lines, points should always be taken on the ridge and valley lines, and at as many intermediate points as may be desirable. There are two general systems of selecting these points. By one system points are chosen approximately in lines or sections cutting the contours about at right angles, the critical points being the tops and bottoms of slopes; while by the other system points are selected nearly in the same contour line,—that is, on the same horizontal plane,—the critical points

being the ridge and valley points, these being the points of maximum and opposite curvature in the contour lines themselves. By the second method one or two principal contours may be followed continuously, the points being taken as nearly as may be on these contour lines. If such principal contours are 50 feet apart, then when these are accurately drawn on the map, any desired number of additional contours may be interpolated between the principal ones.

216. The Final Map.*—The field-sheets are drawn as described above, in pencil, or partly in pencil and partly in ink, or wholly in ink, according to the use to be made of them. If they are simply to serve as the embodiment of the field-survey, to be used only for the construction of the final maps, they are usually left in pencil, a six-H pencil being used. The field-sheets are usually small, about 18x24 inches. The final sheets may be of any desired size. Usually several field-sheets are put on one final sheet, which will be worked up wholly in ink, or color, the scale remaining the same. The work on the field-sheet is then simply transferred to the final sheet by the most convenient means available. Tracing-paper (not linen) may be used. This is carefully tacked or weighted down over the field-sheet, and the principal features, such as triangulation stations, stream and contour lines, roads, buildings, fence lines, etc., are traced in ink. The tracing-paper is then removed and laid upon the final sheet, orienting it by making the triangulation stations on the tracing coincide with the corresponding stations on the final sheet, where they have been carefully plotted from the triangulation reduction. All the matter on the tracing may now be transferred to the paper beneath by passing over the inked lines with a dull point, bearing down hard enough to leave an impression on the paper below. If preferred, the tracing may have its under surface covered with plumbago (soft pencil-scrapings), after the tracing is made, and then with a very gentle pressure of the tracing-point will leave a light pencil line on the final sheet. In either case, when the

* See also Appendix G.

tracing is removed, these lines may be inked in on the final sheet.

If the map is to be photo-lithographed it must be drawn wholly in black, as given in Plates II. and III. If not, it is best to use some color in its execution. The water-lines may be drawn in blue, and the contours in brown on arable land, and in black on barren or rocky land. In this way the character of the surface may be partly given. Where the slopes are very steep the contour lines become nearly coincident, but to further emphasize the uneven character of the ground, cross-hatching, or hachures, may be employed on slopes greater than 45° from the horizontal. All these conventional practices are illustrated on Plate III., except the use of colors, this map having been drawn for the purpose of being photo-lithographed. Plate II. is a photo-lithograph copy of a student's map of the annual field survey of the engineering students of Washington University.

217. Topographical Symbols are more or less conventional, and for that reason given forms should be agreed upon. The forms given in Plate III. were used on all the Mississippi River surveys made under the Commission, and are recommended as being elegant and fairly representative or natural. Evidently the rice, cotton, sugar, and wild-cane symbols would find no place in maps of higher latitudes. The cypress-tree symbols may be used for pine to distinguish them from deciduous growth, and the sugar-cane symbol could be used for corn if desired. It is not important to distinguish between different kinds of cultivated crops, since these are apt to change from year to year, but it is sometimes desirable to do so to give a more varied and pleasing appearance to the map. The grouping of the trees in a large forest is also varied simply for the appearance, to prevent monotony. Colors are sometimes used in place of pen-drawn symbols, but these are necessarily so very conventional as to require a key to interpret them, and besides it makes the map look cheap and unprofessional.

218. Accuracy of the Stadia Method.—In measuring distances by stadia the errors made in reading the rod are as apt to be plus as minus. They therefore follow the law of compensating errors, which is that the square root of the number of errors remains (probably) uncompensated. If the rod was properly graduated, therefore, the only error is that from reading the position of the wires. On inclined sights the distance read on the rod is accurately reduced to the horizontal by means of proper tables or diagrams. There is another peculiarity of this system, and that is that the accuracy depends very largely on the state of the atmosphere. If this is clear and steady the accuracy attainable for given lengths of sight is much greater than when it is either hazy or very unsteady from the effects of heat. Or, for a given degree of accuracy, the lengths of sight may be taken much longer under favorable atmospheric conditions than under unfavorable ones. It is impossible, therefore, to specify any given degree of precision for given lengths of sight for all atmospheric conditions. The results obtained on the U. S. Lake Survey are perhaps a fair average for various conditions. On that service the errors of closure of 141 meandered lines was computed with a mean result of one in 650.* The lengths of sight averaged from 800 to 1000 feet, with a maximum length of about 2000 feet. The official limit of error of closure was one in 300. The average length of the lines run was one and a half miles. If care is taken to shorten up the sights for unsteady atmosphere, and to reduce all readings to the horizontal, it would not be difficult to reduce the error of closure on lines averaging from one to two miles in length, to one in 1000 or one in 1200. Since the absolute error increases as the square root of the length of the line run, it is evident that *the relative error diminishes as the length of line increases.*

* On the Mexican Boundary Survey (1893-4) the average error on fifteen lines, averaging 2½ miles long, was one in 750. On six lines, averaging 14 miles long, the error was one in 1250, as compared with triangulation measurements. The atmospheric conditions were very bad.

CHAPTER IX.

RAILROAD TOPOGRAPHICAL SURVEYING,

WITH THE TRANSIT AND STADIA.

219. **Objects of the Survey.***—Since the transit and stadia are the best means of making a general topographical survey, so they are the means that are best adapted to make a preliminary railroad survey, so far as this is a topographical survey.

The map of a railroad survey may serve two purposes:

First, to enable the engineer to make a better location of the line than could be done in the field.

Second, to give all necessary data relating to right of way, as the drawing of deeds, assessment of damages, etc.

In flat or gently undulating country, it is not advisable to locate by a map; but even here the map is quite as essential for determining questions relating to the right of way.

In either case, therefore, a good topographical map of the line is of prime importance, and all the data for this map may be taken on the preliminary survey.†

Both these ends may be served by the same map. The method of location by contours (sometimes called "paper location") is often absolutely necessary in rough ground, but is still more often judicious in simpler work, inasmuch as a better location can often be made in this way.

220. **The Field-work.**—In this case there would be no △'s or B.M.'s to check on; but the errors in distance and elevation would be no more, probably, than are now made on

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* See also Appendix G.

† By "preliminary survey" is here meant a survey of a belt of country which it is expected will embrace the final line, and not a mere reconnoissance made to determine the feasibility of a line, or which of several lines is the best.

preliminary surveys. In fact, the errors in distance would not be nearly so great, unless the chain be tested frequently for length, and the greatest care taken on irregular ground. If a chain 100 feet long has 600 wearing-surfaces, which most of them have, and if each of these surfaces be supposed to wear 0.01 inch, which it will do in the course of a 200- or 300-mile survey, then the chain has lengthened by six inches, or the error in distance is now 1 in 200 from this cause alone. If we add to this the uncertain errors that come from chaining up and down hill, and over obstructed ground, it is certain that the stadia measures will be much the more accurate.

In the matter of elevations, since the local change of elevation is alone significant, and not the total difference of elevation of points at long distances apart, the line of levels carried by the stadia would be amply sufficient for a preliminary survey.

The following observations are applicable to the preliminary survey for final location, when it is expected the line will be included in the belt of country surveyed:

1st. All data should be taken that will contribute to the solution of all questions of location, such as elevations for contour lines; streams requiring culverts, trestles, or bridges, and the necessary size of each, if possible; all depressions which cross the line, and will require a water-way, together with the approximate size of the area drained; highways and private roads or lanes; buildings of all kinds, fences, and hedges; character of surface, as rock, clay, sand, etc.; character of vegetation, as cultivated, forest, prairie, marsh, etc.; the location of any natural rock that may be used for structures on the line, such as culverts or abutments; high-water marks if in a bottom subject to overflow; and, in fact, all information which will probably prove of value in determining the location, or in making up a report with estimates to the board of directors, or in letting contracts for earthwork.

2d. All data that may be found useful in respect to land titles or right of way, or that may relate to claims for damages, such as section corners, boundaries, fences, buildings, streets, roads, lanes, farm roads, cultivated and uncultivated land, as well as such as may be cultivated, public and private grounds, orchards, forests, together with the value of the forest timber, mineral lands, stone quarries, proximity to villages, etc. Since the bearings and position of all boundary-lines are of great importance in the matter of right of way, every such boundary should have at least two readings upon it in the field; and these should be as far apart as possible.

221. **The Maps.**—Before any plotting is done, two questions of importance must be decided. They are—*first*, whether one set of maps is to serve for both the location and for the further use of the company, or whether a set of contour maps, worked up in pencil, shall serve for the location, and another set for the continuous use of the company; *second*, what shall be the scale of the maps? These will be argued separately.

Whether one or two sets of maps will be decided on, will depend largely on the care that is exercised with the locating-sheets. If these are carefully worked up for the location, and kept clean, they can be utilized for the final maps. If they become too badly soiled by field use, new sheets would probably be substituted for the uses of the company.

If it is expected, at the start, to have a different set of sheets for the final maps, then "protractor sheets" should be used for the location. In this case, plot on these sheets only such of the field-notes as will contribute to the location; and these need only be plotted in pencil. When the location has been made, such features may be transferred from the locating-sheets to the final maps, as may be desired. These would consist mainly in the stadia stations, the contours, and the located line. The rest of the field-notes may then be plotted on the final sheets, and the whole worked up in ink.

If, on the other hand, one set of maps is to serve both purposes, then it would, perhaps, be best to use plain sheets, as the protractor circle would somewhat disfigure the final maps. The protractor sheets would, however, furnish a ready means of taking off the bearings of lines from the final charts, which might be thought to compensate for the slight marring of the map's appearance. If plain sheets are chosen, then they should be divided into squares by lines drawn in ink parallel to the sides of the paper, in the direction of the cardinal points of the compass. Both the stadia stations and the side-readings may then be plotted by means of the auxiliary protractor, this being oriented by the meridian lines on the sheet. Even here, only those readings would at first be plotted that will contribute to the location, and these marked in pencil. After the location has been decided on, and the location notes taken off, as described below, then the stadia stations, contour lines, the located line of road, and such other features as should be preserved on the final map, are inked in, and the map thoroughly cleaned. The rest of the field-notes may now be plotted, and the map finished up.

If the road runs through a settled region, the questions of right of way are among the first things to be settled; so that preliminary maps showing the relation of the road belt to the property lines are essential to the settlement of damages, and to obtaining the right of way from the property-holders. Coincident, therefore, with the making of maps to determine the location must come the construction of preliminary right-of-way maps or tracings. On these latter need be plotted only the boundary-lines, fences, more important buildings, roads, etc., or just sufficient to enable the right-of-way agent to negotiate intelligibly with the property-owners.* Neither the lo-

* For an excellent article on the subject of right-of-way maps and permanent railway-property records, by Charles Paine, see *The Railroad Gazette* of Nov. 14, 1884. Reprinted in book form in "Elements of Railroading."

cating nor the final map should be on a continuous roll. The roll requires more room for storage, is more apt to get dusty, and is much more inconvenient for reference. When sheets are used, the survey plot covers a more or less narrow belt across the map. One of the edges of the sheet, either where the plot enters upon it or disappears from it, should be trimmed straight, and the plot extended quite to this edge. This edge is then made to coincide with one of the parallel or meridian lines of the next sheet; so that when the line is plotted, the sheets may be tacked down in such a way as to show the continuous plot of the survey.

The scale of the map will depend on whether or not separate sets of charts are to serve the purposes of location and of the continuous use of the company. For the purpose of location, a scale of 400 feet to one inch does very well; but for the final detail sheets the scale should be larger. If both purposes are to be served by one set of maps, then the scale should be about 200 feet to one inch,* with 5- or 10-foot contours. The sheets should be about twenty by twenty-four inches.

222. Plotting the Survey.—In case the map is plotted on a protractor sheet, the methods of plotting will be identical with those for general topographical work, except that here there will be no checks, either for distance, azimuth, or elevation, except such as are carried along or independently determined. For distance, there is no check, except the duplicate readings between instrument stations, unless the survey is through a region which has already been surveyed. In this case the section lines may serve as a check on the distances.

The azimuth should be checked at every station by reading the needle, as described on p. 264, and also by independently determining the meridian frequently, either by a solar attachment or by a stellar observation. If the line is not nearly

* Some engineers prefer a scale of 100 feet to one inch for the final charts of the company.

north and south, or, in other words, if it is extended materially in longitude, then the azimuth must be constantly corrected for convergence of meridians, as is shown in Chap. XIV.

The elevations can only be checked by the duplicate readings between instrument stations.* All the greater care should be used, therefore, on readings between stations.

The first plotting, whether there are to be two sets of maps or one, will consist in representing on the sheet only such data as will assist in deciding on the location. These will be mainly contour points, streams, important buildings near the line, principal highways, other lines of railway, villages with their streets and alleys near the proposed location, the lines of demarkation between cultivated and timbered or wild land, etc. From the plotted elevations, aided by the sketches in the notebook, the contour lines are drawn in; if necessary, this may be done on the ground. This is sufficient for determining upon a location.

When this has been done, then the natural features, the contour lines, the stadia stations, and the located line, may be inked in (or transferred by means of tracing-paper, in case the final maps are to be on separate sheets), and the remainder of the notes plotted.

In drawing the contour lines in ink, make those upon barren or rocky land in black, and those on arable land in brown. If they are ten feet apart, make every tenth one very heavy, and every fifth one somewhat heavier than the others. If this be done, only the 50- and 100-foot contours need be numbered. In case a map does not contain at least two of these numbered contours, then every contour which does appear on the map should be numbered, giving its elevation above the datum of the survey.

* It may be observed that the same lack of sufficient checks on the distance, azimuth, and elevation obtains with the ordinary preliminary survey with transit, level, and chain. If preferred, all bearings may be taken from the needle, and then each alternate station only need be occupied by the instrument. See series of articles on this subject by the author in "The Railroad Gazette" for Feb. 3d, Mar. 2d, 9th, and 30th, 1888.

The streams should be water-lined in blue, and an arrow should tell the direction of its flow. The name should also be given when possible.

All fences should be shown, and especial pains taken to represent division fences in their true position; for it is from this map that the deeds for the right of way are to be drawn.

Outhouses may be distinguished from dwellings by diagonal lines intersecting, and extending slightly beyond the outline. The character of the buildings may be shown by colors, as red for brick, yellow for frame, pale sepia for stone; the outlines always being in black.

The stadia stations should be left on the finished sheets; as, in case of a disputed boundary, or for other cause, the map may be replotted if the positions of the instrument stations are left on it. The numbers of the stations should, of course, be appended.

The magnetic bearings of boundary-lines may be given on the map, or they may be determined, as occasion requires, by means of the auxiliary protractor and the true meridian lines when the variation of the needle is known. For this purpose, the magnetic meridian should be drawn on each map, diverging from one of the meridian lines, and the amount of the variation marked in degrees and minutes.

223. Making the Location.—When a preliminary survey is made, as above described, for the purpose of making what is called a "paper location," the location is first made on the map, and then staked out in the field.

Every railroad line is a combination of curves, tangents, and grades; and it is the proper combination of these which makes a good location. If it be assumed that the line is to be included in the belt of country surveyed, then the map contains all the data necessary to enable the engineer to select the best arrangements of curves, tangents, and grades it is possible for him to obtain on this ground. This selection can be made with much more certainty than is possible on the ground, where the view is generally obstructed, and where grades are so deceptive.

It is no part of this treatise to discuss the various problems that enter into the question of a location, but only to show how to proceed to make a location that may satisfy any given set of conditions, by means of the contour map.

The contours themselves will enable the engineer to decide what the approximate grades will have to be. Suppose a grade of 0.5 foot in 100 feet, or 26.4 feet to the mile, has been fixed upon. It is now known that the line should follow the general course of the contours, except that it should cross a 10-foot contour every 2000 feet. Spread the dividers to this distance, taken to scale, and mark off in a rough way these 2000-foot distances as far as this grade is to extend; and do the same for the successive grades along the line. Knowing the grade of the line at the beginning of the sheet, the problem is to extend this line over the sheet so as to give the best location one can hope to get on this ground with the available means.

First, starting from the initial fixed point of line on the map, sketch in a line which will follow the contours exactly, crossing them, however, at such a rate as to give the necessary grade. This is the *cheapest* line, so far as cut and fill are concerned. Of course, where depressions or ridges are to be crossed, the line must cross over from a given contour on one side to the corresponding contour on the other, and then follow along the contour again.

Second, mark out a series of tangents and curves which will follow this sketched line as nearly as it is possible for a railroad to follow it. This will not be the final location, but it is valuable for study. This line will be faulty from having too many and too sharp curves, and too little tangent.

Third, draw in a third line, as straight as possible, and with as low grade of curves as possible consistent with a reasonable amount of earthwork and a proper distribution of the same.

For the purpose of deciding what degree of curve is best suited to the ground for a given deflection angle, it is well to have a series of paper templets made, with the various curves for their outer and inner edges. Of course, these are cut with radii laid off to the scale of the drawing. It is still more convenient to have these curves, laid off to scale, on a piece of isinglass, horn, or tracing-paper (not linen), so that this can be laid upon the map, and the curve at once selected which will follow the contours most economically. Fig. 66 shows such a series of curves drawn to a scale of 1600 feet to the inch.

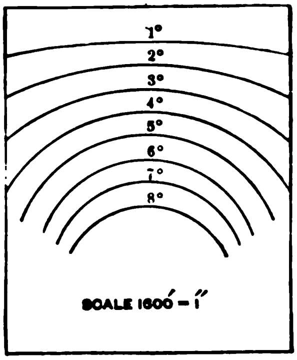

FIG. 66

In this way the line is laid out over the map. The questions of greater or less curvature have been balanced against a less or greater first cost, and greater or less operating expense. The question of shifting it laterally has also been examined, and finally a definite location fixed upon which seems to answer best to the case in hand. When this is done, it only remains to make up the location notes from which the line is to be staked out.

The following is considered a good form for the location notes:

Location Notes for ABC Railroad. From Map No. ....

Line.	Azimuth and Deflection Angles.	Length.	Station.	Remarks.
		ft.		
T	260° 40′	1020	10 + 20	P.C.
3° C.R.	+ 18° 30′	617	16 + 37	P.T.
T	279° 10′	2670	43 + 7	P.C.
4° C.L.	− 12° 20′	308	46 + 15	P.T.S. 46° 30′ W. ⊡ 12 320 ft.
T	266° 50′	680	52 + 95	P.C.

The first column designates the tangents and curves, and gives the degree of the curve, and the direction of its curvature, whether right or left. If it curve toward the right, the azimuth of the next tangent will be increased, and hence its sign is plus, and *vice versa.*

The second column gives the azimuths of the tangents and the deflection-angles of the curves. Each azimuth is seen to be the algebraic sum of the two preceding angles.

The third column gives the lengths of the tangents as measused from the map, and the lengths of the curves as determined by dividing the deflection-angle by the degree of the curve. Thus, $12° \ 20' = 12°.33$, and $12°.33 \div 4 = 308$, which is the length of the curve in feet.*

The fourth column gives the stations and pluses for the P.C.'s and the P.T.'s. These quantities are simply the continued sum of those in the third column.

The first, second, and fourth columns now give all the infor-

* It is a great convenience to have at least one vernier, in railroad work, graduated to read to hundredths of a degree. The case here given is only one of many similar cases; but the principal advantage is in running the fractional parts of curves when the curve chosen is some even degree, as here taken.

mation necessary to stake out the line. The stadia is no longer to be used, but a transit and chain, as is ordinarily done.

The tangents need not be run out to their intersection; but when the P.C. is reached, according to the location notes taken from the map, set up the instrument, and stake out the curve as far as possible, or around to the P.T. In either case, when the instrument is to be moved, make a note of the forward azimuth, and go forward and orient on the last station the same as when moving between two ⊡'s. If the instrument be moved to the P.T. direct, then, after orienting back on the P.C., turn off to the azimuth given for the next tangent, and go ahead. The tangents could be run out to the intersection and the point occupied by the instrument, for a check, if thought desirable. *The telescope is never reversed in laying out the line from the system of notes above given.*

With careful work, the line ought thus to be run out, and the curves put in at once. We have supposed there was no regular line cleared out on the preliminary, so the necessary clearing would all have to be done on the location.

A levelling party follows the transit, and obtains the data for constructing a profile and for determining the exact grades.

The stadia has served its purpose when it has enabled the engineer to select the most favorable position for the line. The transit, chain, and level must do the remainder. It is not improbable that occasional modifications will be introduced in the field, even though the survey and the location have been made with the greatest possible care.

224. **Another Method** of making the preliminary survey from which to determine the final location is as follows:

Run a transit and chain line, setting 100-foot stakes, as nearly on the line of the road as can be determined by eye. Follow this party by a level party which obtains the profile of the transit line. A third party of one or more topographers takes cross-sections at each 100-foot stake by means of a pocket-compass, clinometer, and hand-level. These cross-sections show the ground on either side of the line as far as desirable by slope and distance, these latter being either measured by tape or paced. It is evident that contour lines could be worked out from these data, but these would not be needed if the distances and slopes were well determined, since these give a better cross-section than contours alone could do.

The objections to this method are in the poor means it furnishes for accurate determination of either distances or slopes, and the haste with which it is usually done. There can be no question but that accurate distances and slopes on cross-sections 100 feet apart would give fuller data than even five-foot contours accurately drawn. But to be *accurately* determined the slope would have to change at all points—in other words, it would be a curve. As to whether the slopes and distances *as they would probably be taken* would give a better idea of the ground than five-foot contours determined by the stadia method, and the relative cost of the two systems, are matters of experience. Both systems are competent to give a good location when they are well executed.

NOTE.—The further study of railroad surveying falls within the province of the various railroad field-books, which are printed in pocket form and contain the necessary tables for laying out a line of road. Having learned the construction and use of surveying-instruments, and the general methods of topographical surveying and levelling, the special applications to railroad location given in the field-books are readily mastered. They will therefore not be further considered in this work.

CHAPTER X.

HYDROGRAPHIC SURVEYING.

225. **Hydrographic Surveying** includes all surveys, for whatever purpose, which are made on, or are concerned with, any body of still or running water. Some of the objects of such surveys are the determination of depths for mapping and navigation purposes; the determination of areas of cross-sections, the mean velocities of the water across such sections, and the slope of the water surface; the location of buoys, rocks, lights, signals, etc.; the location of channels, the directions and velocities of currents, and the determination of the changes in the same; the determination of the quantity of sediment carried in suspension, of the volume of the scour or fill on the bottom, or of the material removed by artificial means, as by dredging.

A hydrographic survey is usually connected with an extended body of water, as ocean coasts, harbors, lakes, or rivers. The fixed points of reference for the survey are usually on shore, but sometimes buoys are anchored off the shore and used as points of reference. All such points should be accurately located by triangulation from some measured base whose azimuth has been found. The buoys will swing at their moorings within small circles, these being larger at low tide than at high, but the errors in their positions should never be sufficient to cause appreciable error in the plotted positions of the soundings. Where soundings need to be located with great exactness, buoys could not be relied on. The triangulation work for the location of the fixed points of reference differs in no sense from that for a topographical survey. In fact, a hydrographic survey is usually connected with a topographical survey of the adjacent shores or banks, the triangulation scheme serving both purposes. It is not uncommon, however, to make a hydrographic survey for navigation purposes simply, wherein only the shore-line and certain very prominent features of the adjacent land are located and plotted. This is the practice of the U. S. Hydrographic Office in surveying foreign coasts and harbors. In this case the work consists almost wholly in making and locating soundings for a certain limiting depth, as one hundred fathoms, or one hundred feet, inward to the shore, and along the coast as far as desired. The length and azimuth of a base-line are determined and the latitude observed by methods given in Chapter XIV. The longitude is found by observing for local time, and comparing it with the chronometer time which has been brought from some station whose longitude was known. Whenever telegraphic communication can be obtained with a place of known longitude, the difference between the local times of the two places is found by exchanging chronographic signals. No special description will be here given of the methods used in this part of the work, as they are all fully described in Chapter XIV.

THE LOCATION OF SOUNDINGS.*

226. Methods.—The location of a sounding can be found with reference to visible known points by (1) two angles read at fixed points on shore; (2) by two angles read in the boat; (3) by taking the sounding on a certain range, or known line, and reading one angle either on shore or in the boat; (4) by sounding along a known range, or line, taking the soundings at known intervals of time, and rowing at a uniform rate; (5) by taking the soundings at the intersections of fixed range lines; (6) by means of cords or wires stretched between fixed

* See Appendix F, and foot-note p. 662.

stations, these having tags, or marks, where the soundings are to be taken. These methods are severally adapted to different conditions and objects, and will be described in order.

227. Two Angles read on Shore.—If two instruments (transits or sextants) be placed at two known points on shore, and the angles subtended by some other fixed point, and the boat be read by both instruments, when a sounding is taken, the intersection of the two pointings to the boat, when plotted on the chart containing the points of observation duly plotted, will be the plotted position of the sounding. If three instruments are read from as many known stations, then the three pointings to the boat should intersect in a point when plotted, thus furnishing a check on the observations. The objections to this method are that it requires at least two observers, and these must be transferred at intervals, as the work proceeds, in order to maintain good intersections, or in order to see the boat at all times. While an observer is shifting his position the work must be suspended. If there are long lines of off-shore soundings to be made and there are no fixed points or stations on shore of sufficient distinctness or prominence to be observed by the sextant from the boat, then this method must be used. When the angles are read on shore signals should be given preparatory to taking a sounding, and also when the sounding is made. If, however, the soundings are taken at regular intervals the preparatory signal may be omitted, and only the signal given when the sounding is taken. This usually consists in showing a flag. The instrument may be set to read zero when pointing to the fixed station. This reading need only be taken at intervals to test the stability of the instrument.

228. By Two Angles read in the Boat to three points on shore whose relative positions are known. This is called the "three-point" problem. Let A, C, and B be the three shore points, being defined by the two distances a and b and the angle

C. Let the two angles P and P' be measured at the point P. The problem is to find the distances AP and BP.

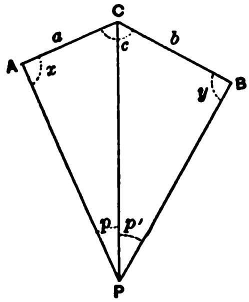

FIG. 68.

(a) *Analytical Solution.*—Let the unknown angle at A be x, and that at B be y. Then we may form two equations from which x and y may be found. For,

$$PC = \frac{a \sin x}{\sin P} = \frac{b \sin y}{\sin P'}. \quad . \quad . \quad . \quad . \quad (1)$$

Also, $$x + y = 360° - (P + P' + C) = R. \quad . \quad . \quad . \quad (2)$$

From (2), $$y = R - x,$$

and $$\sin y = \sin R \cos x - \cos R \sin x.$$

Substitute this value of $\sin y$ in (1), reduce, and find

$$\cot x = \frac{a \sin P' + b \sin P \cos R}{b \sin P \sin R}$$

$$= \cot R \left(\frac{a \sin P'}{b \sin P \cos R} + 1 \right). \quad . \quad . \quad . \quad (3)$$

When x and y are found, the sides AP and BP are readily obtained. This is perhaps the simplest analytical solution of the problem.

(b) *Geometrical Solution.*—The following geometrical solution is of some interest, though it is seldom used:

Let A, C, and B be the fixed points as before, and P and P' the observed angles. Having the points A, B, and C plotted in their true relative positions, draw from A the line AD, making with AB the angle P' (CPB), and from B the line BD, making with AB the angle P (APC), cutting the former line in D. Through A, D, and B pass a circle, and through C and D draw a line cutting the circumference again in P. The point P is the plotted position of the point of observation from which the angles P and P' were measured.

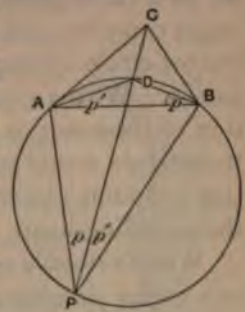

FIG. 69.

For P must lie in the circumference through ADB by construction, otherwise ABD would not be equal to APD, as they are both measured by the same arc AD. The same holds for the angle P'. Also, the line PD must pass through C, otherwise the angle APC would be greater or less than P, which cannot be. The point P is therefore on the line CD, and also on the circumference of the circle through ADB, whence it is at their intersection.

This demonstration is valuable as showing when this method of location fails to locate, and when the location is poor. For the nearer the point D comes to C the more uncertain becomes the direction of the line CD, and when D falls at C—that is, when P is on the circumference of a circle through A, B, and C—the solution is impossible, inasmuch as P may then be anywhere on that circumference without changing the angles P and P'. This is also shown by equation (3), above; for if A, C, B, and P all fall on one circumference, then $x + y = R = 180°$; whence $\cot x = \infty \times 0$, which is indeterminate. For $\cot R = -\infty$, and $\cos R = -1$. Also $a \sin P' = b \sin P$, both being equal to the perpendicular from C on AB. The equation then becomes

$$\cot x = \infty(1 - 1) = \infty \times 0.$$

(*c*) *Mechanical Solution.*—If the three known stations be plotted in position and the two observed angles be carefully set on a three-armed protractor,* then when the three radial edges coincide with the three stations, the centre of the protractor circle corresponds to the position of the point of observation. With a good protractor this method gives the position of the point as closely as the nature of the observations themselves would warrant. It is the common method of plot ting soundings when two sextant angles have been read from the sounding boat.

Wood's double sextant (see p. 113) is designed to read these two angles simultaneously. In the hands of an expert observer this instrument is very valuable for surveys on running water.

(*d*) *Graphical Solution.*—The angles may be laid off on tracing-paper or linen by lines of indefinite length, and this laid on the plot and shifted in position until the three radial lines coincide with the three stations, when their intersection marks the point of observation. This is the most ready method of plotting such observations when no three-armed protractor is available.

The advantages of this method of locating soundings are that it requires but one observer, no time is lost in changing stations, and the party are all together, and hence there can be no misunderstandings in regard to the work. If the soundings are made in running water, so that the boat cannot be stopped long enough to read two sextant angles, two sextants are sometimes used with one observer, he setting both angles and reading them afterwards; or two observers may be employed in the same boat and the angles taken simultaneously.

229. By one Range and one Angle.—The range may be two stations or poles set in line on shore, or it may consist of one point on shore and a buoy set at the desired position off-

* For description, with cut, see p. 167.

shore. If buoys are used they must be located by triangulation from the shore stations. A triangulation system along a rocky or wooded coast may consist of one line of stations on shore and a corresponding line of buoys. The angles are read only from the shore stations, two angles in each triangle being observed. If the buoys are well set and the work done in calm weather, the results will be good enough for topographical or hydrographical purposes. The stations and buoys should be opposite each other, as in the figure, and readings taken to the two adjacent shore stations and to the three nearest buoys from each shore station. If the length of any line of this system be known, the rest can be found when the angles at *A*, *B*, *C*, and *D* are measured. In such a system the measured lines should recur as often as possible, ordinary chaining being sufficient.

FIG. 70.

230. Buoys, Buoy-flags, and Range-poles.—A convenient buoy for this purpose may be made of any light wood, eighteen inches to three feet long in tideless waters, and long enough to maintain an erect position in tide-waters. It should be from six to ten inches in diameter at top, and taper towards the bottom. If the buoy is not too long, a hole may be bored through its axis for the flag-pole, which may then project two or three feet below the buoy and as high above it as desired. The buoy rope is then attached to the bottom end of the pole and made of such length as to maintain the pole in a vertical position in all stages of the tide. The anchor may be any sufficiently heavy body, as a rock or cast-iron disk. If the buoys are liable to become confused on the records, different designs may be used in the flags, as various combinations of red, white, and blue, all good colors for this purpose.

The range-poles should be whitewashed so as to show up against the background of the shore. The ranges are designated by attaching to the rear range-poles slats (barrel-staves would serve) arranged as Roman numerals when read up or down the pole. If range-poles are relied on, they must be very carefully located and plotted, in order to establish accurately a long line of soundings from a very short fixed base.

The observed angle may be either from the boat or from a point on shore. In either case any other range-post of the series may be used either for the position of the observer, if on shore, or for the other target-point if the angle is read from the boat.

231. **By one Range and Time-intervals.**—This is a very common and efficient method, and quite satisfactory where soundings need not be located with the greatest accuracy and where there is no current. A boat can be pulled in still water with great uniformity of speed; and if the soundings be taken at known intervals with the ends of the line of soundings fixed, the time-intervals will correspond almost exactly with the space-intervals. If the ends of the line of soundings are not fixed by buoys or sounding-stations on shore, but the line simply fixed by ranges back from the water's edge, the positions of the end soundings may be fixed by angle-readings and the balance interpolated from the time-intervals.

232. **By means of Intersecting Ranges.**—This method is only adapted to the case where soundings are to be repeated many times at the same places. When the object of the survey is to study the changes occurring as to scour or fill on the bottom it is very essential that the successive soundings should coincide in position, otherwise discrepant results would prove nothing. Such surveys are common on navigable rivers and in harbors. Many systems of such ranges could be described, but the ingenious engineer will be able to devise a system adapted to the case in hand.

233. **By means of Cords or Wires.**—In the case of a fixed

but narrow navigable channel, having an irregular bottom, or undergoing improvement by dredging, it may be found advisable to set and locate stakes on opposite sides of the channel, to stretch a graduated cord or wire between them, and to locate the soundings by this. By such means the location would be the most accurate possible.

MAKING THE SOUNDINGS.*

234. The Lead is usually made of lead, and should be long and slender to diminish the resistance of the water. It should weigh from five pounds for shallow, still water, to twenty pounds for deep running water, as in large rivers. If depth only is required, the lead may be a simple cylinder something like a sash-weight for windows. If specimens of the bottom are to be brought to the surface at each sounding, the form shown in Fig. 71 may be used to advantage. An iron stem, *I*, is made with a cup, *c*, at its lower end. The stem has spurs cut upon it, or cross-bars attached to it, and on this is moulded the lead which gives the requisite weight. Between the cup and the lead is a leather cover sliding freely on the shank and fitting tightly to the upper edges of the cup. When the cup strikes the bottom, it sinks far enough to obtain a specimen of the same, which is then safely brought to the surface, the leather cover protecting the contents of the cup from being washed out in raising the lead. A conical cavity in the lower end of the lead, lined with tallow, is often used, and it is found very efficient for indicating sand and mud. It is often very essential to know whether the bottom is composed of gravel, coarse or fine, sand, mud, clay, hard-pan, or rock, and this knowledge can be obtained with the cup device described above.

FIG. 71.

235. The Line should be of a size suited to the weight of

* See Appendix F for a description of methods — [illegible] — River Survey.

the lead, and made of Italian hemp. It is prepared for use by first stretching it sufficiently to prevent further elongation in use after it is graduated. Probably the best way to stretch a line is to wind it tightly about a smooth-barked tree, securely fasten both ends, wet it thoroughly, and leave it to dry. Then rewind as before, taking up the slack from the first stretching, and repeat the operation until the slack becomes inappreciable. It may now be graduated and tagged. Sometimes it is fastened to two trees and stretched by means of a "Spanish windlass," and then wet. It is quite possible to stretch the line too much, for sometimes sounding-lines have shortened in use after being stretched by this method. Soundings at sea are taken in fathoms. On the U. S. Lake Survey all depths over twenty-four feet (four fathoms) were given in fathoms, and all depths less than that limit were given in feet. On river and harbor surveys it is common to give depths in feet. Channel-soundings on the Western rivers made by boatmen are given in feet up to ten feet, then they are given in fathoms and quarters, the calls being "quarter-less-twain," "mark-twain," "quarter-twain," "half-twain," "quarter-less-three," "mark-three," etc., for depths of $1\frac{3}{4}$, 2, $2\frac{1}{4}$, $2\frac{1}{2}$, $2\frac{3}{4}$, 3, etc., fathoms respectively.

If the line is graduated in feet leather tags are used every five feet, the intermediate foot-marks being cotton or woollen strips. The ten-foot tags are notched with one, two, three, etc., notches for the 10-, 20-, 30-, etc., foot points, up to fifty feet. The fifty-foot tag may have a hole in it, and the 60-, 70-, 80-, etc., foot-marks have tags all with one hole and with one, two, three, etc., notches. The intermediate five-foot points have a simple leather tag unmarked. Sometimes the figures are branded on the leather tags, but notches are more easily read. The zero of the graduation is the bottom of the lead. The leather tags are fastened into the strands of the line: the cloth strips may be tied on. The line should be frequently tested, and if it changes materially a table of corrections should be made out and all soundings corrected for erroneous length of line.

236. Sounding-poles should be used when the depth is less than about fifteen feet. The pole may be graduated to feet simply, or to feet and tenths, according to the accuracy required.

237. Making Soundings in Running Water.—The sounding-boat should be of the "cutter" pattern, with a sort of platform in the bow for the leadsman to stand on. If the current is swift, six oarsmen will be required and two observers and one recorder. One of the observers may act as steersman. If the depth is not more than sixty or eighty feet, the soundings are made without checking the boat, the leadsman casting the lead far enough forward to enable it to reach bottom by the time the line comes vertical. When the depth and the current are such as to make this impossible, the boat is allowed to drift down with the current and soundings taken at intervals without drawing up the lead. The boat is then pulled back up stream and dropped down again on another line, and so on.

In still water a smaller crew and outfit may be used, as the boat may be stopped for each sounding if necessary.

The record should give the date, names of observers, general locality, number or other designation of line sounded, the time, the two angles, the stations sighted, and the depth for each sounding, and the errors of the graduated lengths on the sounding-line.

238. The Water-surface Plane of Reference.—In order to refer the bottom elevations to the general datum-plane of the survey, it is necessary to know the elevation of the water-surface at all times when soundings are taken. In tidal waters the elevation of "mean tide" is the plane of reference for both the topographical and hydrographical surveys, and then the state of the tide must be known with reference to mean tide, This is found from the hourly readings of a tide-gauge (provided it is not automatic), the elevation of the zero of which, with reference to mean tide-water, has been determined. All soundings must then be reduced to what they would have been if made at mean tide before they are plotted.

If the soundings are made in lakes, the datum is usually the lowest water-stage on record; and here also gauge-readings are necessary, as the stage of the water in the lake varies from year to year. In this case the gauge need only be read twice a day.

In rivers of variable stage the datum is either referred to mean or low-water stage, or else to the general datum of the map. If the stage is changing rapidly the gauge should be read hourly when soundings are taken, otherwise daily readings are sufficient. If the soundings are to be referred to the general datum of the map, then the *slope* of the stream must be taken into account. If they are referred to a particular stage of water in the river, then the slope does not enter as a correction, as the slope is assumed to be the same at all stages, although this is not strictly true.

239. **Lines of Equal Depth** correspond to contour lines in topographical surveys; but to draw lines of equal depth with certainty the elevations of many more points are necessary than are needed for drawing contour lines, because the bottom cannot be viewed directly, while the ground can be. Where the ground is seen to be nearly level no elevations need be taken, while for a similar region of bottom a great many soundings would be required to prove that it was not irregular.

240. **Soundings on Fixed Cross-sections in Rivers.**—Where the same section is to be sounded a great many times, and especially when it is desirable to obtain the successive soundings at about the same points, it is best to fix range posts on the line of the section (on both sides if it be a river)

and then fix one or more series of intersecting ranges at points some distance above or below the section on one or both sides of the river. The soundings can then be made at the same points continuously without having to observe any angles at all. Such a system of ranges is shown in Fig. 72. AA' and BB' are range-poles on the section line. O and O' are tall white posts set at convenient points on opposite sides of the river, either above or below the section. I., II., III., etc., are shorter posts set near the bank in such positions that the intersection of the lines O-I., O-II., etc., with the section range

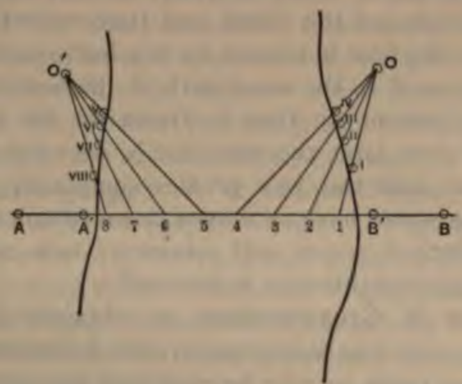

FIG. 72.

BB' will locate the soundings at 1, 2, etc., on this section line. The posts in the banks should be marked by strips nailed upon them so as to make the Roman numerals as given in the figure. Such a system of ranges as the above is useful also for fixing points on a section-line, for setting out floats, or for running current meters for the determination of river discharge.

241. Soundings for the Study of Sand-waves.—In all cases where streams flow in sandy beds, the bottom consists of a series of wave-like elevations extending across the channel. These are very gently sloping on the up-stream side and quite abrupt on the lower side. They are called sand-waves, or sand-reefs. They are constantly moving down-stream from the slow removals from the upper side and accretions on the abrupt lower face. They have been observed as high as ten feet on the Mississippi River, and with a rate of motion as great as thirty feet per day. In order to study the size and motion of these sand-waves, it is necessary to take soundings very near together, on longitudinal lines over the same paths at frequent intervals for a considerable period. The boat is allowed to drift with the current and the lead floats with the boat near the bottom. It is lowered to the bottom every few seconds and the depth and time recorded. About once a minute the boat is located by two instruments on shore or in the boat, and so the exact path of the boat located. A profile of the bottom can then be drawn for the path of the boat. A few days later the same line is sounded again in a similar manner and the two profiles compared. It will be found that the waves have all moved down-stream a short distance, the principal waves still retaining their main characteristics, so that identification is certain.*

242. Areas of Cross-section are obtained by plotting the soundings on cross-section paper, the horizontal scale being about one tenth or one twentieth of the vertical. The horizontal line representing the water-surface is drawn, and the plotted soundings joined by a free-hand line. The enclosed area is then measured by the planimeter. If the horizontal scale is 50 feet to the inch and the vertical scale 5 feet to the inch, then each square inch of the figure represents 250 square feet of area. The planimeter should be set to read the area in square inches, and the result multiplied by 250.†

* It is believed the author made the first successful study of the size and rate of motion of sand-waves, at Helena, Ark., on the Mississippi River, in 1879. See Rep. Chief of Engineers, U. S. A., 1879, vol. iii., p. 1963.

† See p. 143 for a description and theory of the planimeter.

Areas of cross-section are usually taken in running water, and here great care must be taken to get vertical soundings, and to make the proper sounding-line corrections. They should be taken near enough together to enable the bottom line to be drawn with sufficient accuracy.

BENCH-MARKS, GAUGES, WATER-LEVELS, AND RIVER-SLOPE.

243. **Bench-marks** should be set in the immediate vicinity of each water-gauge, and these connected by duplicate lines of levels with the reference-plane of the survey. If the gauge is not very firmly set, or if it is necessary to move it for a changing stage, its zero must be referred again to its bench-mark by duplicate levels, whenever there is reason to suspect it may have been disturbed. Such bench-marks as these are usually spikes in the roots of trees or stumps.

244. **Water-gauges** are of various designs, according to the situation and the purpose in view. For temporary use during the period of a survey, a staff gauge is best, consisting of a board painted white, of sufficient length, graduated to feet and tenths in black. Sometimes it is graduated to half-tenths, but this is useless unless in still water, and there is never any need of graduation finer than this. The gauge may be read to hundredths of a foot if the water is calm enough. It should be nailed to a pile or to a stake driven firmly near the water's edge. It is read twice a day, or oftener, if the needs of the service require.

For the continuous record of tidal stages an automatic, or self-registering, gauge is employed. For rivers with widely varying stage an inclined scantling is fixed to stakes set from low to high water along up the sloping bank. It should be placed at a point where the bank is neither caving away nor growing by filling-in of new deposits. After the scantling is set (the slopes not necessarily the same throughout its length), the foot and tenth graduations are set by means of a level and marked by driving copper tacks. The automatic gauge is described in Chap. XIV. The staff gauge is the one generally used for engineering and surveying purposes.

245. Water-levels.—The surface of still water is by definition a level surface. This fact is used to great advantage on the sea-coast, on lakes, ponds, and even on streams of little slope or on such as have a known slope. Thus, in finding the elevations of the Great Lakes above the sea-level, the elevation above mean tide-water of the zero of a certain water-gauge at Oswego, N. Y., on Lake Ontario, was determined. Then the relative elevations of the zeros of certain gauges at Ports Dalhousie and Colborne, at the lower and upper ends of the Welland Canal respectively, were found by levelling between them, thus connecting Lake Ontario with Lake Erie. Lakes Erie and Huron were joined in a similar manner by connecting a gauge at Rockwood, at the mouth of the Detroit River with one at Lakeport, at the lower end of Lake Huron. Lakes Michigan and Huron were assumed to be of the same level on account of the small flow between them and the very large sectional area of the Straits of Mackinac. Finally, a gauge at Escanaba, on Lake Michigan, was joined by a line of levels with one at Marquette, on Lake Superior. This completed the line of levels from New York to Lake Superior, when sufficient gauge-readings had been obtained to enable *water levels* to be carried from Oswego to Port Dalhousie, on Lake Ontario; from Port Colborne to Rockwood, on Lake Erie; and from Lakeport, on Lake Huron, to Escanaba, on Lake Michigan. It was found that these water-levels were very accurate. Relative gauge-readings were compared for calm days, as well as for days when the wind was in various directions, and a final mean value found which in no case had a probable error as great as 0.1 foot.*

* See Primary Triangulation of the U. S. Lake Survey.

A line of levels run along a lake shore or canal in calm weather should be checked at intervals by reading to the water-surface, and in a topographical survey the stadia-rod should frequently be held at the water-surface, even when the body of water is a stream with considerable slope, as it gives a check against large errors even then, and at the same time gives the slope of the stream. Mean sea-level at all points on the sea-coast is universally assumed to define one and the same level surface. It is probable, however, that this is not strictly true. Wherever a constant ocean current sets steadily against a certain coast, it would seem that the water here must be raised by an amount equal to the head necessary to generate the given last motion. If the current flows into an enclosed space, as the equatorial current into the Gulf of Mexico, or the tides into the Bay of Fundy, the water-surface may rise much higher. There is some evidence that the elevation of mean tide in the Gulf of Mexico is two or three feet higher than that of the Atlantic at Sandy Hook.* The evidence on this point is as yet insufficient to warrant any certain conclusion, however.

246. River Slope is a very important part of a river survey. Sometimes it is desirable to determine it for a given stretch of river with great care, in which case it is well to set gauges at the points between which the slope is to be found and connect them by duplicate lines of accurate levelling. The gauges are then read simultaneously every five minutes for several hours and the comparison made between their mean readings. This is always done in connection with the measurement of the discharge of streams when the object is to find what function the discharge is of the slope. It is now known, however, that in natural channels the discharge is no assignable function of

* See paper by Prof. Hilgard, Supt. U. S. C. and G. Survey, in Trans. Am Asso. Adv. Sciences, 1884, p. 446.

the slope, as is explained in section 259. For ordinary purposes the river slope may be determined with sufficient accuracy by simply reading the level or the stadia-rod at water-surface as the survey proceeds, daily readings of stage being made at permanent gauges at intervals of fifty miles or less along the river.

In all natural channels the local slope is a very variable quantity. It is frequently negative for short distances in certain stages, and over the same short stretch of river it may vary enormously at different stages, and even for the same stage at different times. It is determined by the local channel conditions, and these are constantly changing in streams flowing in friable beds and subject to material changes of stage. Great caution must therefore be exercised in introducing it into any hydraulic formulæ for natural channels. It is usually expressed as a fraction, being really the natural sine of the angle of the surface to the horizon. That is, if the slope is one foot to the mile it is $\frac{1}{5280} = 0.000189$.

THE DISCHARGE OF STREAMS.

247. Measuring Mean Velocities of Water Currents.—This is usually done only for the purpose of obtaining the discharge of the stream or channel, but sometimes it is done for other purposes, as for the location of bridge piers or harbor improvements. In the case of bridge piers the direction of the current at different stages must be known, so that the piers may be set parallel to the direction of the current. For finding the discharge of the stream or other channel the object may be:

(1) To obtain an approximate value of the discharge at the given time and place.

(2) To obtain an exact value of the discharge at the given time and place.

(3) To obtain a general formula from which to obtain sub-

sequent discharges at the given place, or to test the truth of existing formulæ, or to determine the relative efficiency of certain appliances or methods.

It will be assumed that the second object is the one sought, and modified forms of the methods used to accomplish this may be chosen for other cases.

The *mean velocity* of a stream is by definition the total discharge in cubic feet per second divided by the area of the cross-section in square feet. This gives the mean velocity in feet per second. Evidently this is the mean of the velocities of all the small filaments (as of one square inch in area) on the entire cross-section. If the velocities of these filaments could be simultaneously and separately observed and their mean taken, this would be the mean velocity of the stream. It is quite impossible to do this; but the nearer this is approached, the more accurate is the final result. If, however, we could obtain by a single observation the mean velocity of all the filaments in a vertical plane, the number of necessary observations would be diminished without diminishing the accuracy of the result. There are two common methods of measuring the velocities of filaments at any part of the cross-section, and one for obtaining at once the mean velocity in a vertical plane. These are by sub-surface floats and current-meters, and by rod floats, respectively.

248. By Sub-surface Floats.—The ideal sub-surface float consists of a large intercepting area maintained at any depth in a vertical position by means of a fine cord joined to a surface float of minimum immersion and resistance, which bears a signal-flag. As good a form as any, perhaps, for the lower float, or intercepting plane, consists of two sheets of galvanized iron set at right angles, and intersecting in their centre lines, as shown in Fig. 73. There are cylindrical air-cavities along the upper edges and lead weights attached to the lower edges of the vanes. These serve to give the desired tension on the connecting cord and to maintain the float in an upright position, even though the cord is drawn out of the vertical by faster upper currents. The vanes should be from six to fifteen inches in breadth by from eight to twenty inches high, according to the size of the stream. The circular ribs serve simply to hold the vanes in place. The upper float is hollow, cylin-

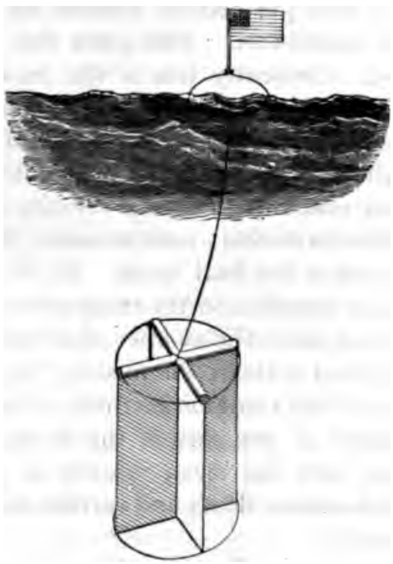

FIG. 73.

drical in plan, and carries a small flag. The tension on the cord should be from one to five pounds, according to the size of the floats. The cord itself should be of woven silk and as small as possible, so as to exercise a minimum influence on the motion of the lower float. Wire is not suitable for this purpose, as it kinks badly in handling. The theory is that the lower float will move with the water which surrounds it, and that the upper float will be accelerated or retarded according as the surface current is slower or faster than that at the submerged float. The velocity of the current at any depth can thus be determined by running the lower float at this depth and observing the time required for the upper float to pass between two fixed range-lines at right angles to the direction of the current about two hundred feet apart. The floats are started about one hundred feet above the upper range-line, and picked up after having passed the lower range. Two transits are usually used for locating and timing the floats, one being set on each range. When the float approaches the upper range the observer on this line sets his telescope on range and calls "ready" as the float enters his field of view. The other observer then clamps his instrument and follows the float with the aid of the slow-motion or tangent screw. When the float crosses the vertical wire of the upper instrument he calls "tick," and the lower observer reads his horizontal angle. He then sets his telescope on the lower range while the upper observer follows the float with his telescope, and the operation is repeated to obtain an intersection on the lower range. One or two timekeepers are needed to note the time of the two "tick" calls, the difference being the time occupied by the float in passing from the upper to the lower range-line. Both these signals are sometimes transmitted telegraphically to a single timekeeper. When the angles are plotted the path of the float is also obtained.

If the channel is not too wide, wires may be stretched across the stream and the float stations marked on these, or the float stations may be determined by means of fixed ranges on shore. The passage of the floats across the section lines may then be noted by a single individual without a transit, using a stop-watch and possibly a field-glass. He starts the watch when the float reaches the upper section, walks to the lower section, and stops the watch when the float passes this range-line. The near range consists of a plumb-line, or wire suspended vertically; and the observer stands several feet back of this, and brings it in line with the range-post on the opposite side of the stream.

If several floats are started a few minutes apart at the same station and at the same depth, they will sometimes vary as

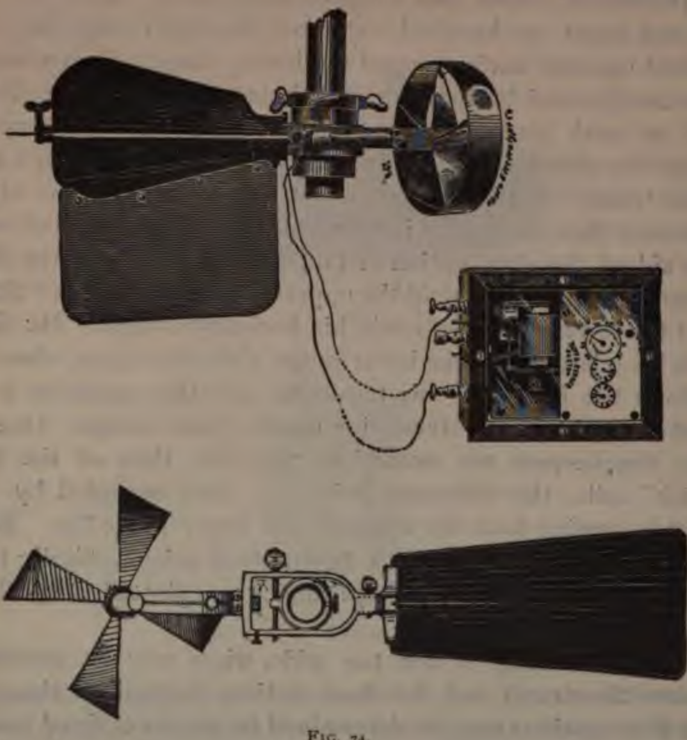

FIG. 74.

much as twenty per cent in their times of passage, showing great irregularity in the velocity of different parts of the same filament. This is due to internal movements in the water, such as "boils," eddies, etc. It is for this reason that great refinement in such observations is useless. A float observation

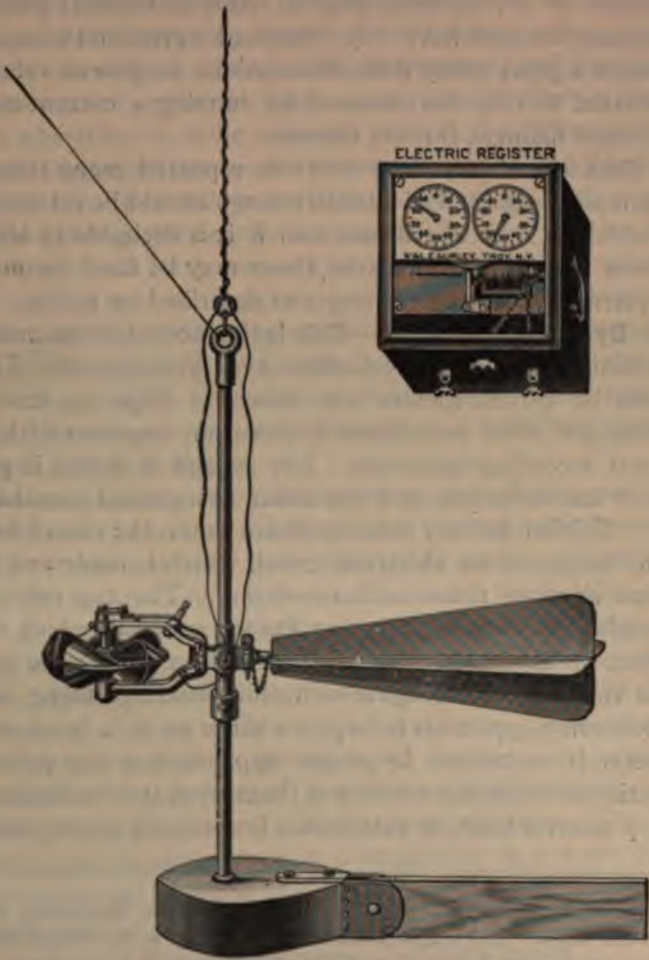

FIG. 75.

gives only the velocity of a given small *volume* of water which surrounds the lower float, while a current-meter observation, as will be seen, gives the mean velocity of a given *filament* of the stream of any required length. And as different portions of the same filament have very different longitudinal velocities, it requires a great many float observations to give as valuable information as may be obtained by running a current-meter in the same filament for one minute.

If discharge observations are to be repeated many times at the same sections, then an auxiliary range should be established from which to start the floats; and if it is desirable to always run them over the same paths, these may be fixed by means of a system of intersecting ranges as described on p. 305.

249. By Current-meter.*—This is the most accurate method of obtaining sub-surface velocities ever yet devised. Three patterns of current-meters are shown in Figs. 74 and 75. The first and third are shown in elevation, together with the electrical recording-apparatus. The second is shown in plan. The first has helicoidal and the other two conical cup-shaped vanes. Neither has any gearing under water, the record being kept by means of an electrical circuit which is made and broken one or more times each revolution. The cup vanes are better adapted to water carrying fibrous materials which tend to collect on the moving parts. The friction can also be made less on the cup meters, agate or iridium bearings being used. The recording-apparatus is kept on shore or in a boat, while the meter is suspended by proper appliances at any point of the section at which the velocity of the current is to be measured. In deep water a boat, or catamaran, is anchored at the desired

* Invented by Gen. Theo. G. Ellis, and first used on the survey of the Connecticut River. The telegraphic attachment is due to D. Farrand Henry of Detroit, Mich. See Report of the Chief of Engineers, U. S. A., 1878, p. 308.

The form shown in Fig. 75 is due to W. G. Price, and was specially designed to be used on the Mississippi River. It is very strong and well protected against floating drift. The first two forms are manufactured by Buff & Berger, of Boston, while the Price meter is made by W. & L. E. Gurley, Troy. See also the Ritchie & Haskell meter for direction and velocity of sub-currents, Art. 255.

point, and a weight attached to the meter, which is then lowered to the requisite depth by means of a windlass. After it is in place the connection is made with the battery, and the record kept for a given period of time, as for two or three minutes. If the operation is to be repeated often at the same section a wire anchorage laid across the stream above the line would be found useful. This wire is anchored at intervals and is used both for holding the boat (or catamaran) in place and for pulling it back and forth across the stream. In large rivers a steam-launch may be required for handling the catamaran.* In this case the record begins and ends when the observer is brought on range, it being impossible to hold up steadily against the current. If only the discharge of the stream is sought, the meter is run at mid-depth at a sufficient number of points in the section.

The *mean velocity in a vertical section* at a given point may be obtained by moving the meter at a uniform rate from surface to bottom and back again, noting the reading of the register for the two surface positions, and also for the bottom position. If the boat was stationary and the rates of lowering and raising strictly constant and equal, the number of revolutions in descending and in ascending should be equal. Either of these registrations, divided by the time, would give the *mean registration per second* of all the filaments in that vertical plane. The mean of the downward and upward results may be used as giving the mean velocity in that vertical plane. This will not be quite accurate, since it is impossible to run the meter very close to the bottom, but the results will be found useful for comparison with the mid-depth results. Such observations are sometimes called *integrations* in a vertical plane.

250. Rating the Meter.—When any kind of current-meter

* For a description of the latest methods used in gauging the Mississippi River see Report of the Miss. Riv. Com. for 1883, Appendix F.

is used for determining the velocity of passing fluids, only the number of revolutions of the wheel carrying the vanes is observed for a given time. Before the velocity of the fluid in feet per second can be found, the relation between the rate of revolution of the wheel and the rate of motion of the fluid must be determined for all velocities that are to be observed. The determination of this relation is called ***rating the meter***. It is usually done by causing the meter to move through still water at

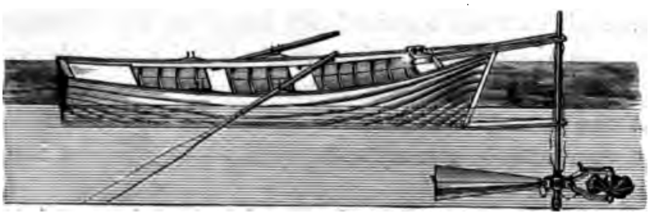

FIG. 76.

a uniform speed, and noting the time occupied and the corresponding number of registrations made in passing over a given distance. It may be attached to the prow of a boat, as shown in Fig. 76, the electric register being in the boat. The distance divided by the time gives the rate of motion or velocity of the meter through the water. The number of registrations (revolutions of the wheel) divided by the time gives the rate of motion of the wheel. The ratio of these two rates is the coefficient by which the registrations of the meter are transformed into the velocity of the current. This ratio is not a constant, but is usually a linear function of the velocity. Thus, if the observations be plotted, taking the number of registrations per second as abscissæ and the velocities in feet per second as ordinates, they will be found to fall nearly in a right line, the equation of which is

$$y = ax + b \quad . \quad . \quad . \quad . \quad (1)$$

Here x and y are the observed quantities, while a and b are constants for the given instrument. If these constants could be found, then the values of y (velocity) could be obtained for all observed values of x (registrations). There are two ways of solving this problem—one graphical and one analytical. Evidently any two observations at different speeds would give values of a and b; but to find the best or most probable values of these constants a great many observations are taken, so that we have many more observations than we have unknown quantities. Each pair of observations would give a different set of values of a and b. The most convenient method of finding the most probable values of these functions, though somewhat approximate, is

(1) *The Graphical Method of Solution.*—This consists simply in plotting the corresponding values of x and y on coördinate paper, and drawing the most probable straight line through the points. Then the tangent of the angle this line makes with the axis of x is a, and the intercept on the axis of y is b. One point on this most probable line is the point $(x_0 y_0)$, x_0 and y_0 being the *mean values* of the coördinates of all the plotted points. This is shown by equation (3). Having determined this point, a thread may be stretched through it and swung until it seems to be in a position of equilibrium, when each point is conceived as an attractive force acting on the line, the measure of the force being the vertical intercept between the point and the line. The arms of these forces are evidently their several abscissæ. Or the forces may be measured by their horizontal intercepts, and then their arms are their several ordinates. For the position of equilibrium the sum of the moments of these forces about the point $(x_0 y_0)$ would be zero.* Such a determination of a and b would be found sufficiently accurate for all practical purposes, but if desired the problem may be solved by

* All this simply means to fix this most probable line by eye, through the point $(x_0 y_0)$, giving greatest weight to the extreme points.

(2) *The Rigid or Analytical Method.*—Equation (1) may be written

$$b + xa - y = 0.$$

Every observation may be written in this form, these being called the observation equations. It is probable that no given values of a and b would satisfy more than two of these observations; and if the most probable values be used, there would, in general, be no single equation exactly satisfied. If we let x_1, x_2, etc., y_1, y_2, etc., and v_1, v_2, etc., be the several values of x, y, and the corresponding residuals for the several observation equations, we would have

$$\left.\begin{array}{l} b + x_1 a - y_1 = v_1; \\ b + x_2 a - y_2 = v_2; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ b + x_m a - y_m = v_m. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since b enters alike in all of them, it is evident that these equations are all of equal value for determining b. Also, since the properly weighted arithmetic mean is the most probable value of a numerously observed quantity, and since in this case the equations (or observations) have equal weight for determining b, we may form from the given series of equations a single standard or "normal" equation which will be the arithmetic mean of the observation equations; put this equal to zero and say this shall give the value of b. If x_0 and y_0 be the mean values of the observed x's and y's, we would then have, by adding the equations all together and dividing by their number

$$b + x_0 a - y_0 = 0, \text{ or } b = y_0 - x_0 a. \quad . \quad . \quad . \quad (3)$$

Substituting this value of b in equation (2), we have

$$\left.\begin{array}{l}(x_1 - x_0)a - (y_1 - y_0) = v_1;\\ (x_2 - x_0)a - (y_2 - y_0) = v_2;\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ (x_m - x_0)a - (y_m - y_0) = v_m.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

We here have a series of equations involving one unknown quantity; but they evidently are not of equal value in determining the unknown quantity a, since its coefficients are very different. In fact, the relative value of these equations for determining a is in direct proportion to the size of this coefficient, so that if this coefficient is twice as large in one equation as in another, the former equation has twice the value of the latter for determining a. In other words, they should all be weighted in proportion to the values of these coefficients, and a convenient way of doing this is to multiply each equation through entire by this coefficient. The resulting multiplied equations then have equal weight, and may then be added together to produce another "normal" equation for finding a. This resulting equation is

$$[(x - x_0)^2]a - [(x - x_0\ (y - y_0)] = 0, \quad . \quad . \quad . \quad (5)$$

where [ ] is a sign of summation. If we had divided this equation by the number of observation equations m, it would in no sense have changed it so far as the value of a is concerned. From equation (5) we can find the mean or most probable value of a, which when substituted in (3) gives the most probable value of b. These values should agree very closely with those found by the graphical method. The analytical method here given is precisely that by least squares, though arrived at through the conception of a properly weighted arithmetic mean, instead of by making the sum of the squares of the residuals a minimum.

The following is an actual example from the records of the Mississippi River Survey:

REDUCTION OF OBSERVATIONS FOR RATING METER A,

taken at Paducah, Ky., June 21, 1882.

W. G. PRICE, *Observer.* L. L. WHEELER, *Computer.**

No.	r	t	x	y	$x - x_0$	$y - y_0$	$(x - x_0)^2$	$(x - x_0)(y - y_0)$	Remarks.
1	100	53	1 886	3.774	+ 0.117	+ 0.245	+ 0.014	+ 0.029	Observations
2	101	44	2.295	4.544	+ 0.526	+ 1.015	+ 0.277	+ 0.534	made with
3	101	41	2.464	4.878	+ 0.695	+ 1.349	+ 0.483	+ 0.938	meter on vertical
4	96	124	0.774	1.613	− 0.995	− 1.916	+ 0.990	+ 1.906	iron rod, five
5	94	152	0.618	1.316	− 1.151	− 2.213	+ 1.325	+ 2.548	feet in front of
6	90	193	0.466	1.036	− 1.303	− 2.493	+ 1.697	+ 3.249	bow of skiff, in
7	91	181	0.503	1.105	− 1.266	− 2.424	+ 1.603	+ 3.069	pond.
8	103	28	3.678	7.142	+ 1.909	+ 3.613	+ 3.644	+ 6.903	
9	100	53	1.886	3 774	+ 0.117	+ 0.245	+ 0.014	+ 0.029	Length of base
10	98	73	1.342	2.740	− 0.427	− 0.789	+ 0.182	+ 0.337	= 200 feet.
11	103	29	3.552	6.896	+ 1.783	+ 3.367	+ 3.178	+ 6.002	

$[x] = 19.464$ $38.818 = [y]$ $[(x-x_0)^2] = 13.407$ $25.544 = [(x-x_0)(y-y_0)]$

$x_0 = 1.769$ $3.529 = y_0$

Normal Equations.

$$b + 1.769a - 3529 = 0; \qquad \text{Whence} \qquad a = 1.905;$$
$$13.407a - 25.544 = 0. \qquad\qquad b = 0.159.$$

Equation for Rating.

$$y = 1.905x + 0.159.$$

Even where the analytical method is to be used it is always well to plot the observations for purposes of study. Then if any observations are especially discrepant, the fact will appear. By consulting column six of the computation it will

* In the original computation the method by least squares was used and the probable errors of a and b found.

be seen that observations of greatest weight were those taken at very high and at very low velocities. If the observations were taken in three groups about equally spaced, an equal number of observations in each group, the members of a group being near together, then the mean of each group could be used as a single observation. The middle group would serve to show whether or not the unknown quantities were linear functions of each other, since, if they were, the three mean observations should plot in a straight line. The value of a could be computed from the two extreme mean observations, and the value of b from the mean of all the observations as before. This would give a result quite as accurate as to treat them separately.

If the observations do not plot in a straight line, draw the most probable line through them, and prepare a table of corresponding values of x and y from this curve. In any case, a reduction table should be used.

The meter should be rated frequently if accurate results are required. In the rating the meter should be fastened several feet in front of the bow of the boat, and in its use it should be run at a sufficient distance from the boat or catamaran to be free from any disturbing influence on the current.

251. By Rod Floats.—These may be either wooden or tin rods, of uniform size, loaded at the bottom, and arranged for splicing if they are to be used in deep water. If the channel were of uniform depth, and the rod reached to the bottom without actually touching, then the velocity of the rod would be the mean velocity of all the filaments in that vertical plane,*

* This is not strictly true, since the pressure of a fluid upon a body moving through it varies as the square of its relative velocity. The rod moves faster than the bottom filaments and slower than the upper filaments, but this difference is greatest at the bottom. Therefore, the retarding action of the bottom filaments will have undue weight, as it were, and so the velocity of the rod will really be about one per cent slower than the mean velocity of the current. See "Lowell Hydraulic Experiments," by James B. Francis.

and this is the value sought. In practice the rod can never reach the bottom, even in smooth, artificial channels, while in natural channels the irregularities are usually such as prohibit its use within several feet of the bottom. The methods of observation are the same as with the double floats, and their velocity is the mean velocity of the water in that plane to the depth of immersion. For artificial channels, and for natural channels not more than twenty or thirty feet deep, rod floats may be advantageously used. Beyond that depth they cannot be made of sufficient length to give reliable results. The method is, therefore, best adapted to artificial channels of uniform cross-section.

The immersion of the rod should be at least nine tenths of the depth of the water, in which case, and for uniform channels, as wooden flumes, Francis found that the velocity of the rod required the following correction to give the mean velocity of the water in that vertical plane:

$$V_m = V_r \; [1-0.116 \, (\sqrt{D} - 0.1)].$$

Where V_m = mean velocity in vertical plane;

V_r = observed velocity of rod;

$$D = \frac{\text{depth of water below bottom of rod}}{\text{depth of water}}.$$

For natural channels, or for a less immersion than nine-tenths of the depth the formula cannot be used with certainty. The rods should be put into the water at least twenty feet above the upper section.

252. Comparison of Methods.—(1) The method by double floats is adapted to large and deep rivers, or rapid currents carrying much drift or impeded by traffic. It may be used in all cases, but it has the disadvantage of registering only the velocity of a small volume of water surrounding the lower float.

(2) The method by meters is adapted to large or small streams. It records the mean velocity of a filament of indefinite length; but it cannot be used where the water carries considerable floating débris, or where the current is too swift to admit of a safe anchorage.

(3) The method by rods is best adapted to small channels of uniform section; it records the mean velocity in a vertical plane to a depth equal to its immersion, and it can be universally used when the law of the velocities in a vertical plane is known, for then a proper coefficient could be derived for any depth of immersion.

(4) One rod observation of sufficient immersion is probably as good as several float observations, and a current-meter observation of two or three minutes is worth as much as twenty float observations for the same filament, provided the meter's rate is constant and well determined.

(5) The rods and floats are cheaper in first cost than the meter; but if the work is to be prosecuted for a considerable period, the excess in the cost of the outfit will be more than balanced by the diminished cost of the work, by using the meter. On the whole, it may be said that the method by current-meter is the most accurate and satisfactory of any yet devised for measuring the velocity of running water.

253. The Relative Rates of Flow in Different Parts of the Cross-section.—(1) *In a horizontal plane.* If the cross-section of a stream were approximately the segment of a circle, then the relative rates of flow of the different filaments in any horizontal plane would be very nearly represented by the ordinates to a parabola, the axis of the parabola coinciding with the middle of the stream. If there should be any shoaling in any part of this ideal section the corresponding ordinates would be shortened, so that when the curve of the bottom is given the curve of velocities in a horizontal plane can be fairly predicted. This applies only to straight reaches. If a portion of the section has a flat bottom line, the velocities over this portion will be about uniform. Where the depth is changing rapidly on the section, there the velocities will be found to change rapidly for given changes in positions across the section.

It follows from this that observation stations should be placed near together where the section has a sloping bottom line, and they may be placed farther apart where the bottom line of the section is nearly flat. They are usually put closer together near the bank than near the middle of the stream.

(2) *In a vertical plane.* A great deal of time and talent has been spent in trying to find the law of the relative rates of flow

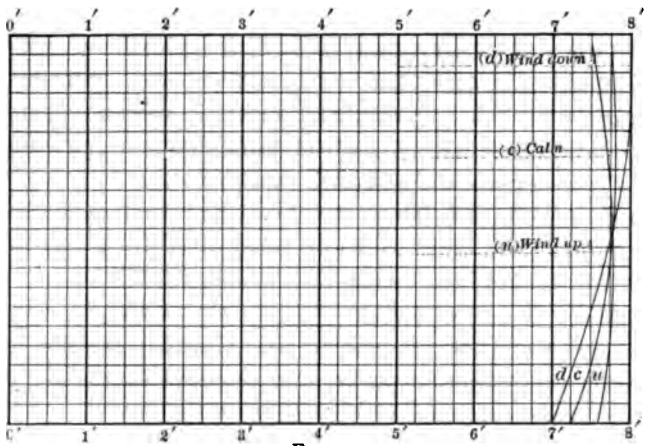

FIG. 77.

in a vertical plane, but there is probably no law of universal application. The curve representing such rates of flow will always resemble a parabola more or less, the axis of which is always beneath the surface except when the wind is down stream at a rate equal to, or greater than, the rate of the current. That is to say, the maximum velocity is always below the surface except where the surface filaments are accelerated by a down-stream wind, and it is generally found at about one third the depth. The cause of this depression of the filament of maximum velocity is partly due to the friction of the air, but mostly to an inward surface flow from the sides toward the centre, which brings particles having a slower motion towards the middle of the surface of the stream. This inward surface flow is probably due to an upward flow at the sides caused by the irregularities of the bank, which force the particles of water impinging upon them in the direction of the least resistance which is vertical.* The curves in Fig. 77 represent the mean vertical curves of velocity observed at Columbus, Ky., on the Mississippi River and given in Humphreys and
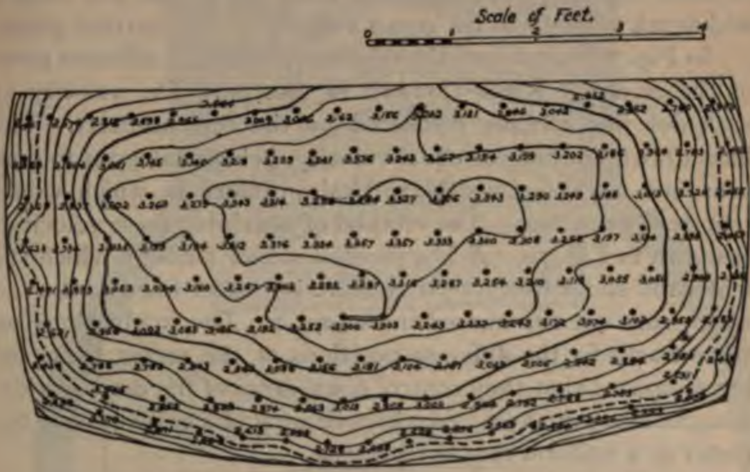
FIG. 78.

Abbot's Report. The left-hand vertical line is the axis of reference, and the curves are found to fall between the seven- and eight-foot lines. That is, the velocity at all depths in this plane was between seven and eight feet per second. In this case double floats were used, and it is probable that the bottom velocities were not very accurately obtained. The effect of the wind is here shown in shifting the axis of the curve. It is to

* See paper by F. P. Stearns before the Am. Soc. Civ. Engrs., vol. xii. p. 331.

be observed that these curves all intersect at about mid-depth. That is to say, the velocity of the mid-depth filament is not affected by wind. This is why the mid-depth velocity should be chosen when the velocity of but a single filament is to be measured, and from this the mean velocity in the vertical section derived. It has also been found that the mid-depth velocity is very near the mean velocity, being from one to six per cent greater, according to depth and smoothness of channel. In general, for channels whose widths are large as compared to their depths, a coefficient of from .96 to .98 will reduce mid-depth velocity to the mean velocity in that vertical plane.

In Fig. 78* are shown the relative velocities in different parts of the Sudbury River Conduit of Boston. The velocity at each dot was actually measured by the current-meter. The lines drawn are lines of equal velocity, being analogous to contour lines on a surface, the vertical ordinates to which would represent velocities. The method of obtaining these velocities is shown in Fig. 79. *B* is a pivoted sleeve through which the meter-rod slides freely. At *A* there is a roller fixed to the rod which runs on the curved tracks *a a a*. The graduations on these tracks fix the different positions of the meter, these being so spaced that they control equal areas of the cross-section. Integrations were here taken in horizontal planes by moving the meter at a uniform rate horizontally.

254. To find the Mean Velocity on the Cross-section.—It is evident that this mean velocity cannot be directly observed. In fact, it can only be found by first finding the discharge per second and then dividing this by the total area of the section. That is to say, the mean velocity is, by definition,

$$v = \frac{Q}{A}.$$

* This and the following figure are taken from the paper by F. P. Stearns, mentioned in foot-note on the previous page.

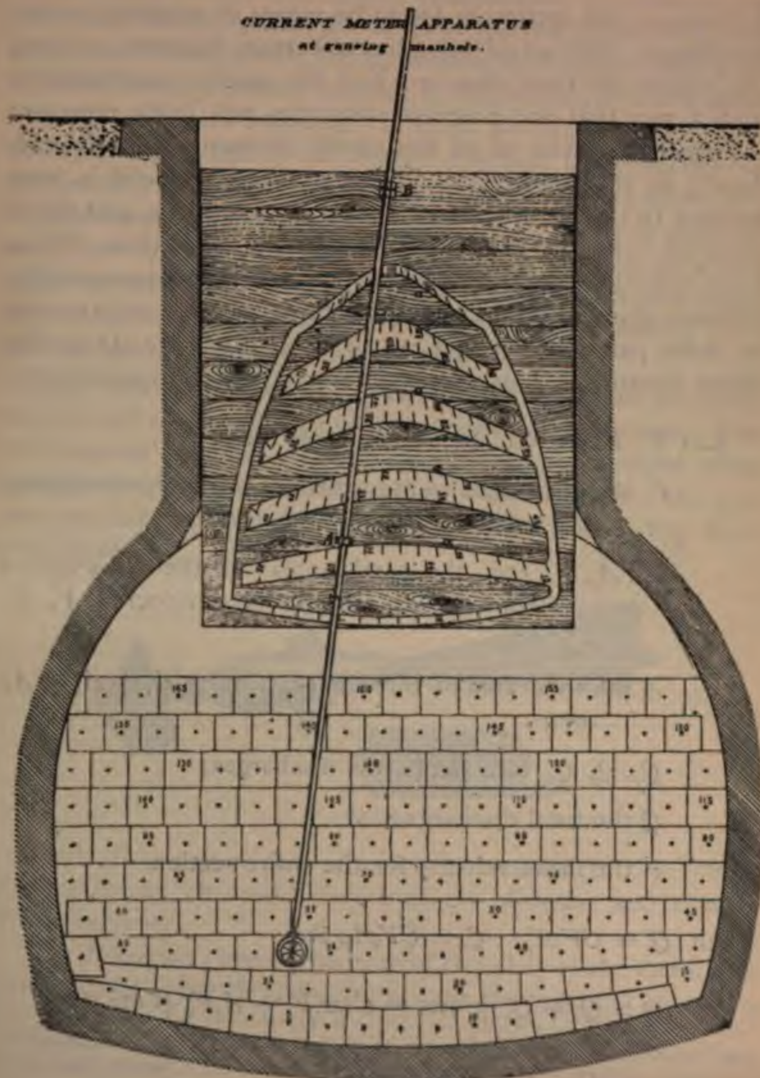

FIG. 79.

The area of the section is found by means of properly located soundings. The actual velocities of certain filaments crossing this section are then observed, and the section subdivided in such a way that the observed velocities will fairly represent the mean velocities of all the similar filaments (usually mid-depth) in that subsection. Each observed velocity is then reduced to the mean velocity in that vertical plane, and this is assumed to be the mean velocity in that subsection. These mean velocities, multiplied by the areas of their corresponding sections, give the discharges across these sections, and the sum of these partial discharges is the total discharge, Q, in the above equation. This may be shown algebraically as follows:

Let V_1, V_2, V_3, etc., be the observed velocities;

C the coefficient to reduce these to the mean velocity in a vertical plane;

A_1, A_2, A_3, etc., the partial areas of the cross-section corresponding to the observed velocities V_1, V_2, V_3, etc.;

A the total area of the cross-section $= A_1 + A_2 + A_3$ etc.;

Q_1, Q_2, Q_3, etc., the partial discharges;

Q the total discharge;

v the mean velocity for the entire section.

Then $Q_1 = CV_1A_1$; $Q_2 = CV_2A_2$, etc.;

$$Q = Q_1 + Q_2 + \text{etc.} = C(A_1V_1 + A_2V_2 + \text{etc.});$$

and $$v = \frac{Q}{A} = \frac{C}{A}(A_1V_1 + A_2V_2 + \text{etc.}).$$

It has been here assumed that observations are made at but one point in any vertical plane. The method is the same, however, in any case, it only being necessary to apply such a coefficient to the observed velocity as will reduce it to the mean velocity in its own sub-area. If these partial areas are made small, as in the case of the Boston Conduit, the observed velocities may be taken as the mean velocities in those areas; and if these areas are all equal, which was also the case in this conduit, then the mean velocity is the arithmetic mean of all the observed velocities. The partial and total areas are best found by means of the planimeter, the cross-section having been carefully plotted on coördinate paper.

255. Sub-currents.—It is often desirable to know the direction as well as the velocity of flow beneath the surface. This is of especial importance in surveys for the improvement of the mouths of tidal rivers and the adjacent harbors. For this purpose the Ritchie-Haskell * Direction-Current Meter, Fig. 80, is well adapted.

FIG. 80.

With this meter the observer is enabled to determine, simultaneously, on dials before him, the *direction* and *velocity* of any current.

The Direction Part.—The central chamber of the meter is a compass, whose needle is free to assume the magnetic meridian

* Invented by E. S. Ritchie and E. E. Haskell, the latter of the U. S. Coast and Geodetic Survey.

at all times. This chamber is filled with oil, giving stability to and preventing rust of the needle and other mechanism. An expansion bag compensates changes in temperature and establishes equilibrium between inside and outside of chamber when immersed.

By the use of an electric current, the angle to the nearest degree between the direction of the current and the magnetic needle or meridian can be measured on the dial shown in the cut.

The Velocity Wheel.—This is of the propeller type, conical in form to prevent the catching of weeds, grass or roots. A pitch has been given to the flukes that makes it extremely sensitive to low velocities, and it is said to register accurately as low as 0.20 of a foot per second. The electric connections are made on the inside through the hub of the wheel, so that they cannot be deranged, and are positive in action. It will record on any counting register or a chronograph.

This instrument has been used on the U. S. Coast Survey with very satisfactory results. It would appear to possess advantages over all other forms of current meter, even when used without the direction part.

256. The Flow of Water over Weirs.*—The most accurate mode of measuring the flow through small open channels is by means of weirs. There are three kinds of weirs with which the engineer may have to deal in measuring the flow of water,—namely, sharp-crested weirs, wide-crested weirs, and submerged weirs.

A sharp-crested weir is one which is entirely cleared by the water in passing over it, as in Fig. 81. A wide crest is shown

* The results given in this and the following article have been mostly taken from a paper by Fteley and Stearns before the Am. Soc. Civ. Engrs., vol. xii. (1883), describing experiments made in connection with the new Sudbury River Conduit, Boston, Mass. The paper was awarded the Norman medal of that society.

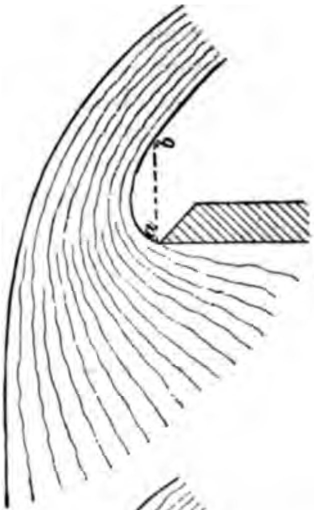

Fig. 81.

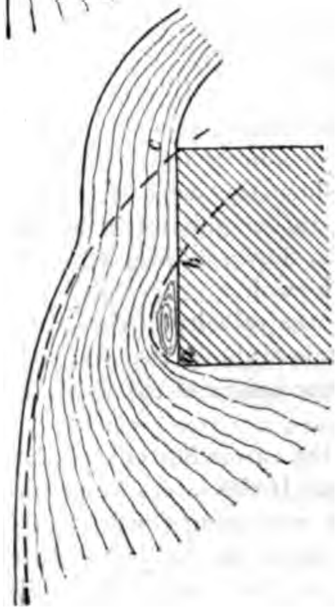

Fig. 82.

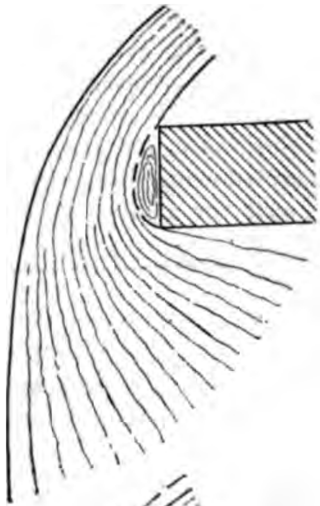

Fig. 83.

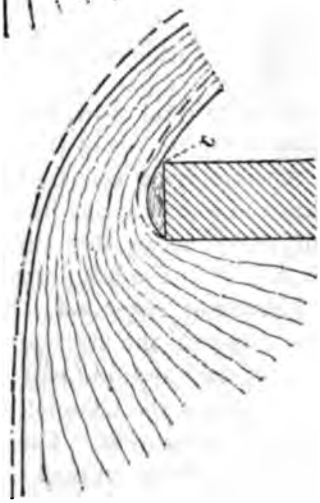

Fig. 84.

in Fig. 82, and its effect in increasing the depth on the weir for a given discharge. If the crest has a width equal to the line *ab* in Fig. 81, then the depth on the weir is unaffected, while if it has a less width, as in Fig. 83, and ***if the air has not free access to the intervening space beneath,*** the water will soon fill this space, and the tendency to vacuum here will depress the overflowing sheet of water, thus diminishing the depth on the weir for a given flow. The dotted lines in Fig. 84 are

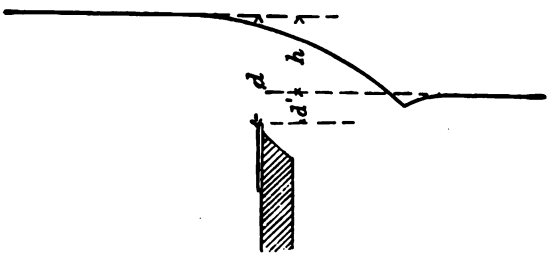

FIG. 85.

those of normal flow, the full lines being the new positions assumed as a result of the partial vacuum below.

A submerged weir is one at which the level of the water below the weir is above its crest, there being, however, a certain definite fall in passing the weir, as shown in Fig. 85. Here $h = d - d'$ is the fall in passing the weir.

Velocity of Approach.—This is the velocity of the surface-water towards the weir at a distance above the weir equal to about two and one half times the height of the weir above the bottom of the channel.

End Contractions.—These are the narrowing effects of the lateral flow at the ends of the weir. If this lateral component of the flow is shut off by a plank extending several feet up stream and from the water's surface to several inches below the top of the weir, then there is no end contraction. This arrangement gives more accurate results, as the correction for end contraction involves some uncertainties.

Depth of Water on the Weir.—This is the principal function of the discharge; it is the difference of elevation between the top of the weir and the surface of the water at a distance above the weir equal to about $2\frac{1}{2}$ times the height of the weir above the bottom of the channel. Evidently this is a quantity which cannot be directly measured. The best way of measuring this quantity is as follows: At a convenient point arrange a closed vertical box which connects by a free opening with the channel at about mid-depth at a point some six feet above the weir. The water will then stand in this box at its normal elevation, unaffected by the slope towards the weir. The elevation of this water-surface is determined by means of a *hook-gauge*, Fig. 86, which consists of a metallic point turned upwards and made adjustable in height by means of a thumb-screw. When the point of the hook comes to the surface of the water it causes a distorted reflection. The elevation of the water-surface can be found in this way with extreme accuracy. The difference of elevation between the point of the hook and the crest of the weir can then be determined with a level and rod. This difference is H in the following formulæ.

FIG. 86.

257. Formulæ and Corrections.—For a simple sharp-crested weir, without end contractions and with no velocity of approach, the discharge in cubic feet per second is

$$Q = 3.31LH^{\frac{3}{2}} + 0.007L, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where L is the length of the weir and H the depth of water upon it, both measured in feet. The weir must have a level crest and vertical ends: it should be in a dam vertical on its up-stream side; the water on the down-stream side may stand even with the crest of the weir if it has considerable depth. The error is not more than one per cent when the water on the down-stream side covers fifteen per cent of the weir area, provided H is then taken as the difference in elevation of the water-surface above and below the weir. In this case two hook gauges would be needed. The crest of the weir should be at a height above the bottom of the channel on the up-stream side equal to at least twice the depth on the weir, to allow for complete vertical contraction.

The following corrections apply to their respective conditions:

For the velocity of approach, the depth on the weir, H in equation (1), is to be increased by $1.5h$, where there is no end contraction, h being the head due to the velocity, or $h = \frac{v^2}{2g}$. At sea-level this correction becomes

$$C = \frac{1.5v^2}{2g} = 0.0234v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is to be added to H in equation (1), v being measured in feet per second.

Where there is end contraction, the correction is

$$C = \frac{2.05v^2}{2g} = 0.032v^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For end contraction, the length of the weir, L in equation (1), is to be shortened by $0.1H$ for each such contraction. This is a mean value, although it varies from $0.07H$ to $0.12H$ for different depths on the weir varying from 1 to 0.3 foot, the smaller correction applying to the greater depth on the weir.

For wide crests the correction to the depth on the weir is sometimes positive and sometimes negative, as shown in figures 82 and 84. The following correction is derived from careful experiments:

$$C = 0.2016\sqrt{y^2 + 0.2146w^2} - 0.1876w, \quad . \quad . \quad . \quad (4)$$

where

C is the correction to be added algebraically to the depth on the wide crest to obtain the depth on a sharp crest which will pass an equal volume of water;
w is the width of the crest;
y is the difference between $0.807w$ and the depth on the crest.

If the crest is narrower than the line *ab*, Fig. 81, then this correction is not to be applied unless the water adheres to the weir as in Fig. 84.

Up-stream edge of the weir rounded. If the up-stream edge of the weir is a small quarter-circle, add seven tenths of its radius to the depth on the weir before applying the general weir formula.

Submerged weir. When the water on the down-stream side rises above the level of the crest, use the formula for a submerged weir, which is

$$Q = cl\left(d + \frac{d'}{2}\right)\sqrt{h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

Q is the discharge in cubic feet per second;
c is to be taken from the following table, its value varying with $\frac{d'}{d}$;
l is the length of the weir in feet;

d is the depth on the weir in feet, measured from still water on the up-stream side;
d' is the depth to which the weir is submerged, measured from still water on the down-stream side;
h is the fall and equals $d - d'$.

The value of d may be corrected for velocity of approach by formulas (2) and (3). There is no known correction for the velocity of discharge below the weir, and hence the formula can only be used for a channel of large capacity below, as compared with the discharge, so that the velocity here will be small

The following are the experimental values of c:

$\frac{d'}{d}$.	c.	$\frac{d'}{d}$.	c.	$\frac{d'}{d}$.	c.	$\frac{d'}{d}$.	c.
0.01	3.330	0.25	3.249	0.55	3.100	0.85	3.150
.05	3.360	.30	3.214	.60	3.092	.90	3.190
.08	3.372	.35	3.182	.65	3.089	.95	3.247
.10	3.365	.40	3.155	.70	3.092	1.00	3.360
.15	3.327	.45	3.131	.75	3.102		
.20	3.286	.50	3.113	.80	3.122		

This table is inapplicable to values of $\frac{d'}{d}$ less than 0.08, unless the air has free access to the space underneath the sheet.

The method of measuring discharge by means of submerged weirs is adapted to channels having very small slope. A fall as low as one half inch will give reliable results if it is accurately measured.

258. The Miner's Inch.—This is an arbitrary standard both as to method and as to volume of water discharged. It rests on the false assumption that the volume discharged is proportional to the area of the orifice under a constant head above the top of the orifice. Its use grew out of the necessities of frontier life in the mining regions of the West, and should now be discarded in favor of absolute units. The miner's inch is the quantity of water that will flow through an orifice one inch square, under a head of from four to twelve inches, according to geographical locality. Even if the head above the top of the orifice be fixed, and a flow of 144 miner's inches be required, the volume obtained would be 3.3, 4.2, or 4.7 cubic feet per second, according as there were 144 holes each one inch square, one opening one inch deep and 144 inches long, or one opening twelve inches square, the tops of all the openings being five inches below the surface of the water. This simply illustrates the unreliable nature of such a unit. In some localities the following standard has been adopted: An aperture twelve inches high by twelve and three-quarter inches wide through one one- and one-half-inch plank, with top of opening six inches below the water-surface, is said to discharge two hundred miner's inches. By this standard the miner's inch is 1.5 cubic feet per minute, or 2160 cubic feet in twenty-four hours. Other standards vary from 1.39 to 1.78 cubic feet per minute.* When the miner's inch can only be defined as a certain number of cubic feet per minute, it is evidently no longer of service and should be abandoned. The method by weirs is more accurate, and could almost always be substituted for the method by orifices.

259. The Flow of Water in Open Channels.—For more than a century hydraulic engineers have labored to find a fixed relation between the slope and cross-section of a running stream and the resulting mean velocity. If such a relation could be found, then the discharge of any stream could be obtained at a minimum cost. It is now known that there is no such fixed relation. There certainly is a relation between the bed of a stream for a considerable distance above and below the section,

* See Bowie's "Hydraulic Mining," p. 126 (John Wiley & Sons, New York).

the surface slope, and the resulting velocity at the section; but as no two streams have similar beds, nor the same stream in different portions of its length, and since the bed characteristics are difficult to determine, and, furthermore, are constantly changing in channels in earth, the function of bed cannot be incorporated into a formula to any advantage except for channels of strictly uniform and constant bed, in which case the cross-section would sufficiently indicate the bed. Again, the slope cannot be profitably introduced into a velocity formula except where it is uniform for a considerable distance above and below the section, for the inertia of the water tends to produce uniform motion under varying slopes, and the effect is that the velocity at no point corresponds strictly to the slope across that section. For uniform bed and slope, however, formulæ may be often used to advantage.

Let A = area of cross-section;
v = velocity in feet per second (= l for one second);
p = wetted perimeter;
r = hydraulic mean radius $= \dfrac{A}{p}$;
s = surface-slope $= \sin i = \dfrac{Z}{l}$;
Z = fall per length l;
Q = quantity discharged in one second;
S = wetted surface in length $l = pl$;
f = coefficient of friction per unit area of S;
ρ = weight of one cubic foot of water = density.

Since the friction varies directly as the density and as the square of the velocity, we have for the frictional resistance on the mass covering the area S,

$$R = f\rho Sv^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the work spent in overcoming this resistance in one second of time is

$$K = Rv = f\rho Sv^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the velocity is constant, which it is assumed to be, then this is also the measure of the work gravity does on this mass of water in pulling it through the height $h' - h'' = Z$, which work is

$$K = \text{weight} \times \text{fall} = Z\rho Q = Z\rho vA\,; \quad . \quad . \quad . \quad (3)$$

$$\therefore Z\rho vA = f\rho Sv^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$Z = \frac{fS}{A}v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But $S = pl$; $\quad \dfrac{A}{p} = r$; $\quad$ and $\quad \dfrac{Z}{l} = \sin i = s$;

$$\therefore s = \frac{fv^2}{r} \quad \text{or} \quad v = c\sqrt{rs}, \quad . \quad . \quad . \quad . \quad (6)$$

where c is an empirical coefficient to be determined. It is evident that c is mostly a function of the character of the bed, and that it can, therefore, have no fixed value for all cases.

Equation (6) is what is known as the Chezy formula. The most successful attempt yet made to give to the coefficient c a value suitable to all cases of constant flow is that of Kutter.* Kutter's formula, when reduced to English foot-units, is

* Kutter's Hydraulic Tables, translated from the German by Jackson, and published by Spon, London, 1876. A revised and enlarged edition has now (1890) been edited by Rudolph Hering and J. C. Trautwine, Jr., and published by John Wiley & Son, New York.

$$v = c\sqrt{rs} = \left[\frac{41.6 + \frac{1.811}{n} + \frac{0.00281}{s}}{1 + \left(41.6 + \frac{0.00281}{s}\right)\frac{n}{\sqrt{r}}}\right]\sqrt{rs}, \quad . \quad . \quad (7)$$

the total coefficient of the radical, in brackets, being the evaluation of c in equation (6). Here v, r, and s are the same as before, and n is a " natural coefficient " dependent on the nature of the soil, character of bed, banks, etc. Although it was the author's intention to make a formula that would be applicable even to natural channels, it cannot safely be applied to such unless they have great uniformity of bed and slope.

The following values of n are given by Kutter:

Planed plank,	$n = 0.008$.
Pure cement,	$n = .009$.
Sand and cement,	$n = .010$ to $.011$.
Brickwork and ashlar,	$n = .012$ " $.014$.
Canvas lining,	$n = .015$.
Average rubble,	$n = .017$.
Rammed gravel,	$n = .020$.
In earth—canals and ditches,	$n = .020$ to $.030$, depending on the regularity of the cross-section, freedom from weeds, etc.
In earth of irregular cross-section,	$n = .030$ to $.040$.
For torrential streams,	$n = .050$.

In the last two cases the results are very uncertain. Kutter's tables are evaluated for $n = 0.025$, $.030$, and $.035$.

The greatest objection to the use of this formula is the labor involved in evaluating the "c" coefficient. To facilitate the use of the formula this coefficient has been evaluated for a slope of 0.001 in Table* IX. This coefficient changes so slowly with a change in slope that the error does not exceed $3\frac{1}{2}$ per cent if the table be used for all slopes from one in ten to one in 5280, which is a foot in a mile. These tabular coefficients may therefore be used in all cases of ditches, pipe lines, sewers, etc. The coefficients are seen to change rapidly for different values of n, so this value must be chosen with care.

For brick conduits, such as are used for water-supply and for sewers, the formula

$$v = 127r^{0.62}s^{0.5}$$

was found to represent the experiments on the Boston conduit, shown in Figs. 78 and 79. This would correspond to a variable value of n in Kutter's formula, being nearly 0.012 however, as given for brickwork. This conduit is brick-lined. Table X.* gives maximum discharges of such conduits as computed by Kutter's formula, n being taken as 0.013. The results in heavy type include the working part of the table for sewers. All values above the heavy-faced type correspond to velocities less than three feet per second when the depth of water is one eighth of the diameter, or when the flow is one fiftieth the maximum. This is as small a velocity as is consistent with a self-cleansing flow in sewers. All values below the heavy-faced type correspond to velocities more than fifteen feet per second when the conduit runs full, and this is as great a velocity as is consistent with safety to the structure. If the velocity is greater than this, the conduit should be lined with stone.

* Taken from a paper by Robt. Moore and Julius Baier in *Journal of the Association of Engineering Societies*, vol. v., p. 349. This table may also be used for tile drains.

The maximum flow does not occur when the conduit runs full, but when the depth is about 93 per cent of the diameter. A conduit or pipe will therefore not run full except under considerable pressure or head. The maximum velocity occurs when the depth is about 81 per cent of the diameter.

The relative mean velocities and discharges of a circular conduit for varying depths is shown by the following table:

Depth of Water.	Relative Velocity.	Relative Discharge.	Depth of Water.	Relative Velocity.	Relative Discharge.
.1	.28	.016	.7	.98	.776
.2	.48	.072	.75	.99	.850
.25	.57	.118	.8	.99	.912
.3	.64	.168	.81	1.00	.924
.4	.76	.302	.9	.98	.992
.5	.86	.450	.93	.96	1.000
.6	.93	.620	1.00	.86	.916

260. Cross-sections of Least Resistance.—From equation (6) of the preceding article it is apparent that for a given channel the velocity varies as the square-root of the hydraulic mean radius, r. But $r = \frac{A}{p}$, hence for a given area of cross-section the velocity is greater as the wetted perimeter is less. The form of cross-section having a minimum perimeter for a given area is the circular, or for an open channel the semicircular. In both cases the hydraulic mean radius is $r = \frac{\pi R^2}{2\pi R} = \frac{R}{2}$, where R is the radius of the circle. Since it is not always convenient to make the cross-section circular in the case of ditches and canals, it is evident that the more nearly a polygonal cross-section coincides with the circular form the less will be the resistance to flow. When a maximum flow is desired for a given slope and cross-section, therefore, the shape should conform as nearly as possible to that of a semicircle. To do this, construct a semicircle to scale of the required area of cross-section. Draw tangents for the sides of the section having the desired slope and join these by another tangent line at bottom, as in Fig. 87. This gives a little larger sectional area, but some allowance should be made for accumulations in the channel. If the slope is very great and it is desirable to reduce the velocity of flow, it may be done by making the channel wide and shallow.

FIG. 87.

261. Sediment-observations.—It is often necessary in surveys of sediment-bearing streams to determine the amount of silt carried by the water in suspension. The work consists of three operations, namely: (1) obtaining the samples of water; (2) weighing or measuring out a specific portion of each, mixing these in sample jars according to some system, and setting away to settle; (3) siphoning off the clear water, filtering, and weighing the sediment. Sometimes a fourth operation is required, which is to examine the sediment by a microscope on a graduated glass plate, and estimate the percentages of different-sized grains. The sedimentary matter carried in suspension may be divided into two general classes,—that in continuous suspension, and that in discontinuous suspension. The former is composed of very fine particles of clay and mud whose specific gravity is about unity, so that any slight disturbance of the water will prevent its deposition. This once taken up by a running stream is carried to its mouth or caught in stagnant places by the way. The matter in discontinuous suspension consists of sand, more or less fine according to the velocity and agitation of the current. This matter is constantly falling towards the bottom and is only prevented by the violent motions of the medium in which they are suspended. These particles are constantly being picked up where the velocity is greater, and dropped again where the velocity is less. A natural channel will therefore carry about the same percentage of fine or continuous matter between two consecutive tributaries, but of the coarser material there will be no uniformity whatever in successive sections in this same stretch of river. In natural channels there are always alternate engorged and enlarged sections for any particular stage of river, and the positions of these engorgements and enlargements are different for different stages. In fact, the engorged sections at high water are usually the enlarged sections at low water, and *vice versa.* If the bed is friable the engorged section is always enlarging, and the enlarged section is constantly filling as a result of the discontinuous movement of sedimentary matter. The cause of these relative changes of position of engorged and enlarged sections is the great variation in width.*

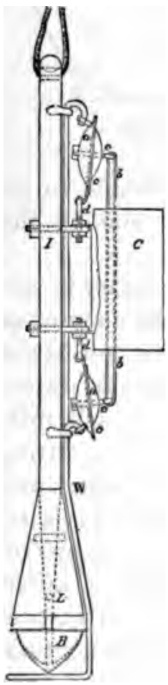

FIG. 88.

It is the discontinuous sediment which is of principal significance to the engineer, for this is the material from which sand-bars are formed which obstruct navigation, and it is also the material from which he builds his great contraction works behind his permeable dikes. The water being partially checked behind these dikes at once drops the heavier sediment, and so artificial banks are rapidly formed. The continuous sediment is of little consequence to the engineer.

* See paper by the author entitled "Three Problems in River Physics," before the American Association for the Advancement of Science, Philadelphia meeting, 1884.

262. Collecting the Specimens of Water.—It is necessary to take samples of water from various parts of the cross-section in order to obtain a fair average. Surface and bottom specimens should always be taken, and if great exactness is required specimens should also be taken at mid-depth. One of each of these should be taken at two or three points on the cross-section. A full set of specimens is collected once or twice a day. A special apparatus is required for obtaining samples from points beneath the surface. The requirements of such an apparatus are very well satisfied by the device shown in Fig. 88, which the author designed and used very successfully in a hydrographic survey of the Mississippi River at Helena, Ark., in 1879.* *C* is a galvanized iron or copper cup; *I* an iron bar one inch square; *L* a mass of lead moulded on the bar at bottom; *B* the bottom cup for bringing to the surface a specimen of the bottom, *l* being a leather cover; *W* the springing wire by which the lids *a a* are released and drawn together by the rubber bands *b b* when the apparatus strikes the bottom, or when this wire is pulled by an auxiliary cord from above; *d d* adjustable hinges allowing a tight joint on the rubber packing-disks *c c* when the lids are closed. In descending, the lids are open and the water in the can *C* is always a fair sample of the water surrounding the apparatus. When the lids are closed the sample is brought securely to the surface. The can when closed should be practically water-tight; if it leaks at bottom some of the heavier sediment is likely to escape, for it settles very quickly. The bottom specimen should be taken about a foot above the bottom to avoid getting an undue portion of sand which is at once stirred up by the apparatus striking the bottom.

263. Measuring out the Samples.—A given portion of each specimen by measure or by weight is selected for deposition.

* See Report of Chief of Engrs., U. S. A., 1879, vol. iii., p. 1963.

Great care must be exercised in obtaining the sample volume. It cannot be poured off, even after violent shaking, for the heavy sand falls rapidly to the bottom. A good way is to draw it from the vessel by an aperture in its side while the water is stirred within; greater accuracy can be attained by weighing the sample of water than by measuring it. All the samples of a given kind are then put together in one jar, which is properly labelled, and set away to settle. Thus, all the surface samples are put into one jar, the mid-depth samples in another. The Mississippi and the Missouri River water requires about ten days' settling to become clear.

264. Siphoning off, Filtering, and Weighing the Sediment.—When the water has become quite clear it is carefully siphoned off, and the residue is filtered through fine filter paper (Munktell's is best). Two papers are cut and made of exactly the same weight. One is used for filtering and the duplicate laid aside. The filter-paper containing the sediment and also its duplicate are then dried in an oven at a temperature not higher than 180°. When quite dry the sediment paper is put in one pan of the balance, and the duplicate in the other and weights added to balance. The sum of the weights is the weight of the sediment. This divided by the weight of the sample of water, usually expressed by a vulgar fraction whose numerator is one, is the proportionate quantity sought.

CHAPTER XI.

MINING SURVEYING.*

265. Definitions.—Mining Surveying, like all other classes of surveying, has for its object the determination of the relative positions of the different portions of the subject of the survey. The same principles which are employed in surveying on the surface govern the engineer in the prosecution of a mining survey. In fact, mining surveying may be considered as an extension of topographical surveying to the accessible portions beneath the surface of the earth, with certain modifications of the adjuncts of surface surveying, necessitated by the nature of the case.

The parts of a mine included in a mining survey are the surface and surface-workings, shafts, tunnels, inclines or slopes, drifts, stopes, winzes, cross-cuts, levels, air-courses, entries, and chambers.

Surface-workings include open cuts, pits, and other excavations of limited extent.

A Shaft is a pit sunk from the surface more or less perpendicularly on the vein or to cut the vein. The inclination of the vein is called the Dip, or Pitch, and its direction across the country is called the Strike.

A Tunnel is a horizontal excavation from the surface along the course of a vein, or across the course of known veins, for purposes of discovery. The approach to a tunnel is called an Adit.

An Incline or Slope is a tunnel run at an angle to the horizontal.

* This chapter written by C. A. Russell, C.E., U. S. Mineral Surveyor of Boulder, Col.

A Drift is a tunnel starting from an underground working such as a shaft. When there are a series of drifts at different depths, they are termed Levels; as first level, second level, or 50-foot level, 100-foot level, etc.

A Stope is the working above or below a level from which the ore is extracted. An overhand or back stope is the working above a level; an underhand stope is the working from the floor.

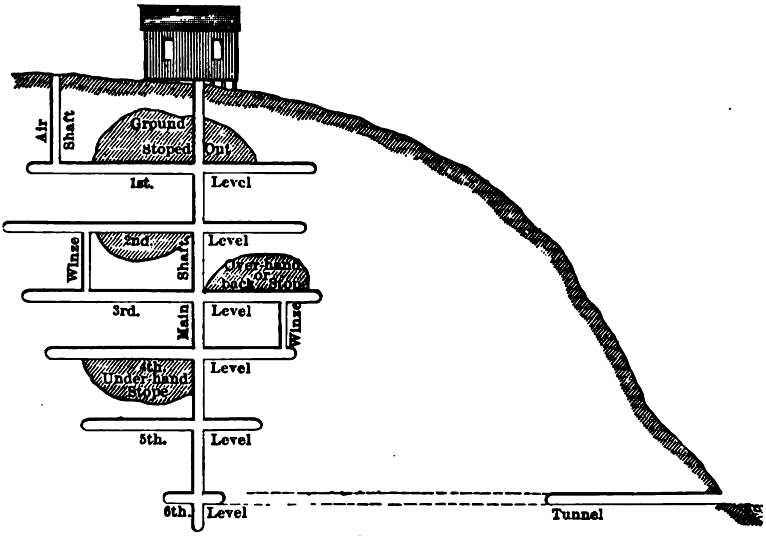

FIG. 89.

A Winze is a shaft sunk from a level.

A Cross-cut is a level driven across the course of a vein.

An Air-course is a tunnel driven for the purpose of ventilation.

An Entry is a passage through a mine.

A Chamber is a large room from which the ore is mined.

The last two terms are used more especially in coal-mines, where the vein lies flat or nearly so.

The operations of a mining survey are conducted like those of a topographical survey. An initial point is selected usually from its importance to the object sought, and all the subsequent stations are connected either directly or indirectly with it, and their positions with reference to it shown on the map of the survey.

266. Stations are occupied by candles or lamps constructed for the purpose, in place of poles, flags, etc., as on the surface. An illuminated plumb-line is a good substitute for a lamp, and gives the observer a greater vertical range, which is helpful in case the station is obscured by intervening objects.

Owing to the peculiar nature of the survey, it is impracticable and sometimes inexpedient to mark stations as on the surface; recourse is therefore had to other devices, which must be employed to suit circumstances.

It would not be advisable, even if it were practicable, to leave a station-mark on the bottom of any portion of a mine, as frequent passing would disturb or obliterate it.

It is better therefore to leave the mark overhead, if accessible and not liable to be disturbed, either by driving a nail in a timber should one be convenient, or by drilling a hole in which may be inserted a wooden plug, properly marked, or simply by cutting a cross or other device on the exposed surface. Another method, where the above cannot be employed, is by marking points on the walls and measuring the respective distances from the station to them.

267. Instruments.—*Steel tapes* only should be used for measuring, being more convenient and less liable to inaccuracy than a chain. These may be of different lengths to suit the work on which they are to be employed; sometimes, as in the case of coal-mine surveys, tapes of several hundred feet in length can be employed to advantage.

The Compass, unless used as an angular instrument to deflect from an established line, should not be employed in mining surveys, as the variation between stations is so inconstant as to render it unreliable when used to deflect from the magnetic meridian. The magnetic needle may be used, however, in connection with the transit as a check.

The Transit alone should be used in important work, and certain additions to it for vertical pointings will be found indispensable. In sighting up or down a shaft the ordinary form becomes useless when the line of sight passes inside the upper plate of the instrument. A prismatic eye-piece, Fig. 90, will overcome this difficulty for upward sights, but the survey cannot be carried downward by its use.

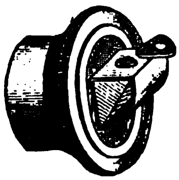

FIG. 90.

An extra telescope attached either to the top or side of the central telescope will overcome this difficulty. The attachment to the top is made as shown in Fig. 91 by coupling-nuts,

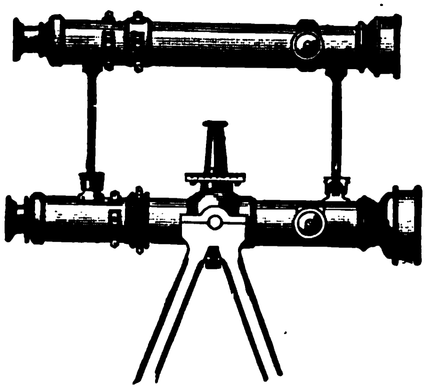

FIG. 91.

which fasten it firmly over the centre of the instrument. The attachment to the side, Fig. 92, is effected by means of a spindle from the attached telescope which fits into the hollow axis of the central telescope and is secured by means of a clip which passes through both the axis and spindle. A counterpoise is similarly attached to the opposite side of the central telescope to preserve the equilibrium.

This latter form of attachment is more compact than the former, the principal objection to it being that a correction must be applied to each reading of a horizontal angle equal to the tangent of an angle whose opposite side is the distance between the centres of the telescopes, and whose adjacent side is the horizontal distance between stations. This objection, how-

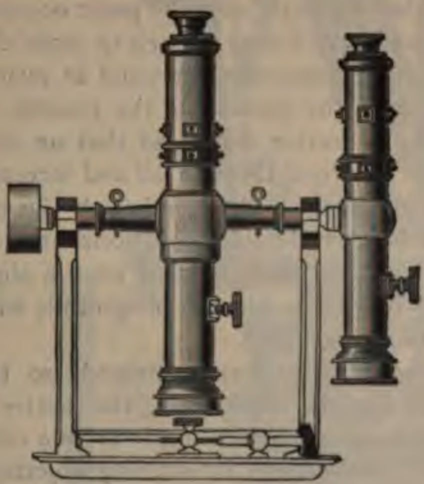

FIG. 92.

FIG. 93.

ever, is removed by a simple device. Two brass tubes, Fig. 93, about two inches long, are connected by an intermediate web, so that the distance between their centres shall exactly equal that between the centres of the telescopes. One of the tubes is of sufficient size to enclose a pike-staff graduated to fractions of a foot, upon which it can be easily moved to any desired height, and the other large enough to contain a candle, and has a light plumb-bob suspended below its centre the better to maintain the staff in a perpendicular position; the staff now being placed over any station and the brass tube and candle set at the height of the instrument on the staff, which is held in a perpendicular position with the line between the tubes parallel to the horizontal axis of the telescope, a reading can be made to the flame of the candle which gives at once the true azimuth of the line and the dip of the shaft.

In using this device the side telescope and tube carrying the candle should always be on the same side of the line. The transit must always be placed exactly over the point occupied by the foot of the staff; and here it may be well to state that the greatest care and accuracy must be exercised in exactly centring the instrument over the station, as the courses are carried forward entirely by deflection angles, so that an error introduced at one station is carried through all and increased at each. Again, in sighting down a shaft, although the perpendicular distance may be considerable, the horizontal distance between stations must be small, so that even a slight error made in a shaft will be of considerable magnitude when carried out in the levels of a large mine.

The transit should have an extension tripod, so that one or more of the legs can be shortened, the better to place it over a station on a steep mountain-side or in a mine, or to lower the instrument to see under intervening objects, or to adapt it to different heights of the workings of the mine.

The Mining Transit should be provided with the Solar Attachment, that all lines of the survey may be referred to the meridian.

In unimportant surveys the pitch of the shaft or the dip of the vein may be determined by the clinometer or by measuring the horizontal and perpendicular distances between any two conveniently located points of the foot or hanging wall and calculating the pitch or dip from the measurements thus obtained.

A plummet-lamp will also be found very convenient.

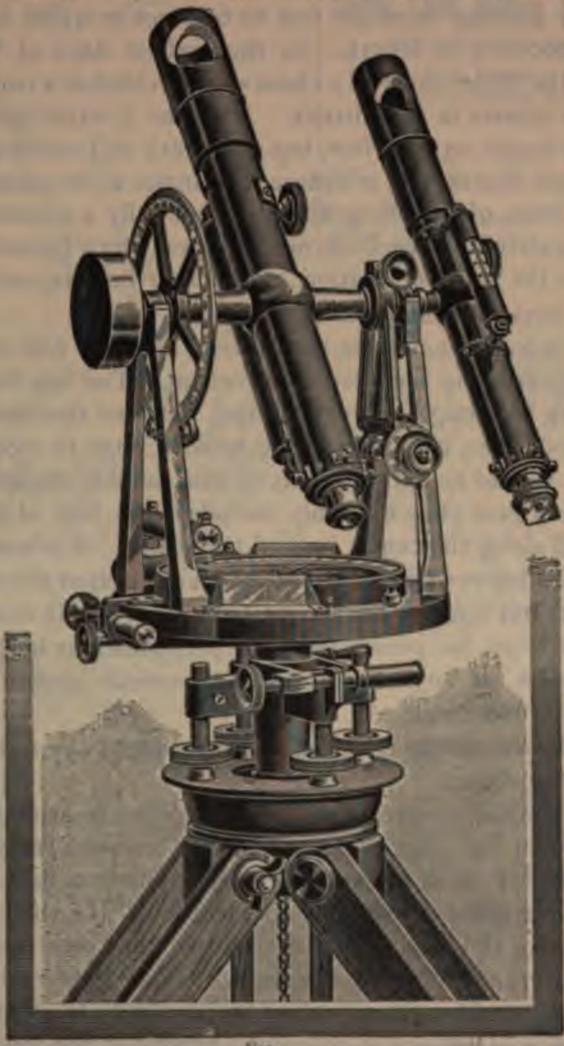

FIG. 94.

268. Mining Claims.—The first work of the surveyor upon a mining claim is its location. Mining claims are of different dimensions according to the local laws and customs of the country; varying from 50 feet to 600 feet in width and from 100 to 3000 feet in length. In the earliest days of Western mining, the dimensions of a claim were decided at a convention of all the miners in the district. Now the United States laws limit the length to 1500 feet, but the width still varies not only in different States, but in different counties in the same State.

The form of a mining claim is essentially a parallelogram, being regulated by the U. S. mining laws, which prescribe that whatever the relative position of the side lines to each other, the end lines must be parallel.

This is to prevent more than fifteen hundred feet of a vein or lode from being included in one claim. The side lines of a claim may be straight lines extending between the ends of opposite end lines, or they may be broken lines to include the vein if it should be curved, so as to pass outside straight lines; but in any case they can only include 1500 feet of the vein measured along the centre line of the claim. A mining claim is included between parallel vertical planes passed through the end lines; but a miner has a right to follow his vein downward, although it so far passes from the perpendicular in its downward course as to extend beyond vertical planes passed through its side lines.

The above are the essential features which govern the shape of a mining claim.

The method of procedure in making the location is as follows: When the discoverer of a mine has sunk his shaft according to law, so as to expose 10 feet of the vein, he is entitled to have his claim surveyed and recorded. He then decides how much of the 1500 feet he desires to extend on either side of his discovery-shaft along the vein. He is governed in this by various considerations, such as his proximity to other claims, the promise of mineral in different portions of the lode, or the nature of the ground. The surveyor begins his survey for

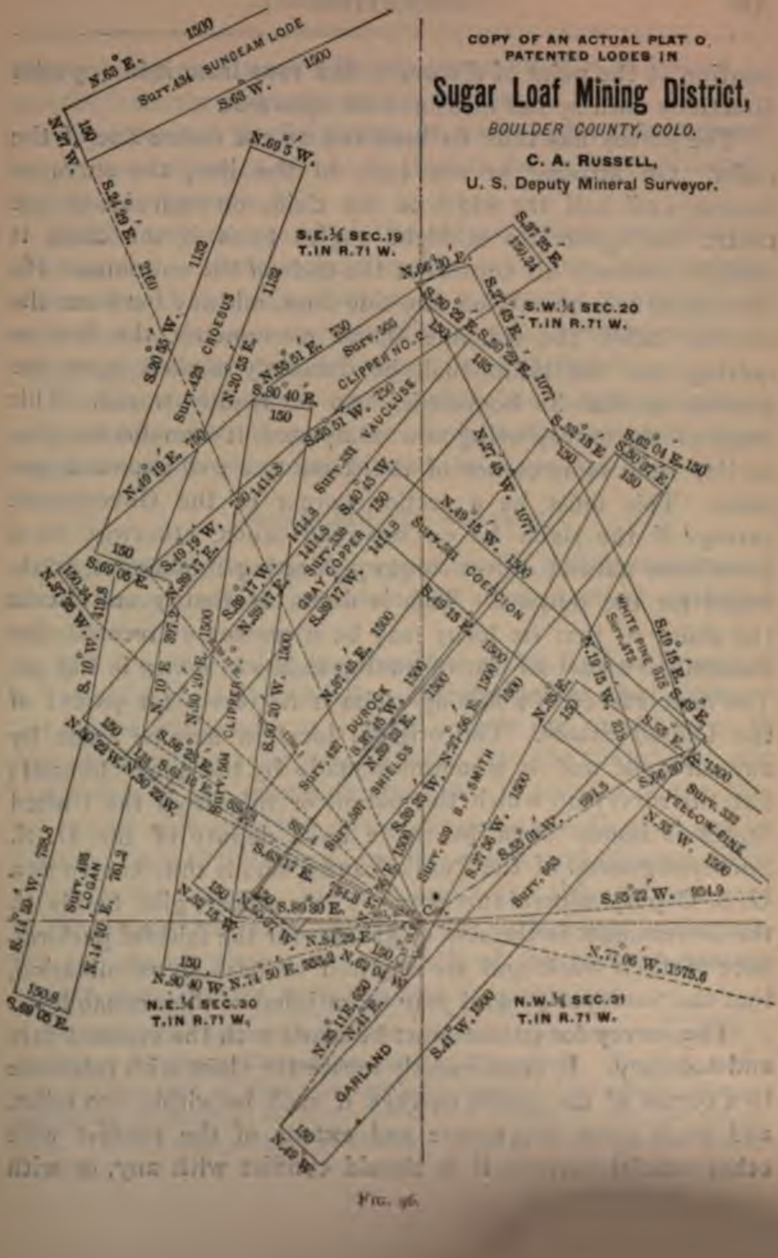
COPY OF AN ACTUAL PLAT O
PATENTED LODES IN
Sugar Loaf Mining District,
BOULDER COUNTY, COLO.
C. A. RUSSELL,
U. S. Deputy Mineral Surveyor.
S.E.¼ SEC.19
T.1N R.71 W.
S.W.¼ SEC.20
T.1N R.71 W.
N.E.¼ SEC.30
T.1N R.71 W.
N.W.¼ SEC.31
T.1N R.71 W.

Fig. 96.

location at the point of discovery, and runs from it in opposite directions until he has measured off 1500 feet.

The survey has thus far been run on the centre line of the claim. On arriving at the ends of the line, the surveyor measures off half the width of the claim on each side of the centre line, generally at right angles to it if the claim is straight, and sets his corners at the ends of the end lines. He also places monuments on the side lines, midway between the corners, called the side-line centre monuments, the law requiring that the claim shall be distinctly marked upon the ground, so that its boundaries can be readily traced. This much of the survey being now completed, it remains to run a tie line from some corner of the claim to a well-known monument. This must be a section corner of the Government surveys if the claim be on surveyed lands, otherwise to a prominent natural object, or to a locating monument established for the purpose. This is done to identify and locate the claim so that its locus may be a matter of record. An example of an actual plat of mining claims is shown in Fig. 96. The next survey of a mining claim is its survey for patent of the United States. The original location may be made by any surveyor, and is sometimes made by the miner himself; but the survey on which the patent or title from the United States is issued must be made by a deputy of the U. S. Surveyor-general of the Public Lands, who is thus known as a U. S. Deputy-mineral-surveyor. These officers give bonds to the Government in the sum of $10,000 for the faithful performance of their work, and are required to pass an examination, that the Surveyor-general may be satisfied of their capability.

The survey for patent must be made with the greatest care and accuracy. It must exactly locate the claim with reference to a corner of the public surveys, if such be within two miles, and must show the nature and extent of the conflict with other official surveys if it should conflict with any, or with other mining claims not officially surveyed if it is desired to exclude from the claim the area in conflict.

A specimen of the field-notes of a survey for patent issued for the instruction of the U. S. Deputy-surveyors of Colorado is given in Appendix B.

The Surveyors-general of the different States and Territories issue instructions to their deputies, and these, with a knowledge of the U. S. mining laws, must govern the surveyor in his work; but as they are more strictly legal than mathematical, it is not important to consider them in this chapter.*

The foregoing surveys are strictly land surveys, and are only mentioned to illustrate the method of staking out a mining claim and to give some idea of the shape and size.

UNDERGROUND SURVEYS.

269. **Mining Surveying proper**, or the underground work of the survey, will be considered in a few practical examples selected from actual cases.

270. **To determine the Position of the End or Breast of a Tunnel and its Depth below the Surface at that Point.**—Set up the instrument at a point outside the tunnel, so as to command as long a sight as possible into the tunnel and also the surface of the mountain above it. If the end of the tunnel can be seen from the station a course and distance can be taken at once to the breast, and this course and distance duplicated on the surface. Vertical angles can then be measured to the points thus determined on the surface and in the tunnel, and the calculation of the depth of the breast below the surface may be made from the data thus obtained.

* Copies of the Instructions can be procured of any Surveyor-general on application. Those for Colorado are given in Appendix B. The U. S. mining laws, together with all State and Territorial laws and local mining rules and regulations are compiled in vol. xiv. of the U. S. Census for 1880, 4to, 705 pp., 1885. This is a most valuable publication. Price $4 if not obtained through an M. C.

In case the breast is not visible from the first station, take as long a sight as practicable to Station No. 2, and before removing the instrument reproduce Station No. 2 upon the sur-

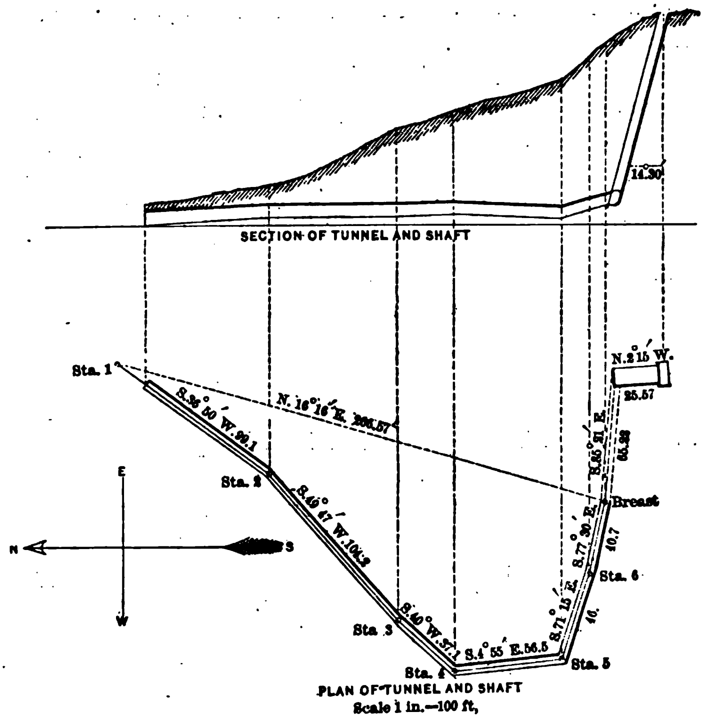

Fig. 97.

face as in the preceding case; thus avoiding a resetting of the instrument at Station No. 1 when the underground work is completed. At the same time measure the vertical angle to Station No. 2 in the tunnel. Set the instrument at Station No. 2, and, after having obtained the back readings to Station No. 1, measure the course, distance, and vertical angle to Station No. 3.

Repeat the above operations at the different stations until the breast is reached, taking any measurements of the dimensions of the work that may be necessary, and leaving station marks for future reference, as described in article 266. Set the instrument over Station No. 2 on the surface and very carefully duplicate the courses and distances measured in the tunnel, at the same time noting the vertical angles between the surface stations. The vertical angles can be measured most easily by sighting to a point on a short staff at a height above the station equal to the height of the instrument.

It is advisable to explore the tunnel before surveying it, as then any difficulties can be provided for and the stations selected more advantageously. Sometimes the course from Station No. 1 to Station No. 2 is assumed as a meridian of the survey and all courses deflected from it, but it is better to use the true solar course between these stations because the field notes can then be placed in the table for calculation without further reduction.

EXAMPLE.—Following is a specimen of field-notes of the survey of a tunnel both underground and surface:

FIELD-NOTES.

STATION.	VERTICAL ANGLES		Course.	Distance.	Remarks.
	in Tunnel.	on Surface.			
1	....	....	S. 36° 50′ W.	19.1 ft.	to mouth of tunnel.
	+ 1° 18′	+10° 35′	S. 36° 50′ W.	99.1	to Sta. No. 2.
2	+ 0° 31′	+15° 43′	S. 49° 47′ W.	104.2	" " 3.
3	+ 0° 45′	+14° 27′	S. 40° 0′ W.	37.1	" " 4.
4	− 0° 34′	+16° 17′	S. 4° 55′ E.	56.5	" " 5.
5	+ 3° 37′	+12° 21′	S. 71° 15′ E.	46.0	" " 6.
6	+ 3° 30′	+13° 56′	S. 77° 30′ E.	40.7	to breast of tunnel.
Breast.		−19° 26′	N. 16° 16′ E.	266.57	from station on surface over breast of tunnel to Sta. No. 1.

The following table shows the method of reducing the survey. The first six columns represent the ordinary method of reducing a traverse to a straight line. The agreement between the resultant and the check course proves the accuracy of the field-work.

Columns 7 and 8 contain the vertical angles in the tunnel and the rise or fall in feet corresponding to them.

Columns 9 and 10 similarly contain the vertical angles of the courses and distances on the surface and the difference of elevations between stations corresponding to them.

The algebraic sum of the vertical heights in the tunnel gives the difference of level between the Station No. 1 and the breast; and the sum of the differences of elevation in column 10 gives the total difference of elevation between Station No. 1 and the point on the surface over the breast.

The difference of columns 8 and 10 shows the depth of the breast of the tunnel below the surface. The tangent of the angle obtained by dividing the sum of the elevations in column 10 by the length of the resultant distance should agree with the vertical angle read to Station No. 1 from the point on the surface over the breast of the tunnel.

The following is the form used in reducing the field-notes:

OFFICE FORM.

Course.	Dist.	Latitude.		Departure.		Vert. Angle.	Rise or Fall.	Vert. Angle.	Rise or Fall.
		N.	S.	E.	W.				
S. 36° 50′ W.	99.1	......	79.32	......	59.41	+ 1° 18′	+ 2.24	+ 10° 35′	+ 18.51
S. 49° 47′ W.	104.2	......	67.28	......	79.57	+ 0° 31′	+ 0.94	+ 15° 43′	+ 29.32
S. 40° 00′ W.	37.1	......	28.42	......	23.84	+ 0° 45′	+ 0.48	+ 14° 27′	+ 9.56
S. 4° 55′ E.	56.5	......	56.29	4.84	......	− 0° 34′	− 0.56	+ 16° 17′	+ 16.50
S. 71° 15′ E.	46	......	14.78	43.56	......	+ 3° 37′	+ 2.91	+ 12° 21′	+ 10.07
S. 77° 30′ E.	40.7	......	8.81	39.73	......	+ 3° 30′	+ 2.49	+ 13° 56′	+ 10.10
Resultant course, N. 16° 20′ E.	265.62	254.90	......	74.69	......	Total	+ 8.50	Total	+ 94.06
		254.90	254.90	162.82	162.82				8.50
								Depth below surface =	85.56

Check. 94.06 ÷ 265.62 = 0.3541 = tan 19° 30′.

271. Required, the Distance that a Tunnel will have to be driven to cut a Vein with a Certain Dip.—Case I. *When the direction of the tunnel is at right angles to the course, or parallel to the pitch of the vein.* The dip having been first ascertained by sighting down a shaft sunk on the vein, or by any other practicable method, set up the instrument on the apex or outcrop of the vein directly over the line of the proposed tunnel and measure the vertical angle and horizontal distance to the mouth of the tunnel.

From the results obtained calculate the depth at which the tunnel will intersect the vein, then from this depth and the angle of the dip, calculate the horizontal distance of the vein from a vertical line through the instrument station.

This distance, added to or subtracted from the horizontal distance between the station and mouth of the tunnel, according as the dip is from or toward the mouth, will give the required distance.

FIG. 98.

EXAMPLE.—A certain vein has a course of N. 45° E. and its pitch is N. 45° W. with a dip of 15° from a vertical line. The horizontal distance to the mouth of a cross-cut tunnel from the apex of the vein is 200 feet S. 45° E. and the vertical angle is — 25°.

At what distance from the mouth will the tunnel intersect the vein? (Fig. 98.)

Depth at which the tunnel will intersect the vein = 200 × tan. 25° = 93.26 ft.

Distance of vein from vertical line at depth of 93.26 ft. = 93.26 × tan. 15° = 24.99 ft.

Adding the last result to the horizontal measured distance, we have 24.99 + 200 = 224.99 ft., the distance from the mouth of the tunnel to its intersection with the vein.

CASE II. *When the direction of the tunnel is oblique to the course of the vein.* Proceed as in Case I. and measure the horizontal distance from the instrument station to the mouth of the tunnel, the vertical angle and the dip, also the angle which the course of the vein and the line of the tunnel make with each other. Calculate the depth at which the tunnel will intersect the vein, and the distance of the vein at the tunnel level from a vertical line through the station, as in the previous case. Multiply this distance by the cosecant of the angle between the courses of the vein and tunnel and apply it to the measured horizontal distance, as in Case I., and we have the required result.

EXAMPLE.—A certain vein has a course of N. 45° E. and its pitch is N. 45° W. with a dip of 15° from a vertical line. The horizontal distance to the mouth of a cross-cut tunnel running due west is 200 ft. due east and the vertical angle is −25°.

At what distance from the mouth will the tunnel intersect the vein? (Fig. 99.)

Depth at which the tunnel will intersect the vein = 200 × tan. 25° = 93.26 ft.

Distance of vein from vertical line at depth of 93.26 ft. = 24.99 ft.

Angle between course of vein and line of tunnel = 45°.

Multiplying, 24.99 × cosec. 45° = 35.34 ft. Add the result to the horizontal measured distance and we have 200 + 35.34 = 235.34 ft., the required distance from the mouth of the tunnel to its intersection with the vein.

272. Required, the Direction and Distance from the Breast of a Tunnel to a Shaft, and the Depth at which it will cut the Shaft.—Make a survey of the tunnel and reproduce it upon the surface, as in the first example. Calculate the depth of the breast below the surface. Set up the instrument at the shaft and measure the vertical angle and horizontal distance to the point on the surface over the breast. Calculate their difference of level from the measurements ob-

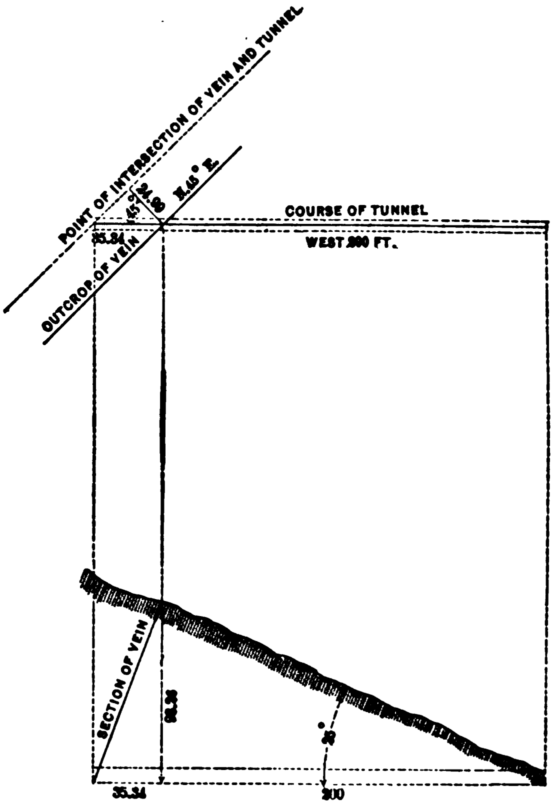

FIG. 99.

tained and add it to or subtract it from the depth of the breast below the surface. The result is the depth of tunnel below the mouth of the shaft.

Survey the shaft to a point whose vertical depth is equal to the depth of the tunnel level. Calculate the horizontal distance and direction of this point from the instrument station at the mouth of the shaft, and mark its position upon the surface. Connect it with the point marking the position of the breast of the tunnel, and we have the line required. From the information thus obtained range-lines can be suspended in the tunnel, to give the direction of the shaft from the breast.

EXAMPLE.—A shaft whose centre at the surface bears S. 65° E. 73 ft., vertical angle + 10° 20', from a point on the surface over the breast of the tunnel in the first example, has a pitch of 14° 30' N. 2° 15' W. from a vertical line. At what direction and distance from the breast of the tunnel will it cut the shaft, and at what depth? (Fig. 97.)

The following is the form of the field-notes:

FIELD-NOTES.

Station.	Vertical Angles.		Course.	Distance.	Remarks.
	In Tunnel.	On Surface.			
1	....	....	S. 36° 50' W.	19.1	To mouth of tunnel.
	+ 1° 18'	+ 10° 35'	S. 36° 50' W.	99.1	" Station No. 2.
2	+ 0° 31'	+ 15° 43'	S 49° 47' W.	104.2	" " " 3.
3	+ 0° 45'	+ 14° 27'	S. 40° 00' W.	37.1	" " " 4.
4	− 0° 34'	+ 16° 17'	S. 4° 55' E.	56.5	" " " 5.
5	+ 3° 37'	+ 12° 21'	S. 71° 15' E.	46.0	" " " 6.
6	+ 3° 30'	+ 13° 56'	S. 77° 30' E.	40.7	" breast of tunnel.
Breast.	....	+ 10° 20'	S. 65° 00' E.	73	" centre of shaft.
Centre of Shaft.	In shaft. − 75° 30'	....	N. 2° 15' W.	In shaft. 102.12	" point in shaft at tunnel level.

The depth of the tunnel at the breast is determined as in the first example. The vertical distance between the point over the breast and the mouth of the shaft is determined by the equation:

$$\text{Difference of elevation} = 73 \times \tan 10° 20' = 13.31 \text{ ft.}$$

Add this to the depth of the tunnel at the breast and we have $13.31 + 85.56 = 98.87$ ft., the vertical depth at which the tunnel will cut the shaft. With this depth and the pitch of the shaft, 14° 30', we obtain the depth (distance along the shaft) at which the tunnel will cut the shaft, measured along the dip, and also the horizontal distance from the instrument station, which is the intersection of the centre line of the shaft with the surface, to the point of intersection, by the following equations:

$$\text{Depth measured on the dip} = 98.87 \times \sec 14^\circ\ 30' = 102.12 \text{ ft.}$$
$$\text{Horizontal distance} = 98.87 \times \tan 14^\circ\ 30' = 25.57 \text{ ft.}$$

Set a stake N. 2° 15' W. 25.57 ft. from the instrument, and connect it with the point over the breast of the tunnel, and we have the course and distance from the breast of the tunnel to the line of survey down the shaft, S. 85° 21' E. 65.22 ft.

273. To survey a Mine, with its Shafts and Drifts.—Set up the instrument at the top of the main shaft, and after having first obtained the meridian, take the bearing, distance, and vertical angle to the point selected for the first station in the shaft. The distance is to be measured on a direct line between the stations, and its horizontal and vertical components afterwards calculated from the data obtained. The stations in the shaft are to be selected with a view to the extension of the survey into the different levels and down the shafts, and, as in case of other underground surveys, it is well to explore the mine ahead of the work, that the stations may be selected advantageously.

The field-notes of the survey of a mine are here given for illustration. The horizontal and vertical components of the distances measured down the shafts can be obtained by the use of a table of natural sines and cosines. (Figs. 100 and 101.)

FIELD-NOTES.

STA.	Course.	Dist.	Vertical Angles.	Horizontal Component.	Vertical Component.	Remarks.
						Begin at Sta. 1 at top of shaft.
1	N. 53° 57′ W.	54.1	− 75° 41′	13.37	52.42	To Sta. 2 at 1st level in shaft.
2	N. 34° 52′ E.	57.6	0° 00′	.....	.....	" air-shaft at end of 1st level.
	S. 25° 13′ W.	60.8	0° 00′	......	......	" centre of bottom of discovery-shaft, 50 ft. deep.
	N. 57° 46′ W.	60.0	− 66° 24′	24.02	54.98	" Sta. 3 at 2d level in shaft.
3	N. 30° 30′ E.	73.0	0° 00′	......	......	" breast of 2d level, running N.E. Second level running S.W. filled with débris, not accessible.
	N. 69° 23′ W.	47.5	− 77° 30′	10.76	46.37	" Sta. 4 at 3d level in shaft.
4	N. 28° 24′ E.	75	0° 00′	......	......	" breast of 3d level, running N.E.
	S. 19° 59′ W.	90	0° 00′	......	......	" breast of 3d level, running S.W.
	N. 71° 15′ W.	55	− 78° 00′	11.43	53.80	" Sta. 5 at 4th level in shaft.
5	N. 37° 20′ E.	75	0° 00′	......	......	" breast of 4th level, running N.E.
	S. 28° 43′ W.	64	0° 00′	......	......	" breast of 4th level, running S.W.
	N. 72° 01′ W.	55.6	− 79° 30′	10.13	54.67	" Sta. 6 at 5th level in shaft. The vein here divides: the shaft follows the portion to the south. The shaft is chambered out at this point, being 10 ft. wide S.W., and 20 ft. long S.E. from Sta. 6.
6	S. 71° 50′ E.	5.1	0° 00′	......	......	" Sta. 7 at top of shaft in chamber.
	S. 71° 50′ E.	8.1	0 00′	.... .	......	" Sta. 8 opposite drift, running S. 24° 15′ W., 102 ft.
7	N. 66° 15′ W.	46.8	− 85° 00′	4.07	46.62	" Sta. 9 at 6th level in shaft.
9	N. 32° 50′ E.	60	0° 00′	......	......	" breast of 6th level, running N.E.
	S. 24° 00′ W.	51	0° 00′	......	......	" breast of 6th level, running S.W.
	N. 55° 03′ W.	40	− 86° 16′	2.25	39.93	" Sta. 10 at 7th level in shaft.
10	N. 34° 15′ E.	39	[illegible]	......	......	" breast of 7th level, running N.E.
	S. 37° 45′ W.	55	0° 00′	......	......	" breast of 7th level, running S.W.
	N. 88° 30′ W.	48.6	− 81° 00′	7.60	48.0	" Sta. 11 at 8th level, at bottom of shaft.
11	N. 34° 00′ E.	54	0° 00′	......	......	" breast of 8th level, running N.E.
	S. 24° 00′ W.	15	0° 00′	......	......	" breast of 8th level, running S.W.

NOTE.—The width of the levels of this mine are about four feet. The dimensions of the shaft are 4' × 10'. The line of survey down the shaft was

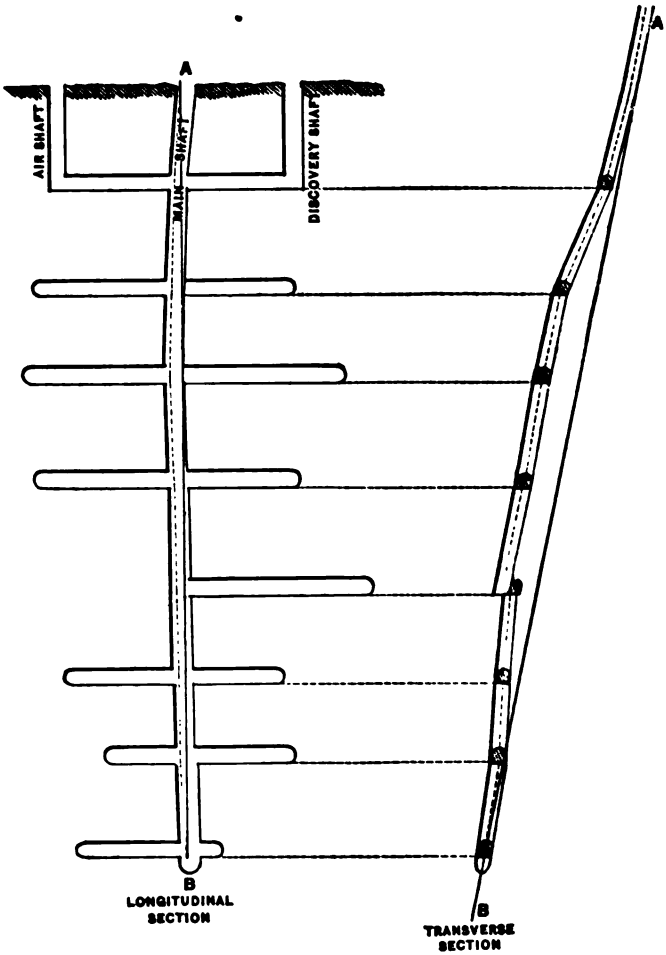

FIG. 100.

run about 2 ft. from the north end of the shaft. In the levels the line of survey was to the centre of the breast.

Fig. 101 shows the plan, and Fig. 100 the longitudinal and transverse sections of the mine as plotted from the field-notes. The plan is plotted from the courses and horizontal components of the measurements in the shaft and levels, as projected upon a horizontal plane.

The longitudinal section is platted from the courses and vertical components of the measurements in the shaft, and the horizontal measurements in the levels as projected upon a vertical plane passing through Station I and at right

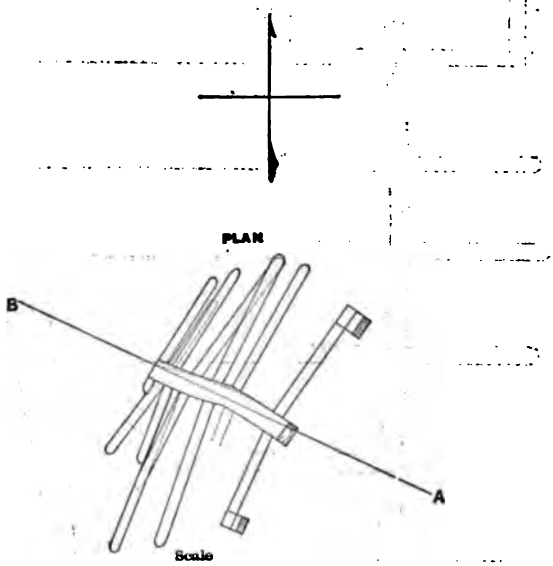

FIG. 101.

angles to a vertical plane passing through Stations I and II, upon which the transverse section is plotted as projected from the vertical angles, courses, and measurements in the shaft.

Thus it will be seen that for the full representation of an underground survey, showing the relative position of the parts to each other, three planes are necessary,—two vertical planes at right angles to each other, and a horizontal plane.

274. Conclusion.—The above examples comprehend some of the more general cases arising in the practice of mining surveying; any other cases which may arise will be found to be modifications or combinations of these. The problem to be considered can be solved by an application of the principles therein embraced, which the surveyor will find useful, also, in solving problems of mining engineering relating to the measurement of ore reserves, development, and systems of working. It has been shown that the following out of the underground workings of a mine corresponds to traversing when elevations are carried by means of vertical angles, as was fully described in the chapter on topographical surveying. The notes are also reduced in the same manner.

It has been the object of this chapter to present the subject of mining surveying in as simple a form as possible, and divest it of all features which, although they may give it a distinctive aspect, serve only to render it more complex and give the reader an idea of difficulties which are only imaginary.

It is useless, also, for the mining surveyor to encumber himself with many paraphernalia. Good work can be done with a mining transit provided with an extra telescope for vertical pointings, one or two short rods, and a reliable steel tape, all of which can be carried by the surveyor on horseback over the rough mountainous roads. Any other adjuncts can be improvised or be found at any well-conducted mine, and would prove more burdensome than useful.

CHAPTER XII.

CITY SURVEYING.*

275. Land-surveying Methods inadequate in City Work. —The methods described in the chapter on Land-surveying are inadequate to the needs of the city surveyor. The value of the land involved in errors of work, with such a limit of error as was there found practicable (see art. 180), is so great as to justify an effort to reduce this limit. Comparing the value of a given area of the most valuable land in large cities with the value of a like area of the least valuable land which a surveyor is ever called upon to measure, the ratio is more than a million to one.

This view is emphasized by the manner of use. On farm lands the most valuable improvements are placed far within the boundary lines, but the owner of the city lot is compelled by his straitened conditions to place the most costly part of his improvements on the limit-line. His neighbor's wall abuts against his own. The surveyor, who should retrace this line and make but a small difference of location, would get his clients and himself into trouble. Both the value of the land and the manner of its use demand increased care. The modifications of the methods used in land-surveying to meet the requirements of work in the city will be treated in this chapter. Much of the work described furnishes data for the solution of engineering problems, but the obtaining of the facts falls entirely within the definition of surveyor's work.

* This chapter written by Wm. Bouton, C.E, City Surveyor, St. Louis, Mo. See also Appendix G.

276. The Transit is used exclusively, but the common pattern may be very materially modified with obvious advantage. Seeing that the magnetic needle* is never precise and seldom correct, it should be wholly discarded in the construction of the city surveyor's transit. The verniers can then be placed under the eye, the bubbles can be removed from the standards and placed upon the plate of the alidade, and the standards themselves can be more firmly braced. By these changes a steadier and more convenient instrument is secured, when the useless and somewhat costly appendage of a needle-box is out of the way. The adjustable tripod head and the levelling attachment are always convenient. For topographical work, the vertical circle, or a sector, and stadia wires are essential, otherwise the methods used must be primitive. The thermometer which is needed in order to make the proper corrections for temperature may be conveniently attached to one of the standards facing the eye-piece of the telescope. The danger of breaking the tube while handling the instrument may be obviated by a guard sufficiently deep to protect the bulb, made open on the side toward the observer.

277. The Steel Tape is generally used for measuring. The legal maxim that "distances govern courses," when interpreted, means that, using customary methods, the intersection of two arcs of circles, centres and radii being known, is a more definite location of a point than the intersection of two straight lines whose origin and direction are likewise known. The fact is, the intersections are not more definite. The maxim grew into authority when the compass was pitted against the chain. With the transit to define directions of courses, and the chain still to measure the distances, such a maxim would not have voiced the results of experience, but would have been sheer nonsense.

* The needle finds its proper place where checks are not so abundant, and in classes of work in which a close and rapid approximation is of more value than precision.

The ordinary chain has too many gaping links, and the brazed chain too many wearing surfaces, to be kept in very close adjustment to standard length. Its weight is such as to make the "normal tension" (see p. 392) impracticable; hence the effect of slight variations of pull is much more marked than if the tape is used. Graduated wooden rods were used until 1860 to 1870. They were unwieldy when twenty feet long, and were still so short that the uncompensated part of their compensating errors was a matter of considerable moment. Every time the pin is stuck or a mark made at the forward end of the tape or rod, the work is a matter of skill and involves an error dependent on the degree of skill attained. When the measure is brought forward, its proper adjustment in the new position is a matter requiring skill. These errors are compensating, but the resultant is not zero. The use of the plumb-line is another source of compensating errors which are reduced by an increase of length in the measure. First, the number of applications varies inversely as the length of the measure; second, using the rod, it was necessary to work to the bottom of ravines and gullies and then work up again; now the long tape spans them at a single application. The minus errors due to imperfect alignment and inaccurate levelling of the two ends have a greater percentage of effect when the measure is short than when it is long. The longer tape brings with it some other sources of error. When used suspended at the ends there is a minus error on account of the sag of the intermediate parts, and a plus error from elongation due to tension; there is also expansion by heat, which produces an error which may be plus or minus as the temperature at the time and place is above or below that for which the tape is tested. The effect of sag increases very nearly as the cube of the length when the tension is constant. When, to counteract this increase, the pull is made greater than a man can apply uniformly under all conditions—at his feet or above his head—there come irregularities from this cause. The limit of length of tape which it is practicable to use will be determined by the conditions of the work, and should be such that the increase of length involves greater error than it eliminates. On account of convenience in keeping tally, 50-foot and 100-foot lengths are generally used. In a level country the 100-foot tape is preferred.

There are tapes made with the purpose to eliminate the errors which arise from the free-hand pull, the inclination of the tape, and the temperature. They carry a spring balance, a bubble adjusted to the desired pull, a thermometer, and a means of adjusting the length to the given pull and temperature. The effort is laudable; but, probably on account of the number and form of the wearing surfaces, they have not yet met with general favor. Further progress may be made in this direction.

LAYING OUT A TOWN SITE.*

278. Provision for Growth.—Cities grow. It is very rare that the considerations which should have governed have been given any place in determining upon the plan of the original town. The considerations first in importance are topographical. What are the natural lines along which business will tend to distribute itself? To what form of subdivision can it adapt itself with the least resistance? Where is the best harbor, the lake or river front, or the railway line? Ordinarily the land immediately adjoining such natural features is not best used when used as a street, but when occupied by private

* For principles governing the laying out of new cities, see a valuable paper entitled *Practical and Æsthetic Principles for the Laying Out of Cities*, by J. Stübben, Commissioner of Public Buildings and Assistant Burgomaster, Cologne, Germany; read before the World's Engineering Congress at Chicago, 1893, and published in the *Trans. Am. Soc. Civ. Eng.*, vol. xxix., p. 632.

docks, or along a railway by warehouses and factories having switching facilities without crossing public streets. The streets parallel to such lines should be of ample width, easy grade, and continuous but not necessarily straight alignment. Much of the heavy hauling will be along such streets. In the business part of the town the cross-streets should be so frequent as to make the blocks approximately square. In the residence portion alternate streets in one direction may with advantage be omitted: this saves the expense of unnecessary streets, and permanently lightens the burden of taxation. Which fronts are on all accounts most desirable in the particular locality will determine in which direction the blocks should be longest.

279. Contour Maps.—Another phase of topography demands attention. The sites of suburban towns may generally be best handled by laying out streets and lot lines in conformity to the undulations of the ground. Additions to the city may also have characteristic features that can be preserved with advantage. For all such cases a contour map is very useful to one who is able to interpret it. The making of all the ground available, and sightly points accessible, and at the same time so locating the streets as to secure economical grades,—in short, the judicious handling of the whole subject is facilitated by the study of the contour map.

280. The Use of Angular Measurements in Subdivisions.—Shall subdivision lines be located by an angle with the street on which the lots front or by distances from the next cross-street? Must distances govern courses, whatever methods are used? Let us suppose, for illustration, that it is required to locate lot 9 in the accompanying sketch (Fig. 102). Suppose, farther, that it is possible to measure each of the lines *ab*

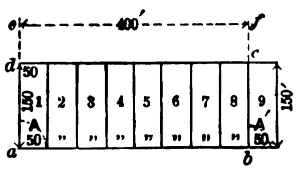

FIG. 102.

and *dc* with a maximum error of 1 in 5000 and that the conditions are such as to produce opposite errors in the two lines. Then, 1st, the resulting error in locating the line *bc*, i.e. $(ab - dc)$ will be $\frac{1}{5000} \times 400 \times 2 = 0.16$ feet. The sine of the angle by which the angle A' differs from A will be $\frac{0.16}{150} = .00107$. Hence the change of direction on account of the errors in measurement is $3\frac{2}{3}$ minutes. 2d, the line *ef* must be distant from *ab* $3\frac{2}{3} \times 150$ feet $= 550$ feet, in order that, under like conditions, if it is measured instead of *dc*, the change in direction shall not exceed one minute. Or the location may be made by measuring the line *ab*, or a line near to it where favorable conditions exist, and then repeating *ba* the same man being fore-chainman; the principle of reversal is thus applied to this measurement. Then measuring $A' = A$ and repeating the angle, reading both verniers, the error is brought within the maximum error in the pointing power of the instrument. In order to locate *bc* from *ab* parallel to *ad*, two monuments marking the line *ab* need to be known. The other method requires also a monument locating the line *ae*. It thus appears that when the side-lines of lots are located perpendicular, or at any other known angle with the street upon which the lot fronts, it is susceptible of more accurate location than by two (front and rear) measurements, unless the usual limit of error can be greatly reduced. While it is not likely that maximum errors of opposite character will fall together affecting the work on the same lot, it is quite as improbable that the maximum error in measuring an angle should vitiate the work of the transit. It is probably quite as easy to reduce the maximum error in measuring an angle to half a minute as it is to keep the maximum error in measuring distances down to 1 in 10,000.

281. Laying out the Ground.—The work of putting the plan upon the ground is a very important one. This is about the worst possible place to do hurried and inaccurate work.

Fences or other styles of marking possession which limit the contour map cannot be relied upon as defining the property-lines. These lines must be accurately located in relation to the streets of the town or of the addition, in order to make practicable such exchanges or sales as may be necessary to adjust property-lines to the new block-lines. This method is preferable to that which adjusts block-lines to the original property-lines.*

As a framework for the whole survey an outline figure, generally a quadrilateral, of sufficient dimensions, and so placed that it can be permanently marked with monuments which will remain accessible when the town is built up, should be located with especial care. All lines should be measured, all angles observed, and all practicable checks introduced. This figure must close absolutely; that is, the record of the work when completed should be mathematically consistent. Unreasonable errors are to be eliminated by retracing the work. In the adjustment which distributes the remaining errors each part of the work should be weighted (art. 174, Rule 2), for it is very rare that a land-survey is completed under such conditions that the man who does the work would be justified, while these conditions are fresh in his mind, in assuming that the probability of error is alike at all points. The angles admit of adjustment independently of the length of the lines. *That distribution of the angular errors which reduces the errors of measurement to a minimum* has such weight that it can be overruled only by the most positive evidence that the cor-

* In some places this idea of the private interest of the proprietor, sometimes private spite, is carried to such an extent that it would seem that each man's farm or garden patch was especially fitted to be a town by itself, laid out with utter disregard to the towns which others are in like manner laying out upon adjacent farms. In this practice the interests of the public for all time are neglected in order to secure a doubtful advantage for one. Where the custom prevails it is better honored in the breach than in the observance.

rections so indicated cannot be the true ones. The distances are then adjusted to the angles so determined. The remainder of the work of the subdivision is checked upon the adjusted outline, reasonable errors being distributed and unreasonable ones retraced.

282. The Plat to be geometrically consistent.—The necessity that the recorded plat should be consistent lies in the use that is to be made of it. A parcel of ground described by reference to the plat of record should have but one location, not any one of two or more possible locations, as is the case when the plat contains errors on its face. In the course of years the lines of such parcels will be retraced probably many times, at one time by one method, at another time by another equally in accord with the plat. If the plat is not consistent with itself and with the monuments upon the ground, this error will be pretty sure to find its way into the lot location. When the fault is with the plat, it matters not how the monuments are placed upon the ground; they cannot mark the chief points and all agree in such a way that if any two remain and the others are lost the relocation will in every case be the same. But this is just what the plat is for—to make a public record of the relation of each part of the subdivision to every other.

283. Monuments.*—How many monuments shall be located, and where shall they be placed? What material shall be used and how set? Answering the first question, it is plain that no more work should be attempted than can be done well. Better one point and an azimuth than points everywhere and no two agreeing either in distance or direction with the relation described by the plat. But so much should be done well that the labor of locating any point in the subdivision from existing monuments shall not be excessive. The points chosen for placing monuments should be such as will continue to be accessible and will not be ambiguous. The centre lines

* See also Arts. 159, 160, 161, and 194 in chapter on Land Surveying.

of intersecting streets are sometimes marked, giving one monument to each intersection; others choose the side-lines, giving four monuments to each intersection of streets. If the blocks are so long that intermediate points are desirable, points on the ridges should be selected.

Stone is more often chosen than any other material; iron bars, gun-barrels, gas-pipe, etc., are sometimes used, driven with a sledge; cedar posts, say 4″ × 4″, are quite durable, and hard-burned pottery is sometimes used. Whatever material is chosen, the foundation, which should be flat—not pointed—must reach below frost; and the centre of gravity is kept as low as possible, so that there shall be no tendency to settle out of place when the ground is soft in the spring. When the tops are much above the surface of the ground, there is a liability that they may be displaced by traffic. Probably the surveyor does not see any traffic, or the prospect of it, when he is doing his work, but the traffic must come before the work of the monument can be spared. It is better to bury the stone wholly and indicate where to dig for it by bearings than to run the risk of losing the whole work through indiscretion in placing the monument that marks it. In situations where every rain storm produces a slight fill it is safe to place the top considerably higher than would otherwise be reasonable. The stones to be set are so placed in the excavation, with the heavy end down, that when the top is in the proper position and before any earth is refilled there is no tendency to fall in any direction; then while the earth is being refilled and thoroughly tamped about the stone. the top is kept in place. It is better that the mark denoting the point for which the stone stands should be cut upon before it is placed in the ground. When this is done, if the mark is worn off by traffic or knocked off by accident, the centre of that portion of the stone which remains is a very close approximation to the original point. A slovenly way of slighting this work is to tumble the stone into the excavation, fill around it pretty much as it happens, push it to one side or another so that the point will come somewhere on the top, and then cut the mark wherever the point comes. Stones set in this way are liable to settle out of place after the first heavy rain, while frost and rain keep up their work till the stone lies flat upon its side. If by chance it should keep its place pretty well and the mark becomes defaced, it might as well be any loose bit of rock as a set stone, for its centre gives no idea of where the mark was placed. No one should be trusted to set corner-stones unwatched who is not familiar with the work and thoroughly reliable.

Points are preserved temporarily by wooden stakes driven flush with the ground. The point, preserved by offsets while the stake is being driven, is marked by a nail. Witness-stakes driven alongside and standing above grass and weeds assist in finding the stakes when wanted. Made of half-decayed soft wood, e.g., old fence-boards, such stakes will hardly last a season; while durable wood, well seasoned, will last much longer than any driven stake can be relied upon, since it does not go below frost, and is liable to be pushed by a passing wheel or be otherwise disturbed when the ground is soft.

284. Surveys for Subdivision.*—The purpose of making a survey before recording a plat of a subdivision is twofold,—first, to get the information which it is desirable to record; second, to leave such monuments as will make it easy to locate any portion when desired. The recorded plat should show sufficient facts to determine the relations of every part to the whole, and these relations should be shown by methods which involve the minimum of error, i.e., giving a location which may be retraced with least possible doubt. The current practice falls short of this standard at some points which are worthy of note.

(*a*) Surveyors seem to have no doubt of the ability of their field-hands to measure a line, but very seriously doubt their

* See also Appendix G.

own ability to measure an angle. Angles are measured during the progress of the work and are used for determining the lengths of lines; these lengths are then made a part of the record, while the angles which determined them are omitted. Apparently some things which are dependent have become more certain and fixed than that upon which they depend. A proper record of angles would show what lines are straight and where deflections are made. Deflections which are sufficient to very seriously affect the position of a brick wall do not show on the scale of the recorded plat. For example, an addition to a town extends from Fifth Street to Twelfth Street; extreme points are well established, but intermediate monuments are missing; and it is required to establish at Eighth Street the line of a street which a ruler applied to the recorded plat suggests is a straight line. Custom approves that in such a case the surveyor should try a straight line, there being a mild presumption in its favor; but if his straight line agrees with one wall and disagrees with two walls and a fence, he had better look further before he comes to a decision. No such doubt could have existed if the recorded plat had been properly made.

(*b*) Very few recorded plats show what stones have been set by the surveyor, or indeed indicate that he has any knowledge that such monuments may ever be useful. If the custom were once established of noting upon the record the location and description of monuments, any monument found during a resurvey, but not shown on the record, would be discredited. As matters now stand it must be proved incorrect to be discredited—a thing not always easy, for a system of quadrilateral blocks whose angles are not recorded and whose street lines are not necessarily straight is not theoretically very rigid.

(*c*) Many plats require measurements to be made along lines which are easily measured while the land is vacant, but which will become inaccessible as soon as the property is built up. The obstacles to be overcome before the result can be reached by the method described on the record will each add to the doubt of the accuracy of that result. There are many ways in which plats are made, which are all justly subject to this criticism. Two examples will suffice. Irregularly shaped blocks are sometimes treated as in the annexed sketch, Fig. 103. The outline is subdivided mechanically, and proportional

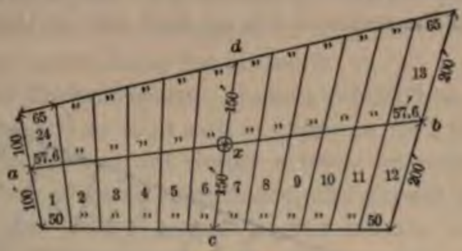

FIG. 103.

distances are given on interior lines which are not consistent with any trigonometrical relation of the exterior lines, much less with that which does exist but is not recorded. The point *x* has nine distinct locations directly from the plat. On the theory that *ab* and *cd* are straight lines, their intersection gives one; *ab* straight, the distances *ax* and *bx* give each one; *cd* straight, the distances *cx* and *dx* give two. Combine the distances *ax* and *cx*, *bx* and *cx*, etc., and get four more. But this is not all, for the point *x* stands related to each of the ten other points along the line *ab*, and each of these has also nine locations which accord with the plat, and our point *x* may be located from either of them or any combination of them when they have been located by any of the methods described.

Besides the difficulty of determining how interior points should be located, this fan-like subdivision wastes ground in each lot which results in wedge-shaped remnants about the buildings, which remnants would be valuable if thrown together into the corners, thus keeping the remaining lots rectangular at the front. The attempt to reach a rectangular front sometimes fails through inattention to very simple matters, as in Fig. 104. Here no angles are recorded. The rear corners of the lots are located along the line *ab* by distances from *a* or *b*; but the record-depths do not fall upon a straight line. The line *ab* should bisect the angle between the block-lines or be parallel to such bisection in order that with a constant distance along *ab* common to the series of lots on each side of that line their

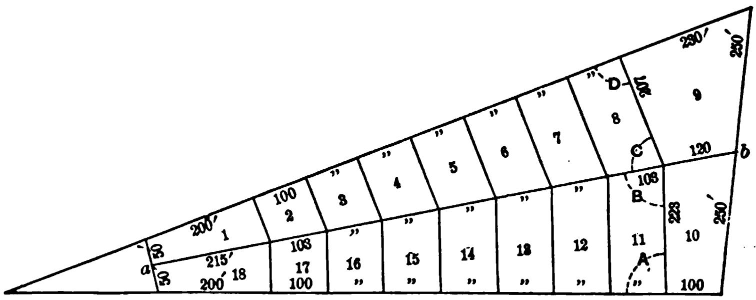

FIG. 104.

angles with their respective fronts may remain constant. In the case given every lot-line has an angle with the block-line upon which it fronts different from that of every other lot-line, and all dependent on some block-angle which is not recorded. If it is not desirable to bisect the block by the line *ab*, its direction may be chosen as desired, the distances along it are fixed by the fronts on one and the angular divergence from that side which is chosen, and the lot fronts on the other side of the block must be correspondingly increased or diminished.

When alleys are laid out in a block so that the interior lines are accessible, it is very rare that after the block is improved these lines can be measured under the same conditions as the fronts. If alleys are not laid out, the difficulties are usually much greater. Location of lot-lines by angle from the front is undoubtedly the most uniform and workmanlike method available to the surveyor. Hence, distances on the rear lines of the corner lots should be omitted from the record, if their presence would leave any doubt as to which method of location is intended. It is not customary, nor is it desirable, that lot-lines or distances should be determined upon the ground before recording a subdivision, but they should be platted by a man who knows at least the first principles of trigonometry, and has an accurately measured basis for his work.

285. The Datum-plane.—Levels referred to a permanent datum are needed as soon as it is apparent that the town is to be a living reality and not simply a town on paper. The datum is a matter of some importance, and should have a simple relation to some natural feature of the locality which will retain a vital interest so long as the town exists. There is an individuality in town-sites which usually determines for each case very definitely what is best. High-water mark indicating the danger of overflow; the lowest available outlet for a drainage system in a flat country; the average sea- or lake-level, as affecting commerce; these are often chosen and may serve as examples. The datum selected has its value accurately determined and marked by a monument as enduring as the granite hills, or, if that is impossible, as near this standard as can be secured; a block of masonry, with a single and durable cap-stone firmly bolted to its place, and bearing the datum, or a known relation to it, definitely marked and secured from abrasion is certainly possible for all.

286. The Location of Streets for which the most economical and practical system of grades may be secured is to be considered when the plat is being prepared. Grades are usually established from profiles taken along the centre lines of the street to be graded. This method is direct and protects the public fund, for the grade, which can be executed at minimum cost, the street being considered by itself, can be determined from such a profile. The method fails from the fact that it treats the fund raised by taxation as the sum total of the public interest. Parties representing abutting property appear before the legislative body which has final action and seek to amend the recommendation of the engineer, claiming that interests which should receive consideration are injured by the grades proposed. It seems plain that whatever is recommended by the city's officer should have the moral weight which attaches to an impartial consideration of all the interests which the city fathers are bound to recognize. But this involves a change of method. The contour map of the district involved seems to offer some help toward a solution. Methods by which a rapid approximation of the amount of cut and fill involved in any proposed grade may be arrived at are discussed in Chapter XIII., on the Measurement of Volumes.

287. Sewer Systems.—A well-devised sewer system touches very closely the public health. The information which is necessary in order to act intelligently involves, if storm-water is to be provided for, the area and slopes of the whole drainage-basin in which lies the area to be sewered. This will enable a close approximation to be made of the work required of the mains at the point of discharge. Each subdistrict involves its own problem. The most economical method of reaching every point where drainage is necessary is learned by studying the details of topography. Borings along the lines of proposed work to determine the character of the soil and the depth of the bed-rock are necessary in order that contractors may bid intelligently. This species of underground topography sometimes modifies the location fixed by surface indications.

288. Water-supply.—The need of a water-supply furnishes new work to the surveyor. The distance and elevation of the source of supply, the topography of the country through which aqueducts or mains must be brought, eligible sites for reservoirs, with their relation in distance and elevation to all points to be supplied, are to be furnished to the hydraulic engineer. The datum-plane for these maps and that of the town should correspond.

289. **The Contour Map,** which is so generally useful from the time the town is first planned until public improvements cease to be considered, if surveyed carefully at first, has no need to be retraced each time such a map is useful. It had best be drawn in sections of sufficient scale for a working-plan, and so arranged that when adjacent sections are placed side by side the contour lines will be continuous. If the contours of the natural surface are drawn in india-ink, and the contours showing the changes made by different kinds of public work be drawn in some color, the map may give a great amount of information without becoming confused.

METHODS OF MEASUREMENT.

290. **The Retracing of Lines*** comes with the private use of lots or blocks and with the execution of public improvements. The demand for this class of work comes not once only, but many times, and never ceases while there is life and growth. The changes to which these forces give rise furnish the main demand for knowing along what lines growth may proceed unchallenged. The man who first fences a lot in the middle of an unimproved block can ill afford to risk being compelled to move his fence for what a survey would cost. But the first attempt to go over any part of a subdivision and locate a lot-line raises the question, how nearly alike can a surveyor measure the same distance twice, or how nearly alike can two surveyors measure the same distance. If the distance noted on the recorded plat was not measured correctly, the resurvey must differ from it, or by chance make a mistake of the same amount. The difference which appears by compar-

* See also Art. 194 in chapter on Land Surveying.

ing results is not the error which exists in either the original or the resurvey; it may be more than either error, it may be less, being the algebraic difference of the two errors. If there is no difference it means that the work is uniform, and may be correct, but both may also be in error a like amount. It has happened in the days of twenty-foot rods and in a city of considerable size that every rod used by surveyors was too long. The change to steel tapes has not set matters wholly right. If a man compares steel tapes bearing the brand of the same manufacturer and offered for sale in the same shop, he soon ceases to be surprised at a very appreciable difference in length.

291. **Erroneous Standards.**—How long is a ten-foot pole or a hundred-foot tape is a pertinent and fundamental question. It cannot be ignored when deeds call for a distance from some other point, as fixing the beginning-point of the parcel conveyed. When the deed describes lot number —, as shown on the recorded plat, there is a theory in accordance with which uniformity is all that is required—a distribution of the distance between monuments in proportion to the figures of the record. Property is often laid out with a view to this theory of surveying. So long as block-boundaries are definitely marked, a degree of precision is very readily secured by this method which is rarely attained when surveyors attempt to measure standard distances. If the surveyor faithfully measures the block through and every time distributes what he finds in proportion to the record, though his block distances may not agree with the record or with themselves, the lot-lines will be much more likely to be the same than if he measures his record distance and stops at the lot. This method assumes that the lots abut one upon another, and reach from one monument to the other. But if this be true, the distances noted must often refer to some empirical standard peculiar to this block and not to the United States standard established by

law. But the courts recognize no standard, so far as the author knows, but that which is established by law. So that when a surveyor comes to mark lot one, finds the corner of the block, and drives his stake by measuring from it the distance which the record assigns to lot one, it is hard to prove that he has not measured according to the subdivision, although he has given no thought to the distance which remains for the other lots. But trouble begins right here, for the theory which is correct for lot one cannot be very wrong for lot two; continue the process to lots six and eight, and give to another surveyor who has been doing the same kind of work at the other end of the block an order to survey lot seven. A conflict in this case is certain unless the surveyor who laid out the subdivision, and each of the others since, knew the length of his tape and knew how to measure.

292. True Standards.—The U. S. Coast and Geodetic Survey Department at Washington standardizes steel tapes for a nominal fee, giving their exact lengths at a given temperature, or the temperature at which the tape is standard. By means of such a standard tape, a standard test bar may be set and graduated, and used as a permanent standard of length. If this bar be of iron or steel, then no attention need be given to the temperature at the time of graduating it, or when subsequent comparisons of steel tapes are made with it, since both will be at the same temperature. In this case the bar becomes a standard at that temperature at which the original tape was found to be standard, by the Coast Survey comparison. For this reason it would be well to require the C. & G. Survey authorities to give the true length of the tape at a given temperature (as 60° F.) and for a given pull.

Where and how to construct a standard rod, and how to care for it so that it may be permanently reliable, each individual had best determine for himself. It should be fastened in its place in such a manner that it can expand and contract freely, i.e., without any strain from its supports. If it is made of separate parts, these should be so joined together that there can be no lost motion between the pieces. The whole requires protection from the weather and to be so supported that it cannot be bent by a blow. The writer has solved this problem for himself in the following way: Bars of tool steel one inch wide and one fourth of an inch thick are joined, as shown in the sketch, to make the desired length 50 feet +; the whole is

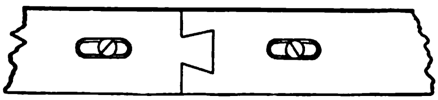

FIG. 105.

fastened to the office floor by screws which hold the middle firmly, but each side of the middle the holes drilled for the screws are slotted sufficiently to allow for any possible change of temperature. The joints are so close that a light blow is necessary to bring the parts to place; the screws were set home and then withdrawn a little, so that they should not cause friction with the floor. After the fastening was completed the standard marks were cut upon the rod.

293. **The Use of the Tape.**—It is one thing to have a tape of correct length; it is another thing to be able to use it. In an improved town with curb-lines clear, perhaps the most obvious method is by a measurement along the grade with the same tension as that at which the tape is tested. It is then necessary to correct for temperature and to note all changes of grade, reducing the observed distance on each grade by the versed sine of the inclination or by the formula given in Chap. XIV. By this method the tape is supported for its entire length, and it is practicable to use a tape two or three hundred feet long to advantage provided there are enough assistants to keep it from being broken. A difficulty arises in the use of

this method from the fact that the town is not made for the convenience of surveyors, and curb-lines are not usually clear where measurements are needed, but are obstructed by piles of building material, bales of merchandise, etc., and in some towns the streets are so dirty that the graduation could not be seen long if a tape were used in this way; it would also be so covered with drying mud that it could not be rolled in the box when out of use, hence would be frequently broken. Tapes that are wound on a reel, and have no graduations to speak of, could be used in the mud, but the other objections mentioned would still make the method of very limited application. It is further to be noted that the laying-out of the town, which is the basis of all later work, has all to be done before the streets are graded or the curbs set. This work must be done by some other method.

The usual method is to keep the ends of the tape horizontal by using a plumb at that end of the tape where the surface is lowest, and often at both ends if the ground is so irregular or so covered with brush and weeds that the tape must be kept off the ground. The tape assumes a curved form, and the horizontal distance is something less than the length of the tape. There is also a tension in the tape which, on account of the elasticity of the metal, somewhat increases its length. As the tension increases the sag diminishes, hence there is a degree of tension such that its effect is equal and opposite to the effect of the sag. Call this the ***normal tension***. If a line is measured with a pull less than the normal tension for the tape used, the tape will sag too much and there will be a minus error due to this excessive sag; if the pull used exceeds the normal tension, there will be a plus error due to this excess. If the pull has been uniform the total error in either case is proportional to the length of the line; but if the pull has not been uniform the error has varied irregularly with each length of tape and can most readily be calculated by retracing the line and using the proper tension. In practice the tape is *tested* with a known tension, and a tension so much above the "normal" is adopted for field use that its plus error is equal to the plus error of the test.

294. To determine the "Normal Tension" in a tape supported at given intervals. The tape forms a catenary curve, since it carries no load but its own weight and is of uniform section.

Let $P =$ horizontal tension (pull);
$w =$ weight of a unit's length of tape;
$e =$ base of Naperian logarithms;
$s =$ length of curve from origin;
$l =$ distance between supports;
$W = wl =$ weight of tape;
x and $y =$ horizontal and vertical coördinates, origin at lowest point;
$x = \frac{1}{2}l$ for cases considered.

Then by mechanics,*

$$y = \frac{P}{2w}\left(e^{\frac{wx}{P}} + e^{-\frac{wx}{P}} - 2\right),$$

and

$$s = \frac{P}{2w}\left(e^{\frac{wx}{P}} - e^{-\frac{wx}{P}}\right).$$

We observe (1), that if $\frac{P}{w}$ is constant y and s are constant for the same length of tape; (2), if P be measured, say ten pounds,

* The discussion here given is rigid, but both the development and the evaluation of the equations are laborious. If the curve be assumed to be a parabola, which it may when the sag is small, the development is much simpler. See the treatment of this subject, Art. 344, Chapter XIV.—J. B. J.

as a working condition, y and s will vary with the weight of every tape used, hence $\frac{Pl}{W} = \frac{P}{w}$ is the ratio which must be constant; (3), if the surveyor can keep y constant, the same conditions keep s constant, and if y varies s must vary; (4), if $x (= \frac{1}{2}l)$ varies, and $\frac{P}{w}$ varies in the same ratio, then $\frac{wx}{P}$ is constant, hence the parts of the equations in parenthesis are constant and y and s vary as l and $\frac{P}{w}$.

TABLES SHOWING NORMAL TENSION AND EFFECT OF VARIABLE TENSION.

l = 100 feet.			x = 50 feet.					
Sag.			Pull.		Resultants ±			
$\frac{P}{w}$.	y.	$(2s - l)$ − error.	$\frac{P}{W}$.	Elongation + error.	Error in l		Error in 1000 ft.	
					−	+	−	+
	ft.	ft.		ft.	ft.	ft.	ft.	ft.
800	1.56	0.065	8	0.010	0.055	....	0.55	......
900	1.39	0.051	9	0.011	0.040	....	0.40	......
1000	1.25	0.040	10	0.012	0.028	....	0.28	......
1100	1.14	0.033	11	0.014	0.020	....	0.20	......
1200	1.04	0.028	12	0.015	0.013	....	0.13	......
1300	0.96	0.023	13	0.016	0.007	....	0.07	......
1400	0.89	0.020	14	0.017	0.002	....	0.02	......
1500	0.83	0.017	15	0.019	....	0.002	......	0.02
1600	0.78	0.014	16	0.020	....	0.006	......	0.06
1800	0.70	0.011	18	0.022	....	0.011	......	0.11
2000	0.62	0.009	20	0.025	....	0.016	......	0.16
2400	0.52	0.007	24	0.030	......	0.022	......	0.22

$l = 50'$. $x = 25'$.								
SAG.			PULL.		RESULTANTS ±			
$\frac{P}{w}$.	y.	$(2s - l)$ − error.	$\frac{P}{W}$.	Elongation + error.	Error in l		Error in 1000 ft.	
					−	+	−	+
	ft.	ft.		ft.				
400	0.78	0.033	8	0.003	0.030	....	0.60	......
500	0.63	0.020	10	0.003	0.017	....	0.34	......
600	0.52	0.014	12	0.004	0.010	....	0.21	......
700	0.45	0.010	14	0.004	0.006	....	0.11	......
800	0.39	0.007	16	0.005	0.002	....	0.04	.....
900	0.35	0.006	18	0.006	....	....	....	......
1000	0.31	0.004	20	0.006	....	0.002	......	0.03
1100	0.28	0.004	22	0.007	....	0.003	......	0.06
1200	0.26	0.004	24	0.008	....	0.004	.....	0.08
1300	0.24	0.003	26	0.008	....	0.005	......	0.10
1400	0.22	0.003	28	0.009	....	0.006	......	0.12
1500	0.21	0.002	30	0.009	....	0.007	......	0.14
1600	0.19	0.002	32	0.010	....	0.008	......	0.16
1700	0.18	0.002	34	0.011	....	0.009	......	0.18
1800	0.17	0.001	36	0.011	....	0.010	......	0.20

Assuming values of $\frac{P}{w}$, the formulas are readily solved for any assumed distance between supports and the results tabulated; seven-place logarithms are best for this work.

The 100′ tape is chosen because it furnishes a ready means of calculating a table for any other length of tape by a decimal reduction of the errors, per 1000′, in proportion to the length desired, and tabulated with values of $\frac{P}{w}$ reduced in the same proportion. There are those who use the 100′ tape free-hand, with 16 to 20 pounds pull, and say they do the work uniformly.

In the ordinary formula for elongation, $\lambda = \frac{PL}{Ek}$,* we have the section k, a multiple of w. The foregoing tables are calculated from the value $w = 3.4k$. The tension in the tape P

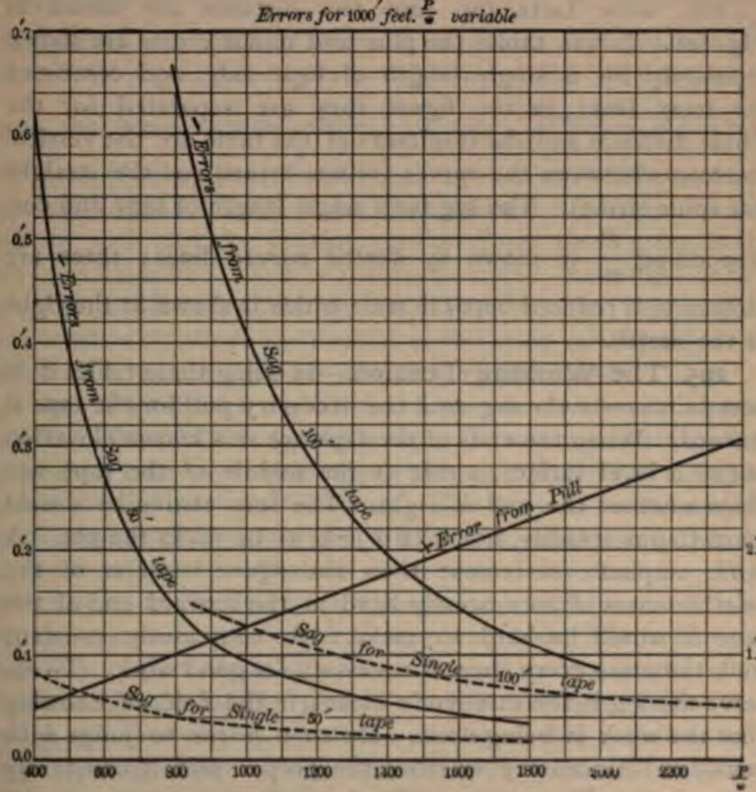

FIG. 106.

* E is the modulus of elasticity in pounds to the square inch, and k is the area of the cross-section in square inches, L being given in the same denomination as λ.

differs from the horizontal tension P, so that $P' = P$ secant i (i = inclination to the horizontal), a second difference which is so small that it may be neglected. Let $E = 27500000$ (see Chapter XIV.), hence $\frac{P'l}{Ek} = \frac{3.4Pl}{27500000w}$, nearly.

The same facts for 1000 feet distance are shown in Fig. 106. In the tables the plus and minus errors are shown separately for a single length of tape only, and combined for 1000′ feet; in the figure they are separated for the whole distance and the resultants of the table are the vertical intercepts between the curves (minus errors) and the straight line (plus errors). The sag for a single length of tape and corresponding $\frac{P}{w}$ is shown by dotted curved lines; these are plotted to a reduced vertical scale which is shown at the right of the sketch.

295. **The Working Tension.**—In using these tables it is best to measure the sag until the necessary pull for the tape is learned. When the ends of the tape are at a known elevation above a level surface, a rule at the middle of the tape will show whether the pull is right. The fore chainman should learn to pull steadily, not with a jerk, as he sticks the pin. A more emphatic statement than the figure itself is of the worthlessness of an unsteady hand at the forward end of the tape it would be hard to make. A consciously constant pull, the same every time, is necessary for good work. To observe the sag is the surveyor's means, in the field, of knowing that the work is being done. He soon learns to judge with considerable accuracy whether the proper pull is constantly maintained. The proper pull is determined by the tension at which the tape is tested; call this p. Then, having weighed the tape, $\frac{pl}{W} = \frac{p}{w}$. Seek the plus error from elongation for this value of $\frac{p}{w}$; then find the same plus error between the curve for

that length of tape and the straight line; the corresponding $\frac{P}{w}$ is right for field use.

For example, a 50′ tape weighs six ounces, and the pull, when tested, was five pounds; $\therefore \frac{p}{w} = \frac{5 \times 50}{\frac{6}{16}} = 666$, and the elongation $= 0'.083$. The curve for a 50′ tape marked − error from sag is distant from the line marked + error from pull the same amount when $\frac{P}{w} = 1233$. Whence $P = 1233 \times \frac{6}{16} \div 50 = 9\frac{1}{4}$ pounds, and the sag $= 0'.25$. When a tape is to be suspended freely in use, the tension at the test, p, should not be such that the working tension P will be so great as to be impracticable; but it is also to be noted that slight variations of pull do not affect the result as much, when the tension is considerably above the normal, as the same variations would affect it if the tension were at or below the normal.

296. The Effect of Wind.—A very moderate wind has a marked effect on the sag of the tape; the wind-pressure on the surface of tape exposed increases the sag and gives it a diagonal instead of a vertical direction. The exposed surface of the tape constantly changes, and this results in vibrations which make it difficult to tell where either end of the tape is. The effect of its action, which is a minus error, varies approximately as the square of the length of tape exposed. The effect of winding up part of the tape so as to use a shorter length is to increase the use of the plumb, which is also affected by the wind, and the result is a loss of a part or all that is gained. A high working tension reduces the effect of the wind. But the only way to eliminate this source of error is to cease from any piece of work when the wind is so high that it cannot be done as it should be done. There are estimates, topography, etc., which do not require a high degree of precision and which can be done when other work cannot.

297. The Effect of Slope.—When the tape is used with its ends at different elevations, if it hangs freely its lowest point would not be in the middle, but nearer the lower end. The corrections for sag and pull still apply, however, with inappreciable error, for all practicable cases. The normal tension, therefore, remains the same as for a level tape. A correction must now be made, however, for the grade, the value of which is l vers. i, where l is the distance measured along the slope, and i is the angle with the horizontal. The measured distance is always too great by this amount.*

The available means by which the tape may be kept level are: (1) The judgment of two field-hands. (2) On difficult lines, the presence of the surveyor standing at one side where his position has some advantages. A distant horizon often very sharply defines the horizontal. (3) Where streets are improved, although it may be impracticable to measure along the slope, the known fall per 100 feet will give the needed information. (4) Where none of these methods are sufficient, test the judgment by plumbing at different heights and correcting the pin if necessary. These methods will eliminate the worst errors; but where it is necessary to measure lengths of five or ten feet, and then plumb from above the head, the uncorrected remnant will be considerable, probably that due an inclination of two per cent on the whole length of such lines, with very careful work to get so near. This difference in the character of lines is to be taken into the account in balancing the survey. Note that the resultant error is always minus.

298. The Temperature Correction.—The temperature of the tape at the time when the work is done affects the result. This is not the temperature in the shade that day, nor the

* This question is fully discussed in Art. 347, Chapter XIV., where the correction is found in terms of the difference in elevation of the two ends.—J. B. J.

reading at the nearest signal station, but is the temperature out on the line, under the conditions which exist there. A grass-covered slope, descending away from the sun, will often show at the same time as much as twenty or thirty degrees lower temperature than a bare hillside inclining toward the sun. The thermometer is needed with the work. If the co-efficient of expansion is not known, use 0.0000065 for 1° F.

It is very desirable in a city-surveyor's work that he be able to apply his corrections at once while in the field. If he goes out to measure any given distance, he must be able to fix his starting-point and drive his stake at the finish. If the weather is hot or cold, he knows what it differs from the temperature at which his tape is tested, and applies the correction at once to the whole distance. He watches that the pull is right, that the tape is kept horizontal, that the work stops when the wind is too severe, and that the checks show the desired accuracy.

299. **Checks.**—Every piece of work should be carried on till it checks upon other work, verifying its accuracy within desired limits. This method ties up every survey at both ends. In order to be prepared to do this expeditiously, the surveyor should lay out general lines which should be joined into a system embracing the town-site. The lines of leading streets and the boundary-lines of additions give most valuable information when made parts of such a system. This borders on the geodetic idea, but it will generally be impracticable to determine the lengths of these lines by triangulation from a measured base, for the stations can very rarely be so chosen that the angles can be measured upon the whole length of the lines, or the diagonals be observed at all. Still, the angles should be measured upon the best base practicable. Permanent buildings and existing monuments showing the lines of intersecting streets should be noted both for line and distance.

MISCELLANEOUS PROBLEMS.

300. The Improvement of Streets involves—(1) The estimation of the earthwork in the grading and shaping of the street. (2) The location of the improvements along the lines of the dedicated streets. City ordinances usually prescribe a certain width of sidewalks and roadway for each width of street. (3) The location of improvements at the grade fixed by ordinance. (4) The estimation of materials furnished by contractors and used in the work. The position of monuments which will be disturbed during the progress of the work is preserved by witness-stakes driven beyond the limits of disturbance. When this precaution is neglected it results in all sorts of angles and offsets in the curb-lines, in cases where there is surplus or deficiency in the original survey. Take a case improved one block at a time, where the first block is established by record distance from the right, the second block by record distance from the left, and a third by running from this last point to a point established at the end of the third block by measuring again from the right, etc. The resulting lines of curb will not give a suggestion of where the street was laid out. Some surveyors are accustomed to replace from their witness-stakes the monuments on the new grade. Such a practice is certainly to be commended; the small cost to the public treasury can well be borne for the public good.

301. Permanent Bench-marks.—In order to secure accuracy and uniformity in elevations throughout a city, bench-marks are established by running lines of levels radiating from the directrix, and checking the work by cross-lines at convenient intervals, these cutting the whole territory into small parcels, so that a standard bench-mark will never be far from any work which must be done.* This work is carried on as far as

* These various lines of levels will form a network, such as that shown in Art. 407, Chapter XIV., which should be adjusted once for all as described in that chapter.

grades are established, and generally as far as the city officers are prepared to propose grades for adoption by ordinance. There is a view of what constitutes or is essential to accurate methods which would make every piece of work start from first principles, so that it may not depend in any way upon errors involved in work previously done. But work done on this plan does not have to be extended very far before the results will show plainly that there is a wide margin between the uniformity attained and the accuracy attempted.

302. **The Value of an Existing Monument** is based (1) on the fact that it corresponds in character and position to a monument described on the recorded plat; (2) on the custom to place monuments upon the completion of a survey, and on the supposition that this monument in question was set in pursuance of such custom, although no monuments are noted on the plat; (3) on recognition by surveyors and owners of land affected by it; (4) on the knowledge that it was placed by a competent surveyor at a time when data were accessible which are not now in existence. The value of the evidence which establishes or tends to establish the reliability of the monument is primarily a question for the judgment of the surveyor. His decision must be reviewed and defended before courts and juries when there is a difference of opinion.

The monument is valueless, or less valuable in all degrees, when there is evidence that it has been disturbed. It sometimes happens that there is no better way to establish a corner than to straighten up a stone which is leaning, but has not been thrown entirely out of the ground. Inquiry often brings out the fact that a stone, after being completely out of the ground, has been reset either by agreement of owners adja-

and so *one* elevation obtained for each bench-mark. It is common for each bench-mark in a city to have numerous elevations differing by several tenths of a foot, and all of about equal credence.—J. B. J.

cent, or by the reckless individual who did the mischief, and is still pointed out as the stone the surveyor set. As a recognized corner such a stone has some value, i.e., it is to be supposed that it is somewhere in the right neighborhood; but if its position can be verified from other points which have not been disturbed the work should be retraced. If the original survey was made in a careless way or the corner-stones were badly set, they may help a careful man to come to an average line which shall correspond with the recorded plat. Monuments are sometimes moved or destroyed maliciously. It is wise for a surveyor to test discreetly everywhere, but to be especially careful where there has been quarrelling about lines.

There is a principle, recognized to some extent by the courts, that the existing monument is the evidence of the original survey, whether or not it is called for by the recorded plat. The custom that the surveyor making the subdivision and the plat for record shall set corner-stones is so far followed that this is generally true, cases of accident, carelessness, and mischief, and such cases as that mentioned below, being somewhat exceptional, but many times very real. It is sometimes attempted to go a step further and affirm that the recorded plat is the record of the survey. This reverses the order of events in most cases, the survey being made in order to mark upon the ground the chief points of a plan already fixed upon; and as to all the main lines, the plat is not altered, however carelessly the survey may be made. There are subdivisions where no monuments were set and where no certain evidence is in existence of how or where the original survey was made, or whether any survey was made at all, and yet there is a recorded plat. A surveyor being called upon to make a survey of some parcel in such a subdivision, sets stones in order to secure recognition for his theory of the proper location. If he does his work carefully he undoubtedly does the public a service. Can any amount of ignorance of when or why these stones were set ever make them evidence of the original survey? In other cases some monuments may be in existence, but more would be convenient,—points are determined from existing monuments in *accordance with the recorded plat* and stones are set. Another surveyor may feel a little nervous about manufacturing this sort of evidence of the original survey, or more likely, may think it too much trouble and a damage to the business, for the more doubt the more work for the surveyor, so drives his stake. Then comes the owner who, desiring to secure a permanent corner, digs a hole about the stake without taking offsets, throws it out, and sets in a stone—an existing monument! This is no fancy sketch, nor are such facts so very rare. The young man who thinks he would like to be a surveyor, but has no eyes nor ears for facts like these, had better turn his attention to some other business. Surveying is an art—not an exact science.*

303. **The Significance of Possession.**—Possession has a value in reëstablishing old lines where all monuments have disappeared. It is a species of perpetuating testimony of their positions. The average of a series of improvements will often give a very close determination of where the corner must have stood. The practised eye accustomed to sharply defined lines, every lot having very nearly its right quantity, which are customary where lines are well established, will notice at once the irregular possession,—gaps between houses, vacant spaces between fences and houses, too little for use, too much for ornament, which may be seen where lines are in doubt and every man expects the next surveyor to make a conflicting survey. Like the men of the present, most men in the past have preferred to be right—have made efforts to be right—have employed surveyors; we can judge where these men in

* Consult Judge Cooley's paper on the Judicial Functions of the Surveyor, Appendix A, and also Art. 194 in chapter on Land Surveying.

the past worked from by seeing where their works are. The legal principle has a bearing here, that "he who would sue to dispossess another must first show a better title." The surveyor who attempts to dispute possession must show better evidence than possession of the right location of the lines he is employed to retrace.

304. Disturbed Corners and Inconsistent Plats.—The work of testing a corner that probably has been disturbed has many points of likeness to the work of reëstablishing corners that have disappeared altogether. The recorded plat is in all cases the basis of the work. When it records the results of a survey it is to be presumed that the surveyor endeavored to do accurate work; hence his work, if not absolutely correct, was probably uniform. Lines which are shown by the plat as straight lines are to be retraced as straight lines. Lines involve less liability to error than measurements, and are first to be considered. Determine as many points as possible by straight lines between existing monuments. Then test the measurements along the extreme lines and the streets which are the basis of the subdivision. If the measurements between undoubted corners agree with the plat so closely, or if they differ so uniformly that the presumption of accurate work is justified, corners that are out of line or out of proportionate distance have the burden of proof against them. He who would claim for them authority must show that they have not been disturbed, and that they are consistent with some rational location. If there was no original survey, that fact is no excuse for careless work at a later time; there is always some place to begin. The case when the recorded plat does not agree with itself presents more difficulties, such as the following: (1) The lines do not give the same points as the distance; (2) The distances disagree among themselves; (3) The monuments disagree with both lines and distances impartially, or agree with one and disagree with the other, while the general character of the work negatives the supposition that they were ever carefully set. The object to be sought is not to perpetuate forever the blunders of the original survey, but to seek the most rational adjustment of all the evidence, so that all parts may be located with a minimum of conflict, and so that no one shall be able to prove your survey wrong, i.e., show a more reasonable location for any part. A consultation of surveyors before too many conflicting interests have developed is often advantageous.

305. **Treatment of Surplus and Deficiency.***—It is generally a simpler problem to determine in which block differences of measurement, whether surplus or deficiency, belong than it is to know what to do with them in the matter of lot-location. There has never been any theory invented for the treatment of either surplus or deficiency which is able to stand the test of the courts against all combinations of circumstances. A few suggestions with the more probable limitations are all the help that can be offered: every case must be investigated for itself. (1) A distribution of the whole front in proportion to the record distances meets general approval, at least in cases of surplus, until it comes in conflict with possession. This is just the time when an owner of ground wants to know what his rights are, and it is also the time when no surveyor can tell him. A compromise, or the verdict of a petit jury, which passes foreknowledge, are the chief alternatives. The courts say that he who would sue for possession must show a better title. An examination shows that each has a better title than any other to so much ground as the plat assigns to his lot, but that no one has a better title than any other one to any part of the surplus. The surveyor does wisely to take note of possession and make, if he can, such a location as is in accordance with the record, and yet not in conflict with possession. When this is not possible, let the map and certificate of survey be made in such a way that they are simply a state-

* See also Art. 196 in chapter on Land Surveying.

ment of the facts. It is not a surveyor's business to decide legal questions or give judgment in ejectment. (2) Because a suit for surplus will not lie, it has been thought that he who first took possession of the surplus would be secure if he were only careful to take it so that every other one might have his ground. Trouble with this view arises because it is not possible to locate the surplus. When one man has appropriated all there is in the block, and the rest but one have appropriated each his proportionate share, then comes the last man. The more surplus in the block the more he is deficient; he wants his ground, and he finds it easier to sue the one man than the twenty. Perhaps, in order to be sure of a case, he had better sue them all. The cases which arise in practice take on an infinite variety of complications and are not usually so simple as these described. (3) The fact is, that the idea that a subdivision ought to have a little surplus is irrational. The work should be so close to the standard that the surveyor who retraces the lines would testify: "According to the best of my knowledge and belief, there is neither surplus nor deficiency there. In retracing my own work, which is carefully executed, I observe as great discrepancies as any which I find in this subdivision, and I conclude that the small difference which I observed in this case was as likely to have been an error in my own work as to attach to the subdivision." (4) Deficiency would seem to be easier to deal with than surplus; for when the last man has not his ground he has a valid claim against the original owner for a rebate on the purchase-price. But the burden of the difficulty in this case falls on the surveyor. When a man brings his deed and asks a survey of lot 9, while 8 and 10 are unsold and lots 1 to 7 are already in possession, he leaves lot 8 its ground and the deficiency in lot 10. Suppose it turns out that lot 10 is next sold, and that the surveyor reports it deficient, the seller, when waited on, may reply, "I have not sold more ground in the block than I owned; the surveyor has made a mistake in locating lot 9." This liability attaches to every location which is made before every lot, between the one located and one corner of the block, is sold. (5) It is practicable for the original owner to so write his deeds as to locate surplus or deficiency. By beginning all deeds at the record distance from one street and continuing this uniformly through the block, the difference goes in the lot farthest from the starting-point; or he may continue the process up to any line which he may choose, and work from the other end of the block in deeding the remaining lots; then the difference falls upon the line chosen and falls to the share of the lot abutting upon that line which is last deeded. But to approve this method is to affirm the practicability of absolute accuracy in work. No one can tell how small a difference may cause trouble.

306. The Investigation and Interpretation of Deeds* for the use of the land-surveyor, dealing with the harmony or conflict of the descriptions, is entirely a different work from that of the investigator of titles, which deals with the legal completeness of the conveyance. In the older parts of a town the deed of the present proprietor frequently does not give information sufficient to fix the correct location. The key may lie in some boundary in an early deed referring to a still earlier conveyance of adjacent property. Or the earlier deeds may give clearly defined locations, while the latter ones say "more or less" at every point. In some cases the deeds are in such a condition that it is impossible to tell what they mean until it is known what the possession is. Skill in this work can only come after considerable experience; local practices must largely determine what is necessary.

307. Office Records.—The surveyor's office when well planned is so arranged that no item of information which promises to be useful shall be lost. The customary methods of indexing, and of block-plats for keeping notes, do not take a very firm hold on general lines or the connections between

* See also Art. 194, Chapter VII.

subdivisions; they fail, in fact, in that part of the work which has the most vital relation to efforts at future improvement. It is advisable to add to the block-plats and indexes a general atlas of the whole town for office use, at a scale of say 100′ to the inch, so that an area nearly half a mile square may appear on the open pages. Such an atlas may show the notes of the general lines and their angles, the base-line measurements, the relation of subdivisions to one another, and a variety of other information which it is difficult to pick out in the widely scattered field-notes which first gathered the information, and which, with their larger scale, the block-plats are not well adapted to show in a connected form.

There are filed in connection with deeds many plats which do not appear on the record plat-books of the recorder's office; these need to be indexed, or, better, abstracted for office use.

The field-notes, when prepared for the surveyor's use in the field, should show in an accessible and portable form all the information which the office contains and which is relevant to the survey in hand. Labor spent beforehand in a thorough preparation of accessible information is labor saved.

308. **The Preservation of Lines** after the monuments have disappeared is accomplished by means of notes on buildings, marks and notes on curbing, paving, fences, etc. Notes on buildings describe not only the character of the building, but the particular part noted, so that another man, years afterward, using the same note would have no doubt of the identity of the part. In a growing town the work of keeping up the notes goes on without ceasing,—buildings are remodelled or rebuilt, streets reconstructed, destroying old marks. The old becomes the new so constantly that the surveyor who would preserve the information which he already has must be constantly employed at the work of renewal. There is no place either in the street or out of it where the surveyor

can place his mark and say to all comers, "Touch not." It follows that whenever it is necessary to use any mark, about the permanence of which there can be a shadow of a doubt, the permanence of the mark must be shown by some practicable test; it is careless to assume it.

309. **The Want of Agreement between Surveyors** arises from differences of information or of judgment, and in a less degree from differences of skill. These are all just as human elements as the lawyer deals with in his work. Testimony is affected by the interests of those who speak, and the judgment varies with the temperament of the individual. Perhaps one of the most difficult lines for a surveyor to draw is that which separates his confidence in his own skill in retracing a survey which was confessedly inaccurate, from his reliance on testimony which is evidently biassed as to the position or disturbance of monuments, and other facts which may help him to form a correct judgment. Errors in execution may be kept within such limits that work which shows differences in closing of 1 in 5000 should be retraced, and the average observed differences in one surveying party's work will not exceed 1 in 20000. Two sets of men working to reach the same standard may err in opposite directions, so that differences between two surveyors may reasonably be expected to be somewhat larger than either would tolerate in his own work.

CHAPTER XIII.

THE MEASUREMENT OF VOLUMES.

310. Proposition.—*The volume of any doubly-truncated prism or cylinder, bounded by plane ends, is equal to the area of a right section into the length of the element through the centres of gravity of the bases, or it is equal to the area of either base into the altitude of the element joining the centres of gravity of the bases, measured perpendicular to that base.*

Let $ABCD$, Fig. 107, be a cylinder, cut by the planes OC and OB, the unsymmetrical right section EF being shown in plan in $E'F'$. Whatever position the cutting planes may have, if they are not parallel they will intersect in a line. This line of intersection may be taken perpendicular to the paper, and the body would then appear as shown in the figure, the line of intersection of the cutting planes being projected at O.

Let $A =$ area of the right section;
$\Delta A =$ any very small portion of this area;
$x =$ distance of any element from O;
then $ax =$ height of any element at a distance x from O.

An elementary volume would then be $ax\Delta A$, and the total volume of the solid would be $\Sigma ax\Delta A$.

Again, the total volume is equal to the mean or average height of all the elementary volumes multiplied by the area of the right section.

The mean height of the elementary volumes is, therefore,

$\frac{\Sigma ax\Delta A}{A} = \frac{a\Sigma x\Delta A}{A}$. But $\frac{\Sigma x\Delta A}{A}$ is the distance from O to the centre of gravity, G, of the right section,* and a times this distance is the height of the element LK through this point. Therefore, the mean height is the height through the centre of

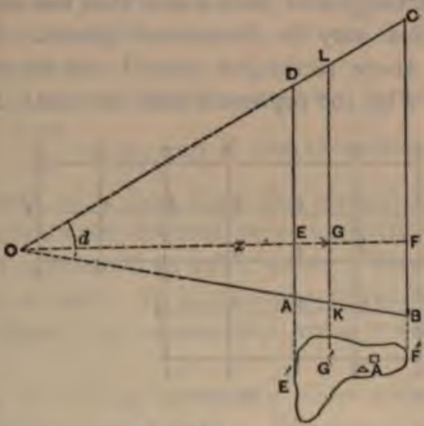

FIG. 107.

gravity of the base, and this into the area of the right section is the volume of the truncated prism or cylinder. The truth of the alternative proposition can now readily be shown.

Corollary. When the cylinder or prism has a symmetrical cross-section, the centre of gravity of the base is at the centre of the figure, and the length of the line joining these centres is the mean of any number of symmetrically chosen exterior elements. For instance, if the right section of the prism be a regular polygon, the height of the centre element is the mean of the length of all the edges. This also holds true for parallelograms, and hence for rectangles. Here the centres of gravity

* This is shown in mechanics, and the student may have to take it for granted temporarily.

of the bases lie at the intersections of the diagonals; and since these bisect each other, the length of the line joining the intersections is the mean of the lengths of the four edges. The same is true of triangular cross-sections.

311. Grading over Extended Surfaces.—Lay out the area in equal rectangles of such a size that the surfaces of the several rectangles may be considered planes. For common rolling ground these rectangles should not be over fifty feet on a side. Let Fig. 108 represent such an area. Drive pegs at

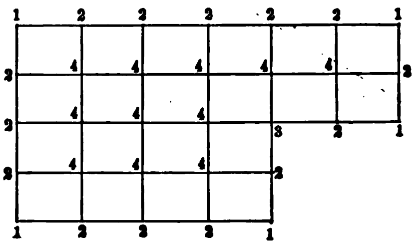

FIG. 108.

the corners, and find the elevation of the ground at each intersection by means of a level, reading to the nearest tenth of a foot, and referring the elevations to some datum-plane below the surface after it is graded. When the grading is completed, relocate the intersections from witness-points that were placed outside the limits of grading, and again find the elevations at these points. The several differences are the depths of excavation (or fill) at the corresponding corners. The contents of any partial volume is the mean of the four corner heights into the area of its cross-section. But since the rectangular areas were made equal, and since each corner height will be used as many times as there are rectangles joining at that corner, we have, in cubic yards,

$$V = \frac{A}{4 \times 27}[\Sigma h_1 + 2\Sigma h_2 + 3\Sigma h_3 + 4\Sigma h_4]. \quad . \quad . \quad (1)$$

The subscripts denote the number of adjoining rectangles the area of each of which is A.

From this equation we may frame a

RULE.—Take each corner height as many times as there are partial areas adjoining it, add them all together, and multiply by one fourth of the area of a single rectangle. This gives the volume in cubic feet. To obtain it in cubic yards, divide by twenty-seven.

If the ground be laid out in rectangles, 30 feet by 36 feet, then $\dfrac{A}{4 \times 27} = \dfrac{1080}{108} = 10$; and if the elevations be taken to the nearest tenth of a foot, then the sum of the multiplied corner heights, with the decimal point omitted, is at once the the amount of earthwork in cubic yards. This is a common way of doing this work. In borrow-pits, for which this method is peculiarly fitted, the elementary areas would usually be smaller.

In general, on rolling ground, a plane cannot be passed through the four corner heights. We may, however, pass a plane through any three points, and so with four given points

FIG. 109.

on a surface either diagonal may be drawn, which with the bounding lines makes two surfaces. If the ground is quite irregular, or if the rectangles are taken pretty large, the surveyor may note on the ground which diagonal would most nearly fit the surface. Let these be sketched in as shown in Fig. 109. Each rectangular area then becomes two triangles, and when computed as triangular prisms, each corner height at the end of a diagonal is used twice, while the two other corner heights are used but once. That is, twice as much weight is given to the corner heights on the diagonals as to the others. In Fig. 109, the same area as that in Fig. 108 is shown with the diagonals drawn which best fit the surface of the ground. The numbers at the corners indicate how many times each height is to be used. It will be seen that each height is used as many times as there are triangles meeting at that corner. To derive

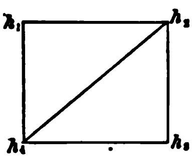

Fig. 110.

the formula for this case, take a single rectangle, as in Fig. 110, with the diagonal joining corners 2 and 4. Let A be the area of the rectangle. Then from the corollary, p. 395, we have for the volume of the rectangular prism, in cubic yards,

$$V = \frac{A}{2 \times 27}\left(\frac{h_1 + h_2 + h_4}{3} + \frac{h_2 + h_3 + h_4}{3}\right)$$

$$= \frac{A}{6 \times 27}(h_1 + 2h_2 + h_3 + 2h_4). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For an assemblage of such rectangular prisms as shown in Fig. 109, the diagonals being drawn, we have, in cubic yards,

$$V = \frac{A}{6 \times 27}[\Sigma h_1 + 2\Sigma h_2 + 3\Sigma h_3 + 4\Sigma h_4 + 5\Sigma h_5$$

$$+ 6\Sigma h_6 + 7\Sigma h_7 + 8\Sigma h_8]; \quad . \quad . \quad . \quad (3)$$

where A is the area of one rectangle, and the subscripts denote the number of triangles meeting at a corner.

As a check on the numbering of the corners, Fig. 109, add them all together and divide by six. The result should be the number of rectangles in the figure. In this case, if the rectangles be taken 36 feet by 45 feet, or, better, 40 feet by 40.5 feet, then the sum of the multiplied heights with the decimal point omitted is the number of cubic yards of earthwork, the corner heights having been taken out to tenths of a foot.

The method by diagonals is more accurate than that by rectangles simply, the dimensions being the same; or, for equal degrees of exactness larger rectangles may be used with diagonals than without them, and hence the work materially reduced. In any case some degree of approximation is necessary.

312. Approximate Estimates by means of Contours.— (*A*) Whenever an extended surface of irregular outline is to be graded down, or filled up to a given *plane* (not a warped or curved surface), a near approximation to the amount of cut or fill may be made from the contour lines. In Fig. 111 the full curved lines are contours, showing the original surface of the ground. Every fifth one is numbered, and these were the contours shown on the original plat. Intermediate contours one foot apart have been interpolated for the purpose of making this estimate. The figures around the outside of the bounding lines give the elevations of those points after it is graded down. The straight lines join points of equal elevation after grading; and since this surface is to be a plane these lines are surface or contour lines after grading. Wherever these two sets of contour lines intersect, the difference of their elevations is the depth of cut or fill at that point. If now we join the points of equal cut or fill (in this case it is all in cut), we obtain a new set of curves, shown in the figure by dotted lines, which may be used for estimating the amount of earthwork. The dotted boundaries are the horizontal projections of the traces on the natural surface of planes parallel to the final graded surface which are uniformly spaced one foot apart vertically. These projected areas are measured by the planimeter and called A_1, A_2, A_3, etc. Each area is bounded by the dotted line and the bounding lines of the figure, since on these

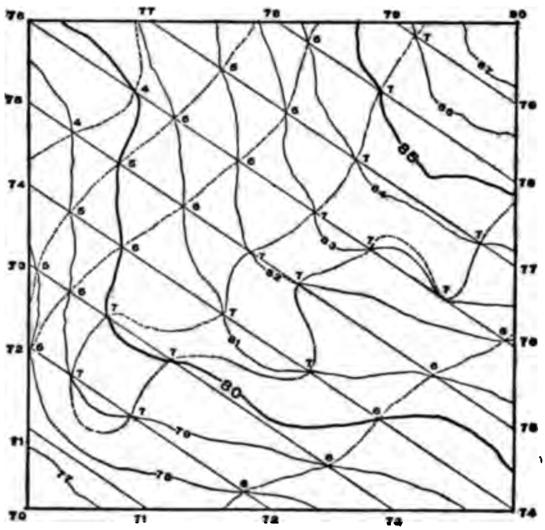

Fig. 111.

bounding lines all the projections of all the traces unite, the slope here being vertical. For any two adjoining layers we have, by the prismoidal formula* as well as by Simpson's one-third rule,

$$V_{1-3} = \frac{h}{3}(A_1 + 4A_2 + A_3), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where h is the common vertical distance between the projected areas.

* For the demonstration of the prismoidal formula see Art. 314.

For the next two layers we would have, similarly,

$$V_{3-5} = \frac{h}{3}(A_3 + 4A_4 A_5) ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or for any even number of layers we would have, in cubic yards,

$$V = \frac{h}{3 \times 27}(A_1 + 4A_2 + 2A_3 + 4A_4 + 2A_5 + \quad . \quad . \quad . \quad . \quad A_n), \quad (3)$$

where n is an odd number, h and A being in feet and square feet respectively.

(*B*) Whenever the final surface is not to be a plane, but warped, undulating, or built to regular outlines like a fortification, a reservoir embankment, or terraced grounds, a different method should be employed.

In the former method the areas bounded by the dotted lines were areas cut out by planes parallel to the final plane surface, passed one foot apart *vertically*. But since the map shows only the *horizontal projections* of these planes, these projections, multiplied by the vertical distance between them, would give the true volumes.

When the final surface is not to be a plane, proceed as follows: First make a careful contour map of the ground. Then lay down on this map a system of contour lines, corresponding in elevation to the first set of contours, but in a different colored ink, which will accurately represent the final surface desired. This second set of contours would be a series of straight lines if a regular surface, composed of plane faces, was to be constructed, but would be curving lines if the ground were to be brought to a final curving or undulating surface.

The closed figures bounded by the two sets of intersecting contours of the same elevation are *horizontal* areas of cut or fill, separated by the common vertical distance between contours. The volumes here defined are oblique solids bounded by horizontal planes at top and bottom, and are a species of prismoid. The volume of one of these prismoids is found by applying the prismoidal formula to it, finding the end areas by means of a planimeter, and taking the length as the

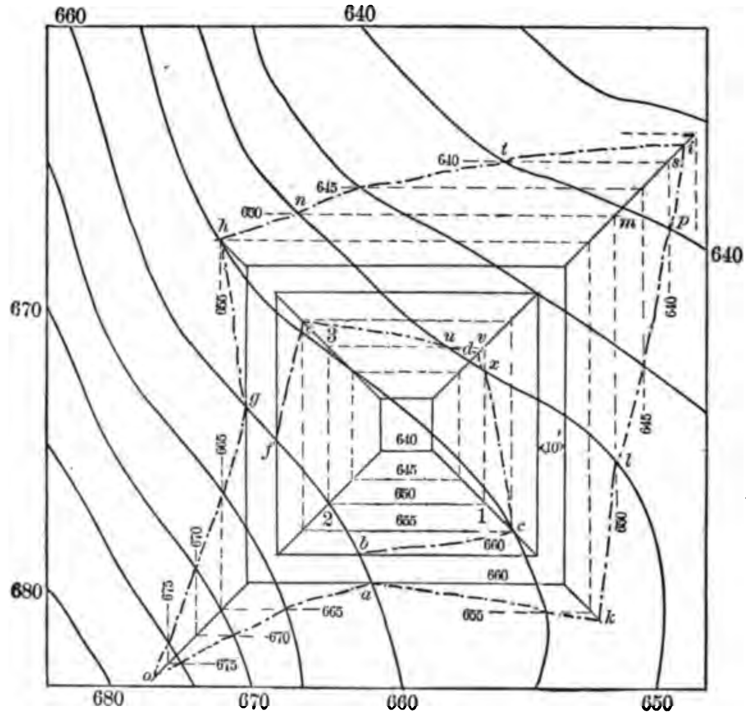

Fig. IIIa.

vertical distance between contours. If the contours be drawn close enough together, then each alternate contour-area may be used as a middle area, and the length of the prismoid taken at twice the vertical distance between contours; or the volume may be computed by either of the formulas (12), (13), (14), or (15) of Appendix C, where the h's would here become the end areas and l the vertical distance between contours.

Example: Let it be required to build a square reservoir on a hillside, which shall be partly in excavation and partly in embankment, the ground being such as shown by the full contour lines in Fig. 111*a*.*

The contours, for the sake of simplicity and brevity, are spaced five feet apart. The top of the wall, shown by the full lines making the square, is 10 feet wide and at an elevation of 660 feet. The reservoir is 20 feet deep, with side slopes, both inside and outside, of two to one, making the bottom elevation 640 feet, and 20 feet square, the top being 100 feet square on the inside. The dotted lines are contours of the finished slopes, both inside and out, at elevations shown on the figure. The areas in fill all fall within the broken line marked *a b c d e f g h i k*, and the cut areas all fall within the broken line marked *a b c d e f g o*. These broken lines are grade lines. The horizontal sectional areas in fill and cut are readily traced by following the closed figures formed by contours of equal elevation, thus—

At 640 foot level sectional area in fill is *p s t*.
" 650 " " " " " *l m n u v x l*.
" 650 " " " " cut is 1 2 3 *u x*.

The other areas are as easily traced. In the figure the lines have all been drawn in black. In practice they should be drawn in different colors to avoid confusion.

This second method should be used in all cases where the graded area is considerable and the final relief form is not a plane. If the contours be carefully determined and be taken

* This figure is taken from a paper describing the method by Prof. William G. Raymond, University of California.

near enough together, the method will give as accurate results as may be obtained in any other way. The volume may be computed by eq. (3) of this article, where the areas are the horizontal sectional areas bounded by contours of equal elevation, and h is the vertical distance between contours.

When these methods are used for final estimates, the contours should be carefully determined, and spaced not more than two feet apart on steep slopes and one foot apart on low slopes.

313. The Prismoid is a solid having parallel end areas, and may be composed of any combination of prisms, cylinders, wedges, pyramids, or cones or frustums of the same, whose bases and apices lie in the end areas. It may otherwise be defined as a volume generated by a right-line generatrix moving on the bounding lines of two closed figures of any shapes which lie in parallel planes as directrices, the generatrix not necessarily moving parallel to a plane director. Such a solid would usually be bounded by a warped surface, but it can always be subdivided into one or more of the simple solids named above.

Inasmuch as cylinders and cones are but special forms of prisms and pyramids, and warped surface solids may be divided into elementary forms of them, and since frustums may also be subdivided into the elementary forms, it is sufficient to say that all prismoids may be decomposed into prisms, wedges, and pyramids. If a formula can be found which is equally applicable to all of these forms, then it will apply to any combination of them. Such a formula is called

314. The Prismoidal Formula.

Let A = area of the base of a prism, wedge, or pyramid;

A_1, A_m, A_2 = the end and middle areas of a prismoid, or of any of its elementary solids;

h = altitude of the prismoid or elementary solid.

Then we have,
For Prisms,

$$V = hA = \frac{h}{6}(A_1 + 4A_m + A_2). \quad . \quad . \quad . \quad . \quad (1)$$

For Wedges,

$$V = \frac{hA}{2} = \frac{h}{6}(A_1 + 4A_m + A_2). \quad . \quad . \quad . \quad . \quad (2)$$

For Pyramids,

$$V = \frac{hA}{3} = \frac{h}{6}(A_1 + 4A_m + A_2). \quad . \quad . \quad . \quad . \quad (3)$$

Whence for any combination of these, having all the common altitude h, we have

$$V = \frac{h}{6}(A_1 + 4A_m + A_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is the prismoidal formula.

It will be noted that this is a rigid formula for all prismoids. The only approximation involved in its use is in the assumption that the given solid may be generated by a right line moving over the boundaries of the end areas.

This formula is used for computing earthwork in cuts and fills for railroads, streets, highways, canals, ditches, trenches, levees, etc. In all such cases, the shape of the figure above the natural surface in the case of a fill, or below the natural surface in the case of a cut, is previously fixed upon, and to complete the closed figure of the several cross-section areas only the outline of the natural surface of the ground at the section remains to be found. These sections should be located so near together that the intervening solid may fairly be assumed to be a prismoid. They are usually spaced 100 feet apart, and then intermediate sections taken if the irregularities seem to require it.

The *area* of the middle section is never the mean of the two end areas if the prismoid contains any pyramids or cones among its elementary forms. When the three sections are similar in form, the *dimensions* of the middle area are always the means of the corresponding end dimensions. This fact often enables the dimensions, and hence the area of the middle section, to be computed from the end areas. Where this cannot be done, the middle section must be measured on the ground, or else each alternate section, where they are equally spaced, is taken as a middle section, and the length of the prismoid taken as twice the distance between cross-sections. For a continuous line of earthwork, we would then have, in cubic yards,

$$V=\frac{l}{3\times 27}(A_1+4A_2+2A_3+4A_4+2A_5+4A_6\ .\ .\ .\ +A_n),\ \ . \quad (1)$$

where l is the distance between sections in feet. This is the same as equation (3), p. 417. Here the assumption is made that the volume lying between alternate sections conforms sufficiently near to the prismoidal forms.

315. Areas of Cross-sections.—In most cases, in practice at least, three sides of a cross-section are fixed by the conditions of the problem. These are the side slopes in both cuts and fills, the bottom in cuts and the top in embankments, or fills. It then remains simply to find where the side slopes will cut the natural surface, and also the form of the surface line on the given section. Inasmuch as stakes are usually set at the points where the side slopes cut the surface, whether in cut or fill, such stakes are called slope-stakes, and they are set at the time the cross-section is taken. The side slopes are defined as so much horizontal to one vertical. Thus a slope of $1\frac{1}{2}$ to 1 means that the horizontal component of a given portion of a slope-line is $1\frac{1}{2}$ times its vertical component, the horizontal component always being named first. The *slope-ratio* is the ratio of the horizontal to the vertical component, and is therefore always the same as the first number in the slope-definition. Thus for a slope of $1\frac{1}{2}$ to 1 the slope-ratio is $1\frac{1}{2}$.

316. The Centre and Side Heights.—The centre heights are found as follows: Place in one column of the note-book the surface elevation of the ground at the centre stake, as given in the level book. Then take off from the profile the elevation of the points of *change of grade* only, and compute the elevation of grade at each station, from the known distance and grade. Place these elevations of grade in a column alongside the first. Then take the differences and put in a third column as the centre heights. These centre heights, together with the width of base and side slopes in cuts and in fills, are the necessary data for fixing the position of the slope-stakes. When these are set for any section as many points on the surface line joining them may be taken as desired. In ordinary rolling ground usually no intermediate points are taken, the centre point being already determined. In this case three points in the surface line are known, both as to their distance out from the centre line and as to their height above the grade line. Such sections are called "three-level sections," the surface lines being assumed straight from the slope-stakes to the centre stake.

317. The Area of a Three-level Section.

Let d and d' be the distances out, and

h and h' the heights above grade of right and left slope-stakes, respectively;

D the sum of d and d', c the centre height, r the slope-ratio, w the width of bed.

Then the area $ABCDE$ is equal to the sum of the four triangles AEw, BCw, wCD, and wED. Or,

$$A = \frac{(d + d')c + (h + h')\frac{w}{2}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This area is also equal to the sum of the triangles FCD and FED, minus the triangle AFB. Or,

$$A = \left(c + \frac{w}{2r}\right)\frac{D}{2} - \frac{w^2}{4r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

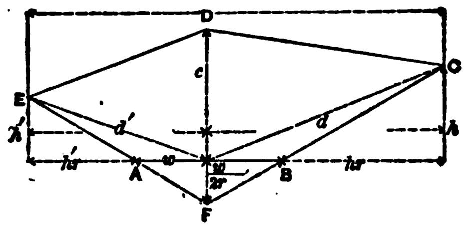

FIG. 112.

Equation (2) can also be obtained directly from equation (1) by substituting for h and h' in (1) their values in terms of d and w, $h = \frac{d - \frac{w}{2}}{r}$, and then putting $D = d + d'$. Equation (2) has but two variables, c and D, and is the most convenient one to use.

318. Cross-sectioning.—It will be seen from Fig. 112 that there are three elevations to be determined above (or below) grade, and two distances out to be determined. A regular line of levels is carried, checking on all pre-established benches. At each position of instrument from which slope-stakes are to be set, the "height of instrument" is taken out, and the difference between this and the "elevation of grade" figured for the several sections, the "elevation of grade" having been taken from the profile, and already entered up for all stations and "grade points." By holding the rod at the section on line and taking a reading, and subtracting this from the "height of instrument," we obtain "elevation of profile" at that section; subtracting this from the "elevation of grade" for that section in fills, and *vice versa* in cuts, we obtain the amount of fill or cut, which can be roughly checked from the profile itself, if desired. The Railroad Field Books usually give forms for keeping these notes. If the ground were level transversely, the distance to the slope-stakes would be

$$d = cr + \frac{w}{2}$$

But this is not usually the case, and hence the distance out must be found by trial. If the ground slopes $\left\{\begin{matrix}\text{down}\\ \text{up}\end{matrix}\right\}$ from the centre line in a $\left\{\begin{matrix}\text{fill}\\ \text{cut}\end{matrix}\right\}$ the distance out will evidently be more than that given by the above equation, and *vice versa*. The rodman estimates this distance, and holds his rod at a certain measured distance out, d_1. The observer reads the rod, and deducts the reading from the height of instrument above grade (or adds it to the depth of instrument below grade), and this gives the height of that point, h_1, above or below grade. Its distance out, then, *should* be $d = h_1 r + \frac{w}{2}$. If this be more than the actual distance out, d_1, the rod is set farther out; if less, it is moved in. The whole operation is a very simple one in practice, and the rodman soon becomes very expert in estimating nearly the proper position the first time.

In heavy work—that is, for large cuts or fills, and for irregular ground—it may be necessary to take the elevation and distance out of other points on the section in order to better determine its area. These are taken by simply reading on the rod at the critical points in the outline, and measuring the distances out from the centre. The points can then be plotted on cross-section paper and joined by straight or by free-hand curved lines. In the latter case the area should be determined by planimeter.

319. Three-level Sections, the Upper Surface consisting of two Warped Surfaces.—If the three longitudinal lines joining the centre and side heights on two adjacent three-level sections be used as directrices, and two generatrices, one on each side the centre, be moved parallel to the end areas as plane directers, two warped surfaces are generated, every cross-section of which parallel to the end areas is a three-level section. These same surfaces could be generated by two longitudinal generatrices, moving over the surface end-area lines as directrices. The surface would therefore be a prismoid, and its exact volume would be given by the prismoidal formula. *The middle area* in this case is readily found, since the center and side heights are the means of the corresponding end dimensions.

The prismoidal formula, giving volumes in cubic yards,

$$V = \frac{l}{6 \times 27}(A_1 + 4A_m + A_2), \quad . \quad . \quad . \quad . \quad (1)$$

could therefore be written

$$V = \frac{l}{12 \times 27}\left[\left(c_1 + \frac{w}{2r}\right)D_1 + \left(c_2 + \frac{w}{2r}\right)D_2 + 4\left(c_m + \frac{w}{2r}\right)D_m\right] - \frac{lw^2}{4 \times 27r}. \quad . \quad . \quad (2)$$

This equation is derived directly from eq. (1) above, and eq. (2), p. 424. The quantity $\frac{w}{2r}$ is the distance from the grade-plane to the intersection of the side slopes, and is a constant for any given piece of road. It would have different values, however, in cuts and fills on the same line.

For brevity, let

$$\frac{w}{2r} = c_0; \quad \text{and} \quad \frac{lw^2}{4 \times 27r} = \frac{lwc_0}{54} = K.$$

Here K is the volume of the prism of earth, 100 feet long, included between the roadbed and side slopes. It is first included in the computation and then deducted. It is also a constant for a given piece of road.

Equation (2) now becomes

$$V = \frac{l}{12 \times 27}[(c_1+c_0)D_1+(c_2+c_0)D_2+4(c_m+c_0)D_m]-K, \quad . \quad (3)$$

where c_m and D_m are the means of c_1c_2 and D_1D_2, respectively.

This equation involves but two kinds of variables, c and D, and is well adapted to arithmetical, tabular, or graphical computation. Thus if $l = 100$; $w = 18$; and $r = 1\frac{1}{2}$; then $c_0 = 6$; and $K = 200$; and equation (3) becomes

$$V = \tfrac{100}{324}[(c_1+6)D_1+(c_2+6)D_2+4(c_m+6)D_m]-200 \quad . \quad (4)$$

If the total centre heights (to intersection of side slopes) be represented by C_1, C_2, and C_m, then eq. (3) becomes, in general,

$$V = K'(C_1D_1 + C_2D_2 + 4C_mD_m) - K, \quad . \quad . \quad . \quad (5)$$

where $K' = \frac{100}{324}$, and is independent of width of bed and of slopes.

For any given piece of road, the constants K, K', and c_0 are known, and for each prismoid the C's and D's are observed, hence for any prismoid all the quantities in eq. (5) are known.

320. Construction of Tables for Prismoidal Computation.—If a table were prepared giving the products $K'CD$ for various values of C and D, it could be used for evaluating equation (3), which is the same as equation (5). The arguments would be the total widths (D_1), and the centre heights (C_1). Such a table would have to be entered three times for each prismoid, first with C_1 and D_1; second with C_2 and D_2; and finally with C_m and D_m. If four times the last tabular value be added to the sum of the other two, and K subtracted, the result is the true volume of the prismoid.

VALUES OF $c_0 \left(= \frac{w}{2r}\right)$ AND $K \left(= \frac{lw^2}{4 \times 27r}\right)$ FOR VARIOUS WIDTHS AND SLOPES.

Width of Road-bed.	Slopes.															
	¼ to 1.		½ to 1		¾ to 1.		1 to 1.		1¼ to 1.		1½ to 1.		1¾ to 1.		2 to 1.	
	C_0	K	C_0	K	C_0	K	C_0	K	C_0	K	C_0	K	C_0	K	C_0	K
10	20	370	10	185	6.7	123	5.0	93	4.0	74	3.3	62	2.9	53	2.5	46
11	22	448	11	224	7.3	149	5.5	112	4.4	90	3.7	75	3.1	64	2.8	56
12	24	533	12	266	8.0	178	6.0	133	4.8	107	4.0	89	3.4	76	3.0	67
13	26	626	13	313	8.7	209	6.5	157	5.2	125	4.3	104	3.7	89	3.2	78
14	28	725	14	363	9.3	242	7.0	181	5.6	145	4.7	121	4.0	104	3.5	91
15	30	833	15	417	10.0	278	7.5	208	6.0	167	5.0	139	4.3	119	3.8	104
16	32	948	16	474	10.7	316	8.0	237	6.4	190	5.3	158	4.6	135	4.0	118
17	34	1070	17	535	11.3	357	8.5	268	6.8	214	5.7	178	4.9	153	4.2	134
18	36	1200	18	600	12.0	400	9.0	300	7.2	240	6.0	200	5.1	171	4.5	150
19	38	1337	19	668	12.7	446	9.5	334	7.6	267	6.3	223	4.4	191	4.8	167
20	40	1481	20	740	13.3	494	10.0	370	8.0	296	6.7	247	5.7	212	5.0	185
21	42	1633	21	816	14.0	544	10.5	408	8.4	327	7.0	272	6.0	233	5.2	204
22	44	1793	22	896	14.7	598	11.0	448	8.8	359	7.3	299	6.3	256	5.5	224
23	46	1959	23	980	15.3	653	11.5	490	9.2	392	7.7	326	6.6	280	5.8	245
24	48	2134	24	1067	16.0	711	12.0	534	9.6	427	8.0	356	6.9	305	6.0	267
25	50	2315	25	1158	16.7	772	12.5	579	10.0	463	8.3	386	7.1	331	6.2	264
26	52	2504	26	1252	17.3	835	13.0	626	10.4	501	8.7	417	7.4	358	6.5	313
27	54	2700	27	1350	18.0	900	13.5	675	10.8	540	9.0	450	7.7	386	6.8	338
28	56	2004	28	1452	18.7	968	14.0	726	11.2	581	9.3	484	8.0	415	7.0	363
29	58	3115	29	1558	19.3	1038	14.5	779	11.6	623	9.7	519	8.3	445	7.2	389
30	60	3333	30	1667	20.0	1111	15.0	833	12.0	667	10.0	556	8.6	476	7.5	417

Table XI.* is such a table, computed for total centre heights from 1 to 50 feet, and for total widths from 1 to 100 feet. In railroad work neither of these quantities can be as small as one foot, but the table is designed for use in all cases where the parallel end areas may be subdivided into an equal number of triangles or quadrilaterals.

EXAMPLE 1. *Three-level Ground having two Warped Surfaces.*—Find the volume of two prismoids of which the following are the field-notes, the width of bed being 20 feet, and the slopes $1\frac{1}{2}$ to 1.

Station 11.	$\frac{28.9†}{+12.6}$	$\frac{0}{+18.6}$	$\frac{43.0}{+22.0}$
Station 12.	$\frac{27.1}{+11.4}$	$\frac{0}{+14.8}$	$\frac{40.3}{+20.2}$
Station 12 + 56.	$\frac{24.3}{+9.5}$	$\frac{0}{+10.3}$	$\frac{34.9}{+16.6}$

From the table, p. 428, giving values of C_0 and K, we find for $w = 20$, and $r = 1\frac{1}{2}$, $C_0 = 6.7$, and $K = 247$.

The computation may be tabulated as follows:

Sta.	Width, $D=d+d'$.	Height, $C=c+c_0$.	Partial Volume.	Volume of Prismoid.
11	71.9	25.3	562	
M	69.6	23.4	$503 \times 4 = 2012$	
12	67.4	21.5	447	
			$3021 - 247$	2774
M	63.3	19.2	$374 \times 4 = 1496$	
12 + 56	59.2	17.0	311	
			$.56\,(2254 - 247)$	1124

* Modeled somewhat after Crandall's Tables, but adapted to give volumes by the Prismoidal Formula at once instead of by the method of mean end areas first and correcting by the aid of another table to give prismoidal volumes, as Prof. Crandall has done.

† The numerators are the distances out, and the denominators are the heights above grade, + denoting cut and − fill.

Entering the table (No. XI.) for a width of 71 and a height of 25, we find 548, to which add 7 for the 3 tenths of height, and 7 more for the 9 tenths in width, both mentally, thus giving 562 cu. yds. for this partial volume. Similarly for the width 67.4, and height 21.5, obtaining 447 cu. yds. The corresponding result for the middle area is 503, which is to be multiplied by 4, thus giving 2012 cu. yds. The sum of these is 3021 cu. yds., from which is to be subtracted the constant volume K, which in this case is 247 cu. yds., leaving 2774 cu. yds. as the volume of the prismoid.

The next prismoid is but 56 feet long, but it is taken out just the same as though it were full, and then 56 hundredths of the resulting volume taken. The data for the 12th station is used in getting this result without writing it again on the page.

EXAMPLE 2. *Five-level Ground having four Warped Surfaces.*—Find the volume of a prismoid of which the following are the field-notes, the width of bed being 20 feet, and the slopes $1\frac{1}{2}$ to 1:

11.	$\frac{28.9}{+12.6}$	$\frac{15.0}{+12.0}$	$\frac{0}{+18.6}$	$\frac{20.0}{+21.0}$	$\frac{43.0}{+22.0}$
12.	$\frac{27.1}{+11.4}$	$\frac{12.5}{+12.0}$	$\frac{0}{+14.8}$	$\frac{18.5}{+19.6}$	$\frac{40.3}{+20.2}$

This is the same problem as the preceding, with intermediate heights added.

To compute this from the table, it is separated into three prismoids, as shown in Fig. 113.

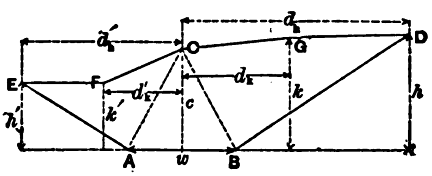

FIG. 113.

Let $ABDGCFE$ be the cross-section. This may be separated into the triangle ABC, and the two quadrilaterals $BCGD$ and $ACFE$. The area of the triangle is $\frac{1}{2}cw$. That of the right quadrilateral is, from Art. 184, p. 209.

$$\tfrac{1}{2}\left[c\left(d_k - \frac{w}{2}\right) + k(d_h - 0) + h\left(\frac{w}{2} - d_k\right)\right] = \tfrac{1}{2}\left[(c-h)\left(d_k - \frac{w}{2}\right) + kd_h\right].$$

Similarly the area of the left quadrilateral is $\tfrac{1}{2}\left[(c-h')\left(d'_k - \frac{w}{2}\right) + k'd'_h\right]$.

The total area of the section then is

$$A = \tfrac{1}{2}\left[(c-h')\left(d'_k - \frac{w}{2}\right) + k'd'_h + cw + kd_h + (c-h)\left(d_k - \frac{w}{2}\right)\right]. \quad . \quad . \quad (1)$$

If the interior side elevations be taken over the edges of the base, then $d'_k - \frac{w}{2}$ and $d_k - \frac{w}{2}$ both become zero, and the first and last terms disappear. Or if the centre and extreme side heights are the same, these terms go out. Experience shows that these terms can usually be neglected without material error. If they are retained, each partial volume will be composed of five terms, while if they are neglected there will be but three. The signs of these terms also must be carefully attended to. When the interior side readings are taken over the edges of the base, therefore, this equation becomes

$$A = \tfrac{1}{2}(k'd'_h + cw + kd_h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The tables are well adapted to compute the prismoidal volume for five-level sections by either of these formulæ. Thus, if the adjacent section also has five points determined in its surface, its area may be represented by an equation similar to one of these, and from these end-area data mean values may be found for the corresponding middle-area points, and the volumes taken out as before. In this case the prism included between the road-bed and side-slopes, whose volume is K', is not included, and hence its volume is not to be deducted from the result. The computation by table XI. of equation (1) would be as follows:

Sta.	h'.	d'_h.	k'.	d'_k.	c.	d_k.	k.	d_h.	h.	Partial Volumes.	Total Volume.
11	12.6	28.9	12.0	15.0	18.6	20.0	21.0	43.0	22.0	+9+108+114+279−10 = 500	
M	12.0	28.0	12.0	13.8	16.7	19.2	20.3	41.6	21.1	4(+6+104+102+260−12)=1840	
12	11.4	27.1	12.0	12.5	14.8	18.5	19.6	40.3	20.2	+3+100+ 90+242−13 = 422	2762

The use of the table is the same as before. First take out from the table the volume corresponding to $(c - h')\left(d'_k - \frac{w}{2}\right)$, which when evaluated for section 11 is $(18.6 - 12.6)(15.0 - 10) = 6.0 \times 5.0$. This is positive, and the volume corresponding to a depth of 6.0 feet and a width of 5.0 feet is 9 cubic yards. Proceed to evaluate the remaining terms of eq. (1) in a similar manner, the last term coming out negative. The dimensions of the mid section are the means of the corresponding end dimensions, as before. If one end-area is a three-level section and the next a five-level section, the included prismoid is computed as a five-level prismoid, the vanishing points in the three-level section corresponding to the interior side elevations on the five-level section being indicated in the field. Partial stations, or prismoids, are first computed as though they were 100 feet long (for which the table is constructed), and then multiplied by their length and divided by 100 as before.

If equation (2) may be used, the work is shortened very much. The columns in h', d'_k, d_k, and h, may be omitted, and there will also be but three terms in each partial product. Thus, if sections 11 and 12 had been taken with the interior elevations, each 10 feet from the centre line, we might have had something as follows:

$$11. \quad \frac{28.9}{+12.6} \quad \frac{10.0}{+15.4} \quad \frac{0}{+18.6} \quad \frac{10.0}{+19.8} \quad \frac{43.0}{+22.0}$$

$$12. \quad \frac{27.1}{+11.4} \quad \frac{10.0}{+12.5} \quad \frac{0}{+14.8} \quad \frac{10.0}{+17.4} \quad \frac{40.3}{+20.2}$$

The computation then, by eq. (2), would have been:

Sta.	d'_h.	h'.	c.	h.	d_h.	Partial Volumes.	Total Volume.
11	28.9	15.4	18.6	19.8	43.0	137 + 114 + 263 = 514	
M	28.0	14.0	16.7	18.6	41.6	4 (121 + 102 + 239) = 1848	
12	27.1	12.5	14.8	17.4	40.3	104 + 90 + 215 = 409	2771

By this method the computation of a five-level section is little more trouble than that of a three-level section, and yet the intermediate points taken at a distance of $\frac{w}{2}$ from the centre, are apt to increase the accuracy considerably on ordinary rolling ground.

321. Three-level Sections, the Surface divided into four Planes by Diagonals.—If the surface included between two three-level sections be assumed to be made up of four planes formed by joining the centre height at one end with a side height at the other end section on each side the centre line (Fig. 114), these lines being called diagonals, an exact computation of the volume is readily made without computing the mid-area. Two diagonals are possible on each side the centre line but the one is drawn which is observed to most nearly fit the surface. They are noted in the field when the cross-sections are taken.

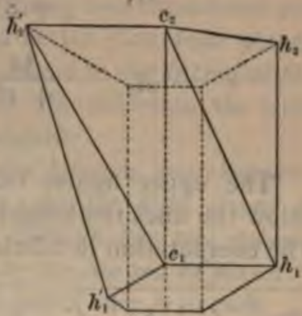

FIG. 114.

The total volume of such a prismoid in cubic* yards is

$$V=\frac{l}{6\times 27}\Big[(d_1+d_1')c_1+(d_2+d_2')c_2+DC+D'C'$$

$$+\frac{w}{2}(h_1+h_2+H+h_1'+h_2'+H')\Big],^* \quad (1)$$

where c_1, h_1, and h_1' are the centre and side heights at one section and d_1 and d_1' the distances out, c_2, h_2', h_2, d_2, and d_2' be-

* For a demonstration of this formula see Henck's Field-Book.

ing the corresponding values for the other end section. C and C' are the centre heights, H and H' the side heights, and D and D' the distances out on the right and left diagonals. Although this formula seems long, the computations by it are very simple. Thus let the volume be found from the following field-notes for a base of 20 feet and side slopes $1\frac{1}{2}$ to 1.

$$A_1 \qquad \frac{22}{+8} \qquad \frac{0}{+8} \qquad \frac{47.5}{+25}.$$

$$A_2 \qquad \frac{34}{+16} \qquad \frac{0}{+4} \qquad \frac{16}{+4}.$$

The upper figures indicate the distances out and those below the lines the heights, the plus sign being used for cuts. The computation in tabular form is as follows:

Sta.	d.	h.	c.	h'.	d'.	$d + d'$.	$(d + d')c$.	DC.	$D'C'$.
1	22	8	8	25	47.5	69.5	556	....	....
2	34	16	4	4	16	50.0	200	88	128

$$h_1 + h_2 = 24 \qquad\qquad 88$$

$$H + H' = 12 \qquad\qquad 128$$

$$\frac{w}{2}\Sigma h\text{'s} = 65 \times 10 \qquad = 650$$

$$6\,)\,162200$$

$$27\,)\,27033$$

1001 cu. yards.

The great advantage of the method consists in the data all being at hand in the field-notes.

Hudson's Tables * give volumes for this kind of prismoid.

* Tables for Computing the Cubic Contents of Excavations and Embankments. By John R. Hudson, C.E. John Wiley & Sons, New York, 1884.

They furnish a very ready method of computing volumes when this system is used.

322. Comparison of Methods by Diagonals and by Warped Surfaces.—Although the surveyor has a choice of two sets of diagonals when this method is used, the real surface would usually correspond much nearer the mean of the two pairs of plane surfaces than to either one of them. That is, the natural surface is curved and not angular, and therefore it is probable that two warped surfaces joining two three-level sections would generally fit the ground better than four planes, notwithstanding the choice that is allowed in the fitting of the planes. More especially must this be granted when the truth of the following proposition is established.

PROPOSITION: *The volume included between two three-level sections having their corresponding surface lines joined by warped surfaces, is exactly a mean between the two volumes formed between the same end sections by the two sets of planes resulting from the two sets of diagonals which may be drawn.*

If the two sets of diagonals be drawn on each side the centre line and a cross-section be taken parallel to the end areas, the traces of the four surface planes on each side the centre line on the cutting plane will form a parallelogram, the diagonal of which is the trace of the warped surface on this cutting plane. Since this cutting plane is any plane parallel to the end areas, and since the warped surface line bisects the figure formed by the two sets of planes formed by the diagonals, it follows that the warped surface bisects the volume formed by the two sets of planes. The proposition will therefore be established if it be shown that the trace of the warped surface is the diagonal of the parallelogram formed by the traces of the four planes formed by the two sets of diagonals. Fig. 115 shows an extreme case where the centre height is higher than the side height at one end and lower at the other. Only the left half of the prismoid is shown in the figure. The cutting plane cuts the centre and side lines and the two diagonals in $efgh$ on the plane, and in $e'f'g'h'$ on the vertical projection. For the diagonal c_1d_2 the surface lines cut out are $e'f'$ and $f'h'$. For the diagonal c_2d_1 they are $e'g'$ and $g'h'$. For the warped surface the line cut out is $e'h'$, this being an

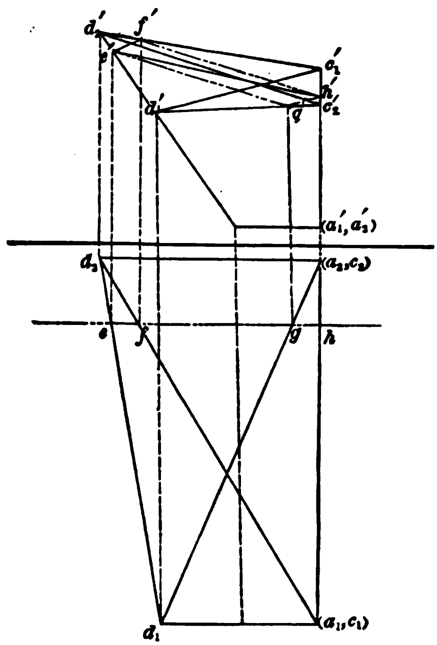

FIG. 115.

element of that surface. It remains to show that $e'f'h'g'$ is a parallelogram.

Since the cutting plane is parallel to the end planes all the lines cut are divided proportionally. That is, if the cutting plane is one n^{th} of l from c_2, then it cuts off one n^{th} of all the lines cut, measured from that end plane. But if the lines are divided proportionally, the projections of those lines are divided proportionally, and hence the points e', f', h', g' divide the sides of the quadrilateral d_2', c_1', c_2', d_1' proportionally. But it is a proposition in geometry that if the four sides of a quadrilateral, or two opposite sides and the diagonals, be divided proportionally and the corresponding points of subdivision joined, the resulting figure is a parallelogram. Therefore $e'f'h'g'$ is a parallelogram, and $e'h'$ is one of its diagonals and hence bisects it. Whence the surface generated by this line moving along c_1c_2 and d_1d_2 parallel to the end areas bisects the volume formed by the four planes resulting from the use of both diagonals on one side the centre line. Q. E. D.

It is probable, therefore, that the warped surface would usually fit the ground better than either of the sets of planes formed by the diagonals. Furthermore, the errors caused by the use of the warped surface (Table XI.) are compensating errors, thus preventing any marked accumulation of errors in a series of prismoids.* There are extreme cases, however, such as that given in the example, Fig. 114, which are best computed by the method by diagonals.

323. Preliminary Estimate from the Profile.—If the cross-sections be assumed level transversely then for given width of bed and side slopes, a table of end areas may be prepared in terms of the centre heights. From such a table the

* The two methods here discussed are the only ones that have any claims to accuracy. The method by "mean end areas," wherein the volume is assumed to be the mean of the end areas into the length, always gives too great a volume (except when a greater centre height is found in connection with a less total width, which seldom occurs), the excess being *one half of the volume of the pyramids involved in the elementary forms of the prismoid.* This is a large error even in level sections, and very much greater on sloping ground, and yet it is the basis of most of the tables used in computing earthwork, and in some States it is legalized by statute. Thus in the example computed by Henck's method on p. 434 the volume by mean end areas is 1193 cu. yards; by the prismoidal formula it is 1168 cu. yards, while by the method by diagonals it was only 1001 cu. yards. This was an extreme case, however, and was selected to show the adaptation of the method by diagonals to such a form.

end areas may be rapidly taken out and *plotted as ordinates from the grade line.* The ends of these ordinates may then be joined by a free-hand curve, and the area of this curve found by the planimeter. The ordinates may be plotted to such a scale that each unit of the area, as one square inch, shall represent a convenient number of cubic yards, as 1000. The record of the planimeter then in square inches and thousandths gives at once the cubic yards on the entire length of line worked over by simply omitting the decimal point. Evidently the scale to which the ordinates are to be drawn to give such a result is not only a function of the width of bed and side slopes, but also of the longitudinal scale to which the profile line is plotted. The area of a level section is

$$A = wc + rc^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where w, c, and r are the width of base, centre height, and slope-ratio respectively.

Now if $h =$ the horizontal scale of the profile, that is the number of feet to the inch, and if one square inch of area is to represent 1000 cu. yards, the length of the ordinate must be

$$y = \frac{hA}{1000 \times 27} = \frac{h(wc + rc^2)}{27{,}000}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If values be given to h, w, and r, which are constants for any given case, then the value of y becomes a function of c only, and a table can be easily prepared for the case in hand. Since y is a function of the second power of c, the second difference will be a constant, and the table can be prepared by means of first and second differences. Thus if c takes a small increment, as 1 foot, then the first difference is

$$\Delta' y = \frac{h}{27{,}000}(w + 2rc + r). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But this first difference is also a function of c, and hence when c takes an increment this first difference changes by an amount equal to

$$\Delta'' y = \frac{h}{27000} \cdot 2r, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

which is constant. An initial first difference being given for a certain value of c, a column of first differences can be obtained by simply adding the $\Delta'' y$ continuously to the preceding sum. With this column of first differences the corresponding column of values of y may be found by adding the first differences continuously to the initial value of y for that column.*

TABULAR VALUES OF y IN EQUATION (2) FOR $w = 20$, $r = 1\frac{1}{2}$, AND $h = 400$.

c	0.'0	0.'1	0.'2	0.'3	0.'4	0.'5	0.'6	0.'7	0.'8	0.'9
	in.	in.	in.	in.	in.	in.	in.	in.	in.	in.
0	0.00	0.03	0.06	0.09	0.12	0.15	0.19	0.22	0.25	0.28
1	.32	.35	.39	.42	.46	.49	.53	.57	.61	.64
2	.68	.72	.76	.80	.84	.88	.92	.96	1.00	1.05
3	1.09	1.13	1.17	1.22	1.26	1.31	1.35	1.40	1.45	1.49
4	1.54	1.59	1.63	1.69	1.73	1.78	1.83	1.88	1.93	1.99
5	2.04	2.09	2.14	2.19	2.24	2.30	2.36	2.41	2.47	3.52
6	2.58	2.63	2.69	2.75	2.80	2.87	2.92	2.98	3.04	3.10
7	3.16	3.22	3.28	3.35	3.41	3.47	3.54	3.60	3.66	3.73
8	3.79	3.86	3.92	3.99	4.05	4.13	4.19	4.26	4.33	4.40
9	4.47	4.54	4.60	4.68	4.75	4.82	4.89	4.97	5.04	5.11
10	5.18	5.26	5.33	5.40	5.48	5.56	5.64	5.72	5.79	5.87
11	5.95	6.03	6.10	6.18	6.26	6.35	6.43	6.51	6.59	6.67
12	6.76	6.84	6.92	7.00	7.09	7.18	7.26	7.35	7.43	7.52
13	7.61	7.70	7.78	7.86	7.96	8.05	8.14	8.23	8.32	8.41
14	8.50	8.60	8.68	8.77	8.87	8.97	9.06	9.16	9.25	9.35
15	9.44	9.54	9.63	9.73	9.83	9.94	10.03	10.13	10.23	10.33
16	10.43	10.53	10.62	10.73	10.83	10.94	11.04	11.15	11.25	11.35
17	11.46	11.56	11.66	11.77	11.88	12.00	12.10	12.21	12.31	12.42
18	12.53	12.64	12.75	12.86	12.97	13.09	13.20	13.32	13.42	13.54
19	13.65	13.77	13.87	13.99	14.10	14.23	14.34	14.47	14.58	14.70
20	14.81	14.93	15.04	15.16	15.29	15.42	15.53	15.66	15.78	15.90

* For a further exposition of this subject, see Appendix C.

The preceding table was constructed in this manner, for $w = 20$ feet, $r = 1\frac{1}{2}$; and $h = 400$ feet to the inch.

324. **Borrow-pits** are excavations from which earth has been "borrowed" to make an embankment. It is generally preferable to measure the earth in cut rather than in fill, hence when the earth is taken from borrow-pits and its volume is to be computed in cut, the pits must be carefully staked out and elevations taken both before and after excavating. The methods given in art. 311 are well suited to this purpose, or they may be computed as prismoids by the aid of Table XI., if preferred. To use the table it is only necessary to enter it with such heights and widths as give twice the elementary areas (triangles or quadrilaterals) into which the end sections are divided, and then multiply the final result by the length and divide by 100. The table is entered for both end-area dimensions and also the mid-area dimensions, four times this latter result being taken the same as before.

325. **Shrinkage of Earthwork.**—Excavated earth first increases in volume, when removed from a cut and dumped on a fill, but it gradually settles, or shrinks, until it finally comes to occupy a less volume than it formerly did in the cut. Both the amounts, initial increase, and final shrinkage depend on the nature of the soil, its condition when removed, and the manner of depositing it in place. There can therefore be no general rules given which will always apply. *For ordinary clay and sandy loam, dumped loosely, the first increase is about one twelfth, and then the settlement about one sixth of this increased volume, leaving a final volume of about nine tenths of the original volume in cut.**

Thus for 100 cubic yards of settled embankment 111 cubic yards in cut would be required. But a contractor should have

* See paper by P. J. Flynn in Trans. Tech. Soc. of the Pacific Coast, vol. ii. p. 179, where all the available experimental data are given.

his stakes or poles set one fifth higher than the corresponding fill, so that when filled to the tops of these, a settlement of one sixth will bring the surface to the required grade.

These changes of volume are less for sand and more for stiff, wet clay.

For rock the permanent increase in volume is from 60 to 80 per cent, the greater increase corresponding to a smaller average size of fragment.

326. **Excavations under Water.**—It is often necessary to determine the volume of earth, sand, mud, or rock removed from the beds of rivers, harbors, canals, etc. If this be done by soundings alone, it is likely to work injustice to the contractor, as he would receive no pay for depths excavated below the required limit; and besides, foreign material is apt to flow in and partially replace what is removed, so that the material actually excavated is not adequately shown by soundings within the required limits. It is common, therefore, to pay for the material actually removed, an inspector being usually furnished by the employer to see that no useless work is done beyond the proper bounds. The material is then measured in the dumping scows or barges. The unit of measure is the cubic yard, the same as in earthwork. There are two general methods of gauging scows, or boats. One is to actually measure the inside dimensions of each load, which is often done in the case of rock, and the other is to measure the displacement of the boat, which is the more common method with dredged material. When the barge is gauged by measuring its displacement, the water in the hold must always be pumped down to a given level, or else it must be gauged both before and after loading and the depth of water in the hold observed at each gauging. A displacement diagram (or table) is prepared for each barge, from its actual external dimensions, in terms of its mean draught. There should always be four gaugings taken to determine the draught, at four symmetrically located points

on the sides, these being one fourth the length of the barge from the ends. Fixed gauge-scales, reading to feet and tenths may be painted on the side of the barge, or if it is flat-bottomed, a gauging-rod, with a hook on its lower end at the zero of the scale, may be used and readings taken at these four points. Any distortion of the barge under its load, or any unsymmetrical loading, will then be allowed for, the mean of the four gauge-readings being the true mean draught of the boat.

To prepare a displacement diagram, the areas of the surfaces of displacement must be found for a series of depths uniformly spaced. This series may begin with the depth for no load, the hold being dry. They should then be found for each five tenths of a foot up to the maximum draught. If the boat has plane vertical sides and sloped ends these areas are rectangles, and are readily computed. If the boat is modelled to curved lines, the water-lines can be obtained from the original drawings of the boat, or else they must be obtained by actual measurement. In either case they can be plotted on paper, and their areas determined by a planimeter. These areas are analogous to the cross-sections in the case of railroad earthwork, and the prismoidal formula may be applied for computing the displacement. Thus,

Let A_0, A_1, A_2, A_3, etc., be the areas of the displaced water surfaces, taken at uniform vertical distances h apart. Then for an even number of intervals we have in cubic yards

$$V = \frac{h}{3 \times 27}(A_0 + 4A_1 + 2A_2 + 4A_3 + \ldots . A_n). \quad . \quad (1)$$

If the total range in draught be divided into six equal portions, each equal to h, then Weddel's Rule * would give a

* For the derivation of this rule see Appendix C.

nearer approximation. With the same notation as the above we would then have, in cubic yards,

$$V = \frac{3h}{10}[A_0 + A_2 + A_4 + A_6 + 5(A_1 + A_3 + A_5) + A_3] \,.\,. \quad (2)$$

These rules are also applicable to the gauging of reservoirs, mill-ponds, or of any irregular volume or cavity.

After the displaced volume of water is found, the corresponding volume of earth or rock is found by applying a proper constant coefficient. This coefficient is always less than unity, and is the reciprocal of the specific gravity of the material. This must be found by experiment. In the case of soft mud it is nearly unity, while with sand and rock it is much more. When rock is purchased by the cubic yard, solid rock is not implied, but the given quality of cut or roughly-quarried rock, piled as closely as possible. When rock is excavated, solid rock is meant. A *measured volume* of any material put into a *gauged scow* will give the proper coefficient for that material. Thus if the measured volume V' give a displacement of V, then $\frac{V'}{V} = C$ is the coefficient to apply to the displacement to give the volume of that material.

CHAPTER XIV.

GEODETIC SURVEYING.*

327. **The Objects of a Geodetic Survey** are to accurately determine the *relative* positions of widely separated points on the earth's surface and the directions and lengths of the lines joining them; or to accurately determine the *absolute* positions (in latitude, in longitude from a fixed meridian, and in elevation above the sea-level) of widely separated points on the earth's surface and the directions and lengths of the lines joining them.

In the first case the work serves simply to supply a skeleton of exact distances and directions on which to base a more detailed survey of the intervening country; in the second, the results furnish the data for computing the shape and size of the earth, in addition to their use in more detailed surveys.

It is usually desirable also to have some knowledge of the latitude and longitude of the points determined in the first case, but a very accurate knowledge of these would not be essential to the immediate objects of the work.

In both cases the points determined form the vertices of a series of triangles joining all the points in the system. One or more lines in this system of triangles and all of the angles are very carefully measured, and the lengths of all other lines in the system computed. The azimuths of certain lines are also determined, and, if desired, the latitudes and longitudes of some of the points. From this data it is then possible to compute the latitudes and longitudes of all the points in the system and

* See Appendix F for a discussion of many subjects considered in this chapter.

the lengths and azimuths of all the connecting lines. The work as a whole is denominated *triangulation.*

The measured lines are called base-lines, the points determined are triangulation-stations, and those points (usually triangulation-stations) at which latitude, longitude, or azimuth is directly determined are called respectively latitude, longitude, or azimuth stations. The latitude of a station and the azimuth of a line are determined at once by stellar observations at the point. The longitude is found by observing the difference of time elapsing between the transit of a star across the meridian of the longitude-station and the meridian of some fixed observatory whose longitude is well determined. An observer at each station notes the time of transit across his meridian, and each transit is recorded upon a chronograph-sheet at each station. This requires a continuous electrical connection between the two stations. This difference of time, changed into longitude, gives the longitude of the field-station with reference to the observatory.

328. Triangulation Systems are of all degrees of magnitude and accuracy, from the single triangle introduced into a course to pass an obstruction, up to the large primary systems covering entire continents, the single lines in which are sometimes over one hundred miles in length.

The methods herein described will apply especially to what might be called secondary and tertiary systems, the lines of which are from one to twenty miles in length, and the accuracy of the work anywhere from 1 in 5000 to 1 in 50,000. Although the methods used are more or less common to all systems, yet for the primary systems, where great areas are to be covered and the highest attainable accuracy secured, many refinements, both in field methods and in the reductions, are introduced which would be found useless or needlessly expensive in smaller systems.

If it is desired to connect two distant points by a system

[redacted]

of triangulation at the least expense, then use system I., shown in Fig. 116. This system is also adapted to the fixing of a double row of stations with the least labor.

If such distant points are to be joined, or such double system of stations established, with the greatest attainable accuracy, then system III. should be used. This system is also best adapted to secondary work, where it is desired to simplify the work of reduction. Each quadrilateral is independently reduced.

If the greatest area is to be covered for a given degree of accuracy or cost, then system II. is the one to use.

System I. consists of a single row of simple triangles, sys-

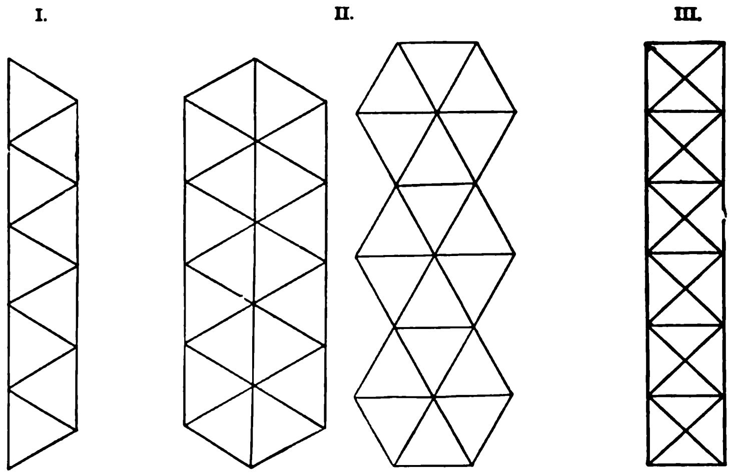

Fig. 116.

tem II. of a double row of simple triangles or of simple triangles arranged as hexagons, and system III. of a single row of quadrilaterals. A quadrilateral in triangulation is an arrangement of four stations with all the connecting lines observed. This gives six lines connecting as many pairs of stations, over which pointings have been taken from both ends of the line.

For the same maximum length of lines we have the following comparison of the three systems: *

System.	Composition.	Distance Covered.	No. of Stations.	Total Length of Sides.	Area Covered.	No. of Conditions.
I.	Equilateral triangles.	5	11	19	4.5	$n - 2 = 9$
II.	Hexagons.	5.2	17	34	9	$\frac{7n - 14}{5} = 21$
III.	Quadrilaterals (squares).	4.95	16	29	3.5	$2n - 4 = 28$

Thus, for the same distance covered, the number of stations to be occupied and the total length of lines to be cleared out are about one half more for systems II. and III. than for system I. The area covered by system II. is twice that by system I., but the number of conditions is much greater in system III. than in either of the others. Since almost all the error in triangulation comes from erroneous angle-measurements, the results will be more accurate according to our ability to reduce the observed values of the angles to their true values. The "conditions" mentioned in the above table are rigid geometrical conditions, which must be fulfilled (as that the sum of the angles of a triangle shall equal 180°), and the more of these geometrical conditions we have, the more nearly are we able to determine what the true values of the angles are. The work will increase in accuracy, therefore, as the number of these conditions increases, and this is why system III. gives more accurate results than systems I. and II. This will be made clear when the subject of the adjustment of the observations is considered.

329. The Base-line and its Connections.—The line whose length is actually measured is called the base-line. The

* Taken from the U. S. C. and G. Survey Report for 1876.

lengths and distance apart of such lines depend on the character of the work and the nature of the ground. Primary base-lines are from three to ten miles in length, and from 200 to

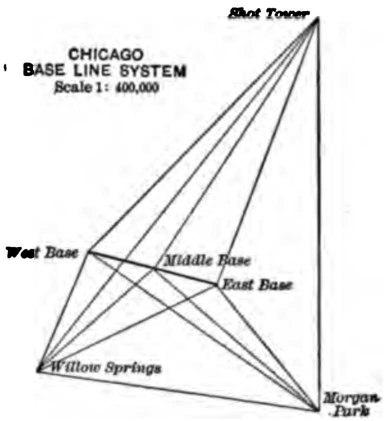

FIG. 117.*

600 miles apart. In general, in primary work, the distance apart has been about one hundred times the length of the base. Secondary bases are from two to three miles in length,

FIG. 118.*

and from fifty to one hundred and fifty miles apart, the distance apart being about fifty times the length of base. Tertiary bases are from one half to one and a half miles in length

* Taken from professional papers, Corps of Engineers U. S. Army, No. 24, being the final report on the Triangulation of the United States Lake Survey.

and from twenty-five to forty miles apart, the distance apart being about twenty-five times the length of base.*

The location of the base should be such as to enable one side of the main system to be computed with the greatest accuracy and with the least number of auxiliary stations for a given length of base. In flat open country the base may be chosen to suit the location of the triangulation-stations in the main system; but in rough country some of the main stations must often be chosen to suit the location of the base-line. In Fig. 117 the location of the base-line is almost an ideal one, being taken directly across one of the main lines of the system. By referring to Fig. 118 it will be seen that the line Willow Springs—Shot Tower is one of the fundamental lines of the main system, and the base is located directly across it. Here the ground is a flat prairie, and the base was chosen to suit the stations of the main system.

The station at the middle of the base is inserted in order to furnish a check on the measurements of the two portions as well as to increase the strength of the system by increasing the number of equations of conditions. Sometimes it is necessary to use one or more auxiliary stations outside the base before the requisite expansion is obtained. Thus suppose the stations Morgan Park and Lombard were the extremities of the line of the main system whose length was to be computed from this base, then the stations Willow Springs and Shot Tower might have been occupied as auxiliary stations from which the line Morgan Park—Lombard could be computed.

330. The Reconnaissance.—A system of triangulation having been fixed upon, of a given grade and for a given pur-

* These intervals between bases are in accordance with the practice that has hitherto been followed. The new method of measuring base-lines with a steel tape, described in Art. 339, will probably change this practice by causing more bases to be measured, leaving much shorter intervals to be covered by angular measurement.

pose, the first thing to be done is to select the location of the base-line and the position of the base-stations. The base should be located on nearly level ground, and should be favorably situated with reference to the best location of the triangulation-stations. These stations are then located, first for expanding from the base to the main system, and then with regard to the general direction in which the work is to be carried, and to the form of the triangles themselves.

No triangle of the main system should have any angle less than 30° nor more than 120°. Although small angles can be measured just as accurately as large ones, a given error in a small angle, as of one second, has a much greater effect on the resulting distances than the same error in an angle near 90°. In fact, the errors in distance are as the tabular differences in a table of natural sines, for given errors in the angles. These tabular differences are very large for angles near 0° or 180°, but reduce to zero for angles at 90°. The best-proportioned triangle is evidently the equilateral triangle, and the best-proportioned quadrilateral is the square. In making the reconnaissance the object should be to fulfil these conditions as nearly as possible.

The most favorable ground for a line of triangles is a valley of proper width, with bald knobs or peaks on either side. Stations can then be selected giving well-conditioned triangles, with little or no clearing out of lines, and with low stations. In a wooded country the lines must be cleared out or else very tall stations must be used. In general, both expedients are resorted to. Stations are built so as to avoid the greater portion of the obstructions, and then the balance is cleared out.

So much depends on the proper selection of the stations in a system of triangulation, as to time, cost, and final accuracy, that the largest experience and the maturest judgment should be made available for this part of the work. The form of the triangles; the amount of cutting necessary to clear out the lines and the probable resulting damage to private interests;

the height and cost of stations, and the accessibility of the same; the avoidance of all sources of atmospheric disturbance on the connecting lines, as of factories, lime- or brick-kilns, and the like, which might either obstruct the line by smoke or introduce unusual refraction from heat; the freedom from disturbance of the stations themselves during the progress of the work, and the subsequent preservation of the marking-stones—these are some of the many subjects to be considered in determining the location of stations.

It is the business of the reconnaissance party not only to locate the stations, but to determine the heights of the same. A station that has been located is temporarily marked by a flag fastened upon a pole, and this made to project from the top of a tall tree in the neighborhood. In selecting a new station it is customary to first select from the map the general locality where a station is needed, and then examine the region for the highest ground available. When this is found, the tallest trees are climbed and the horizon scanned by the aid of a pair of field-glasses to see if the other stations are visible. If no tree or building is available for this purpose ladders may be spliced together and raised by ropes until the desired height is obtained.

331. **Instrumental Outfit.**—The reconnaissance party requires a convenient means of measuring angles and of determining directions and elevations. For measuring angles a pocket sextant would serve very well, provided the stations are distinct or provided distinct range-points in line with the stations may be selected by the aid of field-glasses. A prismatic pocket-compass will often be found very convenient in finding back stations which have been located and whose bearings are known. An aneroid barometer is desirable for determining approximate relative elevations. For methods of using it in such work, see Chapter VI., p. 136. If to the above-named instruments we add field-glasses, and creepers for climbing trees, the instrumental outfit is fairly complete.

332. The Direction of Invisible Stations.—It often happens that one station cannot be seen from another on account of forest growth, which may be cleared out. In such a case the station may be located and the line cleared from one station or from both, the direction of the line having been determined. This direction may always be computed if two other points can be found from each of which both stations and the other auxiliary point are visible. Thus in Fig. 119 let AB be the line to be cleared out, and let C and D be two points from which all the stations may be sighted. Measure the two angles at each station and call the distance CD unity. Solve the triangle BCD for the side BC, and the triangle ADC for the side AC. We now have in the triangle ABC two sides and the included angle to find the other angles. When these are found the course may be aligned from either A or B. It will often happen that either C or D or both can be taken at regular stations. Of course a target must be left at either C or D to be used in laying out the line from A or B. The above is a modification of the problem given in art. 110, p. 107. A use of this expedient will often greatly facilitate the work.

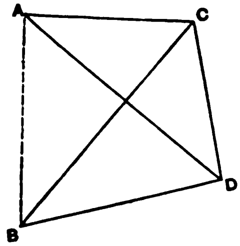

FIG. 119.

333. The Heights of Stations depend on the relative heights of the ground at the stations and of the intervening region. If the surface is level, then the heights of stations depend only on their distance apart. In any case the distance apart is so important a function of the necessary height that it is well to know what the heights would have to be for level, open country.

The following table* gives the height of one station when the other is at the ground level, for open, level country:

* Taken from Report of U. S. Coast and Geodetic Survey for 1882.

DIFFERENCE IN FEET BETWEEN THE APPARENT AND TRUE LEVEL AT DISTANCES VARYING FROM 1 TO 66 MILES.

Distance, miles.	Difference in feet for—			Distance, miles.	Difference in feet for—		
	Curvature.	Refraction.	Curvature and Refraction.		Curvature.	Refraction.	Curvature and Refraction.
1	0.7	0.1	0.6	34	771.3	108.0	663.3
2	2.7	0.4	2.3	35	817.4	114.4	703.0
3	6.0	0.8	5.2	36	864.8	121.1	743.7
4	10.7	1.5	9.2	37	913.5	127.9	785.6
5	16.7	2.3	14.4	38	963.5	134.9	828.6
6	24.0	3.4	20.6	39	1014.9	142.1	872.8
7	32.7	4.6	28.1	40	1067.6	149.5	918.1
8	42.7	6.0	36.7	41	1121.7	157.0	964.7
9	54.0	7.6	46.4	42	1177.0	164.8	1012.2
10	66.7	9.3	57.4	43	1233.7	172.7	1061.0
11	80.7	11.3	69.4	44	1291.8	180.8	1111.0
12	96.1	13.4	82.7	45	1351.2	189.2	1162.0
13	112.8	15.8	97.0	46	1411.9	197.7	1214.2
14	130.8	18.3	112.5	47	1474.0	206.3	1267.7
15	150.1	21.0	129.1	48	1537.3	215.2	1322.1
16	170.8	23.9	146.9	49	1602.0	224.3	1377.7
17	192.8	27.0	165.8	50	1668.1	233.5	1434.6
18	216.2	30.3	185.9	51	1735.5	243.0	1492.5
19	240.9	33.7	207.2	52	1804.2	252.6	1551.6
20	266.9	37.4	229.5	53	1874.3	262.4	1611.9
21	94.3	41.2	253.1	54	1945.7	272.4	1673.3
22	322.9	45.2	277.7	55	2018.4	282.6	1735.8
23	353.0	49.4	303.6	56	2092.5	292.9	1799.6
24	384.3	53.8	330.5	57	2167.9	303.5	1864.4
25	417.0	58.4	358.6	58	2244.6	314.2	1930.4
26	451.1	63.1	388.0	59	2322.7	325.2	1997.5
27	486.4	68.1	418.3	60	2402.1	336.3	2065.8
28	523.1	73.2	449.9	61	2482.8	347.6	2135.2
29	561.2	78.6	482.6	62	2564.9	359.1	2205.8
30	600.5	84.1	516.4	63	2648.3	370.8	2277.5
31	641.2	89.8	551.4	64	2733.0	382.6	2350.4
32	683.3	95.7	587.6	65	2819.1	394.7	2424.4
33	726.6	101.7	624.9	66	2906.5	406.9	2499.6

$$\text{Curvature} = \frac{\text{square of distance}}{\text{mean diameter of earth}};$$

Log curvature = log square of distance in feet − 7.6209807;

Refraction $= \dfrac{K^2}{R}m$, where K represents the distance in feet, R the mean radius of the earth ($\log R = 7.3199507$), and m the coefficient of refraction,* assumed at .070, its mean value, sea-coast and interior.

$$\text{Curvature and refraction} = (1 - 2m)\frac{K^2}{2R}.$$

Or, calling h the height in feet, and K the distance in statute miles, at which a line from the height h touches the horizon, taking into account terrestrial refraction, assumed to be of the same value as in the above table (.070), we have

$$K = \frac{\sqrt{h}}{.7575}, \qquad h = \frac{K^2}{1.7426}.$$

The following examples will serve to illustrate the use of the preceding table:

I. *Elevation of Instrument required to overcome Curvature and Refraction.*—Let us suppose that a line, A to B, was 18 miles in length over a plain, and that the instrument could be elevated at either station, by means of a portable tripod, to a height of 20 or 30 or 50 feet. If we determine upon 36.7 feet at A, the tangent would strike the curve at the distance represented by that height in the table, viz., 8 miles, leaving the curvature (decreased by the ordinary refraction) of 10 miles to be overcome. Opposite to 10 miles we find 57.4 feet, and a

* See discussion on refraction, Arts. 396–8.

signal at that height erected at B would, under favorable refraction, be just visible from the top of the tripod at A, or be on the same apparent level. If we now add 8 feet to tripod and 8 feet to signal-pole, the visual ray would certainly pass 6 feet above the tangent point, and 20 feet of the pole would be visible from A.

II. *Elevations required at given Distances.*—If it is desired to ascertain whether two points in the reconnaissance, estimated to be 44 miles apart, would be visible one from the other, both elevations must be at least 278 feet above mean tide, or one 230 feet and the other 331 feet, etc. This supposes that the intervening country is low, and that the ground at the tangent point is not above the mean surface of the sphere. If the height of the ground at this point should be 200 feet above mean tide, then the natural elevations should be 478 or 430 and 531 feet, etc., in height, and the line is barely possible. To insure success, the theodolite must be elevated at both stations to avoid high signals.

Since the height of station increases as the square of the distance, it is evident that the minimum aggregate station height is obtained by making them of equal height. Or, if the natural ground is higher at one station than the other, then the higher station should be put on the lower ground—that is, when the intervening country is level. If, however, the obstruction is due to an intervening elevation, the higher station should be the one nearer the obstruction.

Sometimes a very high degree of refraction is utilized to make a connection on long lines. Thus on the primary triangulation of the Great Lakes three lines respectively 100, 93, and 92 miles in length were observed across Lake Superior, which could not have been done except that the refraction was found sometimes to exceed twice its average amount. The line from station Vulcan, on Keweenaw Point, to station Tip-Top in Canada, was 100 miles in length. The ground at station

Vulcan was 726 feet above the lake, and the observing station was elevated 75 feet higher, making 801 feet above the surface of the lake. The station at Tip-Top was 1523 feet above the lake, the observing tripod being only 3 feet high. From the above table we find that the line of sight from Vulcan would become tangent to the surface of the lake at a distance of 37.4 miles, and that from Tip-Top at a distance of 51.5 miles, thus leaving a gap of about eleven miles between the points of tangency, for ordinary values of the refraction. If this interval were equally divided between the two stations and these raised to the requisite height, we would find from the table that Tip-Top would have to be elevated some 340 feet and Vulcan some 260 feet. Since this was not done, we must conclude that an occasional excessive value of the refraction was sufficient to bend these rays of light by about these amounts in addition to the ordinary curvature from this source. In other words, the actual refraction when one of these stations was visible from the other must have been more than double its mean amount.

The following is a synopsis of the heights of the stations built for the observation of horizontal angles in the primary triangulation of the Great Lakes:

Total number of stations *	243
Combined height of stations	14,100 feet
Average height of stations	58 "
Average height of stations from Chicago to Buffalo	81.3 "
Number of stations less than 10 feet high	22
" " from 10 feet to 24 feet in height	18
" " " 25 " 49 " "	50
" " " 50 " 74 " "	71
" " " 75 " 99 " "	47
" " " 100 " 109 " "	18
" " " 110 " 119 " "	15
" " " 120 " 124 " "	2

* Only stations built expressly for the work are here included. Sometimes buildings or towers were utilized in addition to these.

The heights above given are the heights at which the instrument was located above the ground. The targets were usually elevated from 5 to 30 feet higher.

The excessive heights of the stations from Chicago to Buffalo are due to the country being very heavily timbered, and the surface only gently rolling. In the vicinity of Lake Superior they averaged only about 35 feet high, while from Buffalo to the eastern end of Lake Ontario they averaged 51 feet in height.

334. **Construction of Stations.**—If it is found necessary to build tall stations, two entirely separate structures must be

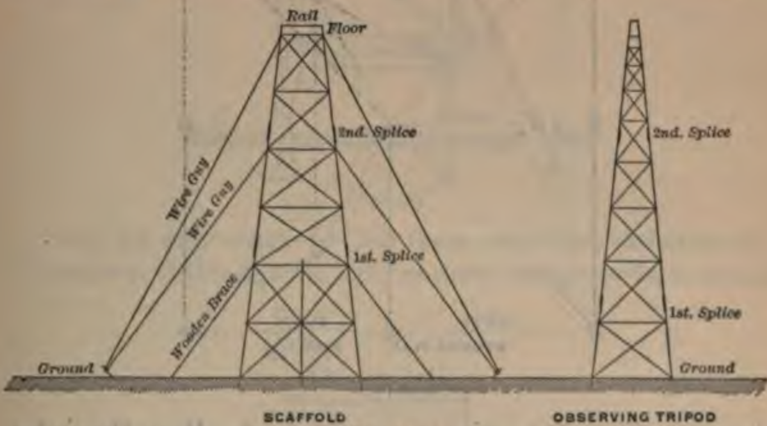

FIG. 120.

erected, one for carrying the instrument and one for sustaining the platform on which the observer stands. These should have no rigid connection with each other. These structures are shown in plan and elevation in Figs. 120 and 121. The inner station is a tripod on which the instrument rests; this is surrounded by a quadrangular structure, shown separately in elevation to prevent confusion. Both structures are built entirely of wood, the outer one being usually carried up higher than the tripod (not shown in the drawing), and the target fixed to its apex. This upper framework serves also to support an awning to shade the instrument from the sun. For lower stations a simpler construction will serve, but the observer's platform must in all cases be separate from the instrument tripod. The wire guys and wooden braces shown in Fig. 120 were not used on the U. S. Lake Survey stations.

For stations less than about 15 feet in height the design

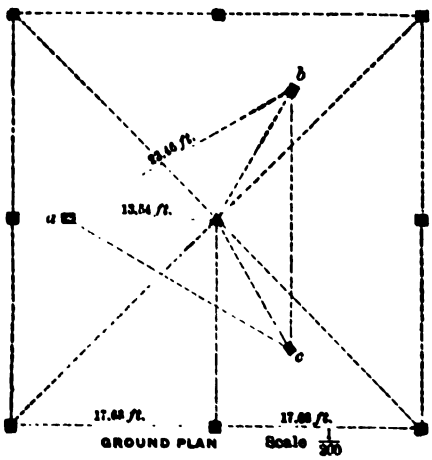

FIG. 121.

shown in Figs. 122 and 123 may be used. Here the outer platform on which the observer stands is entirely separate from the tripod which supports the instrument. For ground stations a post firmly planted serves very well, or a tree cut off to the proper height. The common instrument tripod will seldom be found satisfactory for good work. Sometimes extra heavy and stable tripods of the ordinary pattern have given excellent results.

335. Targets.—The requisites of a good target are that it shall be clearly visible against all backgrounds, readily bisected, rigid, capable of being accurately centred over the station, and so constructed that the centre of the visible portion, whether in sun or in shade, shall coincide with its vertical axis.

FIG. 122.

It is not easy always to fulfil these conditions satisfactorily. To make it visible against light or dark backgrounds, it is well

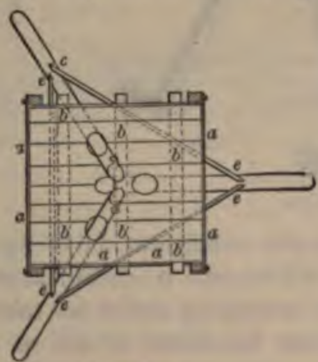

FIG. 123.

to paint it in alternating black and white belts. For ready bisection it should be as narrow as possible for distinctness. This is accomplished by making the width subtend an angle of from two to four seconds of arc. Since the arc of one second is three tenths of an inch for one-mile radius, an angle of four seconds would give a target one tenth of a foot in diameter for one-mile distances, or one foot in diameter for ten-mile distances. Something depends on the magnifying power of the telescope used. The design shown in Fig. 124 will satis-

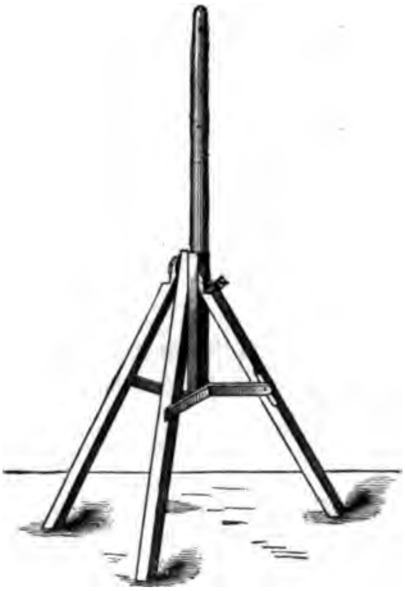

FIG. 124.

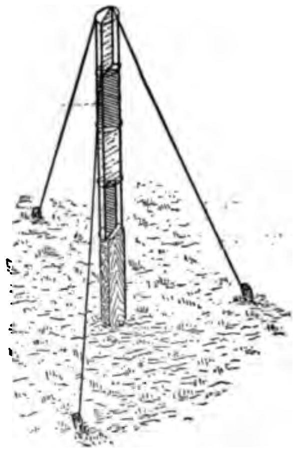

FIG. 125.

factorily satisfy the conditions as to rigidity and convenience of centering. Of course it should stand vertically over the station so that a reading could be taken on any part of its height. The last condition is not so easily satisfied. If a cylinder or cone be used the illuminated portion only will appear when the sun is shining, and a bisection on this portion may be several inches to one side of the true axis.

The target is then said to present a phase, and corrections for this are sometimes introduced. It is much better, however, to use a target which has no phase. If the target is to be read mostly from one general direction, a surface, as a board, may be used; but if the target is to be viewed from various points of the compass, then from those stations which lie nearly in the plane of the target it would not be visible, from its width being so greatly foreshortened.

In this case two planes could be set at right angles, one above the other. One or both would then be visible from all points, and since their axes are coincident, either one could be used. The objection to this would be that the upper disk would cast its shadow at times on the lower one, leaving one side in sun and the other in shade, thus giving rise to the very evil it is sought to eliminate. A very satisfactory solution of this problem was made on the Mississippi River Survey by means of the following device (Fig. 125): Four galvanized-iron wires, about three-sixteenths inch in diameter, are bent into a circle of, say, four inches in diameter, and soldered. To these four circles are attached four vertical wires about one fourth inch in diameter and four feet long, as shown in the accompanying figure. All joints to be securely soldered, the size of the wire increasing with the size of the target. The target is now divided into a number of zones by stretching black and white canvas alternately and in opposite ways between the opposing uprights, making diametral sections. If there are more than two zones, those marked by the same color should have the canvas crossing in different ways, so that if one plane is nearly parallel to any line of sight the other plane of this color will be nearly at right angles to it. This target has no phase, is visible against any background, and readily mounted. A wooden block may be inserted at bottom, with a hole in the axis of the target. This may then be set over a nail marking the station. The target is held at top by wire guys leading off to stakes in the ground. Such a target could be mounted on top of the pole shown in Fig. 124, if it should be found necessary to elevate it.

336. Heliotropes.—When the distance between stations is such that, owing to the distance, the state of the atmosphere, or the small size of the objective used, a target would appear indistinct, or perhaps not be visible at all, the reflected rays of the sun may be made to serve in place of a target. This limiting distance is usually about twenty miles. Any device for accomplishing this purpose may be called a heliotrope. In Figs. 126 and 127 are two forms of such an instrument. That

FIG. 126.

FIG. 127.

shown in Fig. 126 is a telescope mounted with a vertical and horizontal motion. This is turned upon the station occupied by the observer, and is then left undisturbed. On the telescope are mounted a mirror and two disks* with circular openings. The mirror has two motions so that it can be put into any position. Its centre is coincident with the axis of the disks, in all positions. The mirror may be turned so as to

* The disk next to the mirror is unnecessary.

throw a beam of light symmetrically through the forward disk, in which position the reflected rays are parallel to the axis of the telescope, and hence fall upon the distant point.

The heliotrope shown in Fig. 127 is to be used in conjunction with a single disk, which may be a plain board mounted on a plank with the mirror. The silvering is removed from a small circle at the centre of the mirror. The disk has a small hole through it as high above its base as the clear space on the mirror is above the plank. The operator points the apparatus by sighting, through the clear spot on the mirror and the opening in the disk, to the distant station. If the plank be fastened in this position the attendant now has only to move the mirror so as to keep the cone of reflected rays symmetrically covering the opening in the disk, and the light will be thrown to the distant station.

Since the cone of incident rays subtends an angle of about thirty-two minutes, the cone of reflected rays subtends the same angle. The base of this cone has a breadth of about fifty feet to the mile distance, or at a distance of twenty miles the station sending the reflection is visible over an area in a vertical plane 1000 feet in diameter. The alignment of the heliotrope need not, therefore, be very accurate. This alignment may vary as much as fifteen minutes of arc on either side of the true line. This is nearly 0.01 of a foot in a distance of two feet. If the bearing, or direction, of the distant station is once determined, it may be marked on the station by some means within this limit, and a very rude contrivance used for sending the reflected ray, or flash, as it is called. Thus, a mirror and a disk with the requisite movements may be mounted on the ends of a board or pole from five to twenty feet long, and when this is properly aligned it serves as well as any other more expensive apparatus. The hole in the disk should usually subtend an angle at the observer's station of something less than one second of arc, which is a width of three-tenths of an inch to the mile distance. On the best work with large instruments it should subtend an angle of less than one half a second, the minimum effective opening depending almost wholly on the condition of the atmosphere.*

Whatever form of heliotrope is used, an attendant is required to operate the apparatus. Evidently it can be used only on clear days, whereas cloudy weather is much better adapted to this kind of work, since the atmosphere then transmits so much clearer and steadier an image.

The heliotrope can be used as a means of communication between distant stations by some fixed code of flashing signals, and it has been so used very often with great advantage to the work. The attendant on the heliotrope, usually called a flasher, can thus know when the observer is reading his signals, when he is through at that station, and, in general, can receive his instructions from his chief direct from the distant station.

337. **Station Marks.**—If the triangulation is to serve for the fixing of points for future reference, then these points must be marked in some more or less permanent manner. In this case the station has been chosen with this in view, so that if possible it has been provided that even the surface for a few feet around the station shall remain undisturbed. To insure against disturbance from frost or otherwise, the real mark is usually set several feet underground. Many different means are employed to mark these points. The underground mark is to serve only when the superficial marks have been disturbed, there being always left a mark of some kind projecting above ground. On the U. S. Lake Survey, "the geodetic point is the centre of a $\frac{1}{4}$-inch hole drilled in the top of a stone

* Reflected sunlight has been seen a distance of sixty miles, through an opening one inch in diameter, which then subtended an angle of but one eighteenth of one second of arc at the instrument. This would require a very clear atmosphere.

two feet by six inches by six inches, sunk two and one-half feet below the surface of the ground. When the occupation of the station is finished, a second stone post, rising eight inches above the ground, is placed over the first stone. Three stone reference-posts, three feet long, rising about a foot above the ground, are set within a few hundred feet of the station, where they are the least likely to be disturbed. A sketch of the topography within a radius of 400 metres about the station is made, and the distances and azimuths of the reference-marks are accurately determined."

When the station is located in natural rock a copper bolt may be set to mark the geodetic point.

On the Mississippi River survey, stations had to be set on ground subject to overflow. These were to serve both for geodetic points and for bench-marks, both their geographical position and their elevation being accurately determined. Both the rank growth and the sedimentary deposits from the annual overflows would soon obliterate any mark which was but slightly raised above the surface. After much study given to the subject, the following method of marking such points was adopted: A flat stone eighteen inches square and four inches thick, dressed on the upper side, has a hole drilled in the centre, into which a copper bolt is leaded, the end projecting a quarter of an inch above the face of the stone. The stone is marked thus, $\frac{U \cdot S}{B \cdot M}$, and is placed three feet under ground. On this stone, and centred over the copper bolt, a cast-iron pipe four inches in diameter and five feet long is placed, and the dirt tamped in around it. The pipe is large enough to admit a levelling rod. The top is closed with a cap, which is fastened to the pipe by means of a bolt. The elevations of both the top of the pipe and of the stone are determined.

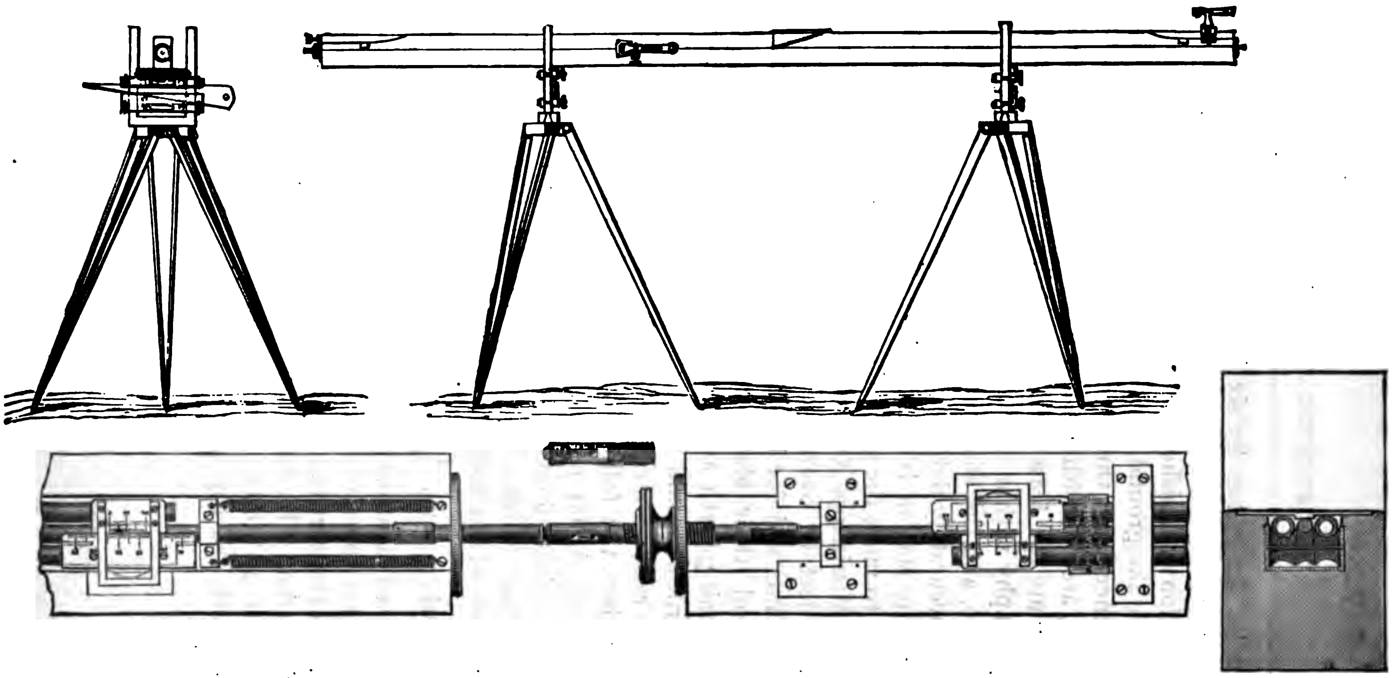

FIG. 128.—FOUR-METRE CONTACT-SLIDE BASE APPARATUS, AFTER THE DESIGN OF PROF. J. E. HILGARD, SUPERINTENDENT U. S. C. AND G. SURVEY.

MEASUREMENT OF THE BASE-LINE.

338. **Methods.**—The methods heretofore employed in measuring a base-line have depended on the degree of accuracy requisite. If an accuracy of one in one million was desired, then the most elaborate primary apparatus has been used, such as may be found described in the U. S. Coast and Geodetic Survey Reports for 1873 and 1882, or in the Primary Triangulation of the U. S. Lake Survey.* For an accuracy of one in fifty thousand or one in one hundred thousand, more simple appliances have been used, such as that shown in Fig. 128. This apparatus is fully described and illustrated in the U. S. Coast and Geodetic Survey Report for 1880, Appendix No. 17. It consists essentially of a four-metre steel bar, with zinc tubes on either side of it. One of these zinc tubes is attached to the steel bar at one end and the other at the other end. Since the expansion of zinc is about two and a half times that of steel, it is evident that the corresponding ends of the zinc bars will have a relative motion with reference to each other as the temperature changes. This relative motion is observed by means of the vernier scales attached to the ends of the zinc tubes. When the absolute length of the steel bar and the coefficients of expansion of both the steel and zinc bars are determined, and the readings of the vernier scales for a given temperature, then any other temperature will be indicated by the scale-readings. This combination thus becomes a metallic thermometer, from which the temperature of the steel rod may be accurately known while in use in the field. This assumes that the steel and zinc rods are at the same temperature at all times, and that the changes in length due to changes in temperature occur simultaneously with the tempera-

* This is a large quarto volume of 920 pp. and 30 plates, describing the methods and results of the geodetic work of the U. S. Lake Survey. It is a most valuable contribution to the science of geodesy, and is No. 24 of the Professional Papers of the Corps of Engineers of the U. S. Army, 1882.

ture changes. Unfortunately, this latter condition is not fulfilled in the case of zinc.

From elaborate observations on the relative expansions of steel and zinc bars on the United States Lake Survey, it was found that zinc is like glass in that its volume-change is not wholly coincident with its corresponding temperature-change, a residual portion of its change of volume requiring a considerable time for its completion. In other words, the volume-change lags behind the temperature-change, so that its volume is not truly indicated by its temperature, it being rather a function of the changes in temperature for an indefinite previous period. Zinc is, therefore, not a fit metal to use in the most accurate measurements, although it is sufficiently reliable for a secondary apparatus.

When two combinations of bars described above are properly protected from sun, wind, and from too sudden and variable temperature-changes, and when they are mounted in such a way as to enable them to be aligned both horizontally and vertically, with suitable provision for making exact contacts between the ends of the steel bars, they then form a base apparatus.

Sometimes simple wooden or iron rods have been used in this way, but then the great source of error is in not knowing the mean temperature (and hence length) of the rods at any time. If mercurial thermometers are used, these may be many degrees warmer or cooler than the bar, since the mercury bulb is so much smaller in cross-section than the bar, and therefore responds more quickly to changes in temperature. The steel-zinc combination is an ideal one, and would be practically perfect if zinc were as reliable a metal as steel. The best metals for metallic thermometers are probably steel and brass, the coefficient of expansion of the latter being about 1.5 times the former.*

* Mr. E. S. Wheeler, U. S. Asst. Engr., who has had a very large experience in the measurement of primary base-lines on the U. S. Lake Survey,

The Steel Tape furnishes the most convenient, rapid, and economical means for measuring any distance for any desired degree of accuracy up to about one in three hundred thousand, and if the most favorable times are chosen, an accuracy of 1 in 1,000,000 may be attained. It is probable, therefore, that all engineering measurements, even including primary base-lines, will yet be made by the steel tape or by steel and brass wires. The conditions of use depend on the accuracy required. Let us suppose the absolute length, coefficient of expansion, and modulus of elasticity have been accurately determined. Any distance can then be measured in absolute units within an accuracy of one in one million, by taking due precautions as to temperature and mechanical conditions. The length of the tape for city work is usually fifty feet, and its cross-section about $\frac{1}{2}$ inch by $\frac{1}{64}$ inch. That used in New York City is $\frac{3}{32}$ inch wide by $\frac{1}{64}$ inch thick. For mining, topographical, and railroad surveying a length of one hundred feet, with a cross-section of about $\frac{1}{5}$ by $\frac{1}{40}$ inch, is most convenient. For base-line measurement the length should be from three hundred to five hundred feet, and its cross-section from two to three one-thousandths of a square inch. For an accuracy of one in five thousand the tape may be used in all kinds of weather, held and stretched by hand, the horizontal position and amount of pull estimated by the chainmen. The temperature may be estimated, or read from a thermometer carried along for the purpose. On uneven ground, the end marks are given by plumb-line.

For an accuracy of one in fifty thousand the mean temperature of the tape should be known to the nearest degree Fahrenheit, the slope should be determined by stretching over stakes, or on ground whose slope is determined, and the pull

recommends the use of a single bar packed in ice, with micrometer microscopes mounted on iron stands to mark the end positions of the bar. By this means a constant length of standard can be obtained. This has never yet been done, however.

should be measured by spring balances. The work could then be done in almost any kind of cloudy weather. For an accuracy of one in five hundred thousand, extreme precautions must be taken. The mean temperature must be determined to about one fifth of a degree F., the slope must be accurately determined by passing the tape over points whose elevations above a given datum are known, the pull must be known to within a few ounces, and all friction must be eliminated. The largest source of error is apt to be the temperature. On clear days, the temperature of the air varies rapidly for varying heights above the ground, and, besides, the temperature of the tape would neither be that of the air surrounding it, nor of the bulb of a mercurial thermometer. In fact, there is no way of determining by mercurial thermometer, even within a few degrees, the mean temperature of a steel tape lying in the sun, either on or at varying heights above the ground. The work must then be done *in cloudy weather, and when air and ground are at about the same temperature.*

There should also be no appreciable wind, both on account of its mechanical action on the tape, and from the temperature-variations resulting therefrom.

339. Method of Mounting and Stretching the Tape.— To eliminate all friction, the tape is suspended in hooks about two inches long, these being hung from nails in the sides of "line-stakes" driven with their front edges on line. These stakes may be from twenty to one hundred feet apart. The nails may be set on grade or not, as desired; but if not on grade, then each point of support must have its elevation determined. A low point should not intervene between two higher ones, or the pull on the tape may lift it from this support. "Marking-stakes" are set on line with their tops about two feet above ground, at distances apart equal to a tape-length, say 300 feet. Zinc strips about one and one half inches wide are tacked to the tops of these stakes, and on these the tape-lengths are marked with a steel point. These strips remain undisturbed until all the measurements are completed, when they are preserved for future reference. In front of the marking-stake three "table-stakes" are driven, on which to rest the stretching apparatus, and in the rear a "straining-stake" to which to attach the rear end of the tape. These auxiliary stakes are set two or three feet away from the marking-stake, and enough

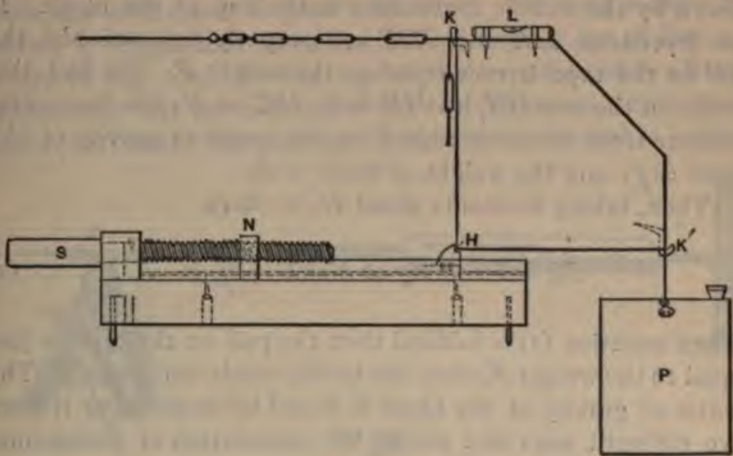

FIG. 129.

lower to bring the tape, when stretched, to rest on the top of the marking-stake.

The stretching apparatus is shown in Fig. 129.* A chain is attached to the end of the tape, and this is hooked over the

* This figure, and the method here described, are taken from the advance-sheets of the Report of the Missouri River Commission for 1886. The work was in charge of Mr. O. B. Wheeler, U. S. Asst. Engr., who first used this method on the Missouri River Survey in 1885. The author had previously developed and used the general method, except that he stretched his tape by a weight hung by a line passing through a loop which was kept at an angle of 45° with the vertical, and his end marks were made on copper tacks driven into the tops of the stakes. He had also used spring balances for stretching the tape.

staple K which is attached to the block KHK'. This block is hinged on a knife-edge at H, and is weighed at K' by the load P. The hinge bearing at H is attached to a slide which is moved by the screw S working in the nut N. The whole apparatus is set on the three table-stakes in front of the marking-stake, the proper link hooked over the staple, and the block brought to its true position by the screw. This position is shown by the bubble L attached to the top of the block. If the lever-arms HK and HK' are properly proportioned, the pull on the tape is now equal to the weight P. To find this length of the arm HK, let $HK = k$; $HK' = k'$; the horizontal distance from the knife-edge H to the centre of gravity of the block $= g$; and the weight of block $= B$.

Then, taking moments about H, we have

$$Pk = Pk' + Bg \text{ or } k = k' + \frac{B}{P}g. \quad . \quad . \quad . \quad . \quad (1)$$

When equation (1) is fulfilled then the pull on the tape is just equal to the weight P, when the bubble reads horizontally. The centre of gravity of the block is found by suspending it from two different axes and noting the intersection of plumb-lines dropped from these axes.

At the rear end the tape is held by a slide operated by an adjusting screw similar to that shown in Fig. 129. This slide rests on the straining-stake, and the rear-end graduation is made to coincide exactly with the graduation on the zinc strip which marked the forward end of the previous tape-length. The rear observer gives the word, and the forward end is marked on the next zinc strip. The thermometers are then read, and the tape carried forward.

The measurement is duplicated by measuring again in the same direction, the zinc strips being left undisturbed.

In obtaining a profile of the line the level rod is held on the suspension nails and on a block, equal in height to the length of the hooks, set on top of the marking-stakes.

For transferring the work to the ground, or to a stone set beneath the surface, a transit is mounted at one side of the line and the point transferred by means of the vertical motion of the telescope, the line of sight being at right angles to the base-line.

340. **M. Jäderin's Method.**—Prof. Edward Jäderin, of Stockholm, has brought the measurement of distances by wires and steel tapes to great perfection. He uses a tape 25 metres in length, and stretches it over tripods set in line, as shown in Fig. 130. On the top of the tripod head is a fixed graduation. At the rear end of the tape there is a single graduation, but at the forward end a scale ten centimetres in length

FIG. 130.

is attached to the tape, this being graduated to millimetres on a bevelled edge. The middle of this scale is 25 metres from the graduation at the other end of the tape. The tripods are set as near as may be to an interval of 25 metres, but it is evident that the reading may be taken on them if this interval is not more than 5 centimetres more or less than 25 metres. The reading is taken to tenths of millimetres, the tenths being estimated. The tape is stretched by two spring balances, a very stiff spring being used at the rear end and a very sensitive one at the forward end. The rear balance simply tells the operator here when the tension is approximately right, the measure of this tension being taken on the forward balance, which is shown in the figure.

If a single steel wire or tape be used, Mr. Jäderin also finds that the work must be done in cloudy and calm weather, or at night, if the best results are to be obtained. But he finds that if two wires be used, one of steel and the other of brass, he can continue the work during the entire day, even in sunshine and wind, and obtain an accuracy of about one in one million in his results.* The wires are stretched in succession over the same tripods, by the same apparatus, one wire resting on the ground while the other is stretched. More accurate results could doubtless be obtained if both wires are kept off the ground constantly, the wire not in use being held by two assistants, or if stakes and wire hooks are used, both wires might be stretched at once in the same hooks. The two wires form a metallic thermometer, the difference between the readings of the same distance by the two wires determining the temperature of both wires, when their relative lengths at a certain temperature and their coefficients of expansion are known. This method is similar in principle to that of the Coast Survey apparatus, where steel and zinc bars are used, shown in Fig. 128. In such cases the true length of line is found by equation (5), p. 481.

At least three thermometers should be used on a 300-foot tape, and they should be lashed to the tape or suspended by it at such points as to have equal weight on determining its temperature. Thus if the tape is 300 feet long the thermometers should be fastened at the 50, 150, and 250 foot marks. They should of course have their corrections determined by comparison with some absolute standard or with other standardized thermometers.

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* See "Geodätische Längenmessung mit Stahlbänden and metalldrähten," von Edv. Jäderin, Stockholm. 1885. 57 pp. Also, "Exposé elémentaire de la nouvelle Methode de M. Edouard Jäderin pour la mesure des droites géodésiques au moyen de Bandes d'Acier et de Fils métalliques," par P. E. Bergstrand, Ingénieur au Bureau central d'Arpentage, à Stockholm. 1885. 48 pp.

If the appliances above outlined be used with a single tape or wire, and the work be done on calm and densely cloudy days, or at night, or with two wires used even in clear weather, it is not difficult to make the successive measurements agree to an accuracy of one in five hundred thousand. There still remains, however, the errors in the absolute length, in the coefficient of expansion, in the modulus of elasticity, in the measure of the pull, and in the alignment, none of which would appear in the discrepancies between the successive measurements.

341. **The Absolute Length** is the most difficult to determine. The best way of finding it would be to compare it with another tape of known length. The U. S. Coast and Geodetic Survey now make comparisons of steel tapes up to 100 feet in length for a small fee.*

If an absolute standard is not available, then the length may be found by measuring a known distance, as a previously measured base-line, and computing the temperature at which the tape is standard. Or the tape may be compared with a shorter standard, as a yard or metre bar, by means of a comparator furnished with micrometer microscopes.†

* The absolute length of the 300-foot steel tape belonging to the Mississippi River Commission, the coefficient of expansion and the modulus of elasticity of which the author himself determined in 1880, has now been obtained. This was done by measuring a part of the Onley Base Line with this tape, using the method herein outlined. This base is situated in Southern Illinois, and forms the southern extremity of U. S. Lake Survey primary triangulation-system. The probable error in the length of the base, from the original measurements, was about one one-millionth. The recent tape-measurements are remarkably accordant, so the length of this tape is now very accurately known. A similar tape belonging to the engineering outfit of Washington University has been compared with this one at different temperatures, and its absolute length and coefficient of expansion found. The 50-foot subdivisions have also been carefully determined.

† Such an apparatus is used in the physical laboratory of Washington University, which, in conjunction with a standard metre bar which has been compared with the European standards, enables absolute lengths to be determined to the nearest one-thousandth of a millimetre.

342. The Coefficient of Expansion may be taken anywhere from 0.0000055 to 0.0000070 for 1° F.* If the tape is used at nearly its standard temperature, then the coefficient of expansion plays so small a part that its exact value is unimportant. If it is used at a temperature of 70° F. from its standard temperature, and if the error in the coefficient used be twenty per cent, the resulting error in the work would be one in ten thousand. This is probably the extreme error that would ever be made from not knowing the coefficient of expansion, some tabular value being used. If nothing is known of the coefficient of expansion, probably 0.0000065 would be the best value to use. It is evident. however, that for the most accurate work the coefficient of expansion of the tape used must be carefully determined.

* The author made a series of observations on a steel tape 300 feet long, the readings being taken at short intervals for four days and three nights. The tape was enclosed in a wooden box, and supported by hooks every sixteen feet. The observations were taken on fine graduations made by a diamond point. there being a single graduation at one end, but some fifty graduations a millimetre apart at the other end. The readings were made by means of micrometer microscopes mounted on solid posts at the two ends. The range of temperature was about 50° F., and the resulting coefficient of expansion for 1° F. was 0.00000699 ± 3 in the last place. The coefficient for the Washington University tape is 0.00000685. Prof. T. C. Mendenhall found from six or eight experiments on steel bands used for tapes, a mean coefficient of 0.0000059. Steel standards of length have coefficients ranging from 0.0000048 to 0.0000066.

Mr. Edward Jäderin, Stockholm, has obtained a mean value of 0.0000055, from a number of very careful determinations, both from remeasuring a primary base-line, and from readings in a water-bath. Several steel wires were tested, and their coefficients all came very near the mean as given above.

For brass wires he found a mean coefficient of 0.0000096 F. The 15-foot standard brass bar of the U. S. Lake Survey has a coefficient of 0.0000100, while tabular values are found as high as 0.0000107 F.

There is some evidence that *cold-drawn wires have a less coefficient of expansion than rolled bars and tapes.*

Coefficients of expansion have seldom been found with great accuracy, the coefficients of the "Mètre des Archives," the French standard, having had an erroneous value assigned to it for ninety years

343. **The Modulus of Elasticity** is readily found by applying to the tape varying weights, or pulls, and observing the stretch. The correction for sag will have to be applied for each weight used, in case the tape is suspended from hooks, which should be done to eliminate all friction.

Let P_1 be the maximum load in pounds;
P_0 " " minimum load in pounds;
a " " increased length of tape in inches due to the increased pull;
L " " length in inches for pull P_0, or the graduated length of tape;
S " " cross-section in square inches;
E " " modulus of elasticity;
d " " distance between supports;
w " " weight of one inch of tape in pounds;
s " " shortening effect of the sag for the length L;
v " " sag in inches midway between supports.

Then we have

$$E = \frac{(P_1 - P_0)L}{Sa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But for the pull P_1, the shortening from sag is much less than for the pull P_0. We must therefore find the effect of the sag in terms of the pull.

344. **Effect of the Sag.**—Where the sag is small, as it always is in this work, the curve, although a catenary, may be considered a parabola without an appreciable error.

If we pass a section through the tape midway between supports, and equate the moments of the external forces on one side of this section, we obtain, taking centre of moments at the support,

$$Pv = \frac{wd}{2} \cdot \frac{d}{4} = \frac{wd^2}{8},$$

or

$$v = \frac{wd^2}{8P}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the length of a parabolic curve be given by an infinite series, and if all terms after the second be omitted, which they may when $\frac{v}{d}$ is small, then we may write—

$$\text{Length of curve} = d\left(1 + \frac{8}{3}\frac{v^2}{d^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we now substitute for v its value as given in equation (1), we have

$$\text{Length of curve} = d\left\{1 + \frac{1}{24}\left(\frac{wd}{P}\right)^2\right\}.$$

If we call the excess in length of curve over the linear distance between supports the *effect of the sag*, we have

$$c = \frac{d}{24}\left(\frac{wd}{P}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for one interval between supports. If there are n such intervals in one tape-length, then $nd = L$, and the effect of the sag in the entire tape-length is

$$C_s = \frac{L}{24}\left(\frac{wd}{P}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If S_1 and S_2 be the effects of the sag for the pulls P_1 and P_2 ($S_1 < S_2$ for $P_1 > P_2$), then the total movement at the free end due to the pull being increased from P_2 to P_1 would be $a + (S_2 - S_1)$. If this total movement be called M, then we would have

$$E = \frac{(P_1 - P_2)L}{S(M - S_2 + S_1)} = \frac{P_1 - P_2}{S\left(\frac{M}{L} - \frac{(wd)^2}{24}\left(\frac{P_1^2 - P_2^2}{P_1^2 P_2^2}\right)\right)} \quad . \quad . \quad (5)$$

EXAMPLE.

Let $P_1 = 60$ pounds;
$P_0 = 10$ pounds;
$w = 0.00055$ pound per inch of tape;
$d = 300$ inches $= 25$ feet;
$S = 0.002$ square inch;
$M = 3.2$ inches;
$L = 3600$ inches $= 300$ feet.

To find E.

From equation (5) we have

$$E = \frac{50}{0.002\left\{\frac{3.2}{3600} - \frac{0.027}{24}\left(\frac{3500}{360000}\right)\right\}} = 28,500,000.$$

From the same data, we find from eq. (4) the effect of the sag to be 0.040 inch for the ten-pound pull, and 0.001 inch for the sixty-pound pull.

Evidently, if the tape is stretched by the same weight when its absolute length is found, and when used in measuring, the stretch, or elongation from pull, would not enter in the computation, and so the modulus of elasticity would be no function of the problem.

Again, the stretch per pound of pull may be observed for the given tape, and then neither E nor S, the cross-section, would enter in the computation.

345. Temperature Correction.—If mercurial thermometers are used, their field-readings must first be corrected for the errors of their scale-reading, each thermometer having, of course, a separate set of corrections. Then the *mean* of the corrected readings may be taken for all the whole tape-lengths in the line measured, and the correction for the entire line obtained at once. Thus,

let L = length of line;
T_0 = temperature at which the length of the tape is given for the standard pull P_0, this usually being the temperature at which its true length is its graduated length for that standard pull;
T_m = the *mean corrected* temperature of the entire line;
α = coefficient of expansion for 1°;
C_t = correction for temperature.

Then $$C_t = + \alpha(T_m - T_0)L. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The temperature correction for a part of a tape-length is computed separately.

If the value of α for the tape used is not known, it may be taken at 0.0000065.

If a metallic thermometer is used, as a brass and a steel wire, or a brass and a steel bar as in the U. S. C. and G. S. apparatus shown on p. 466, then we have the following:

346. Temperature Correction when a Metallic Thermometer is used.

Let l = length of wire or tape used, as 300 feet;
l_s = absolute length of the steel wire at the standard temperature of, say, 32° F.;
l_b = same for brass wire;
L = total length of line for whole tape-lengths ($= nl$ approximately);
n = number of lengths of the standard measured;
r_s = *mean value* of all the scale-readings on steel wire for the entire line $\left(= \frac{\Sigma r_s'}{n}\right)$;
r_b = same for scale-readings on brass wire;
α_s = coefficient of expansion for the steel wire;
α_b = " " " " " brass "
t_0 = *mean temperature* for the entire line.

Then we have

$$\left.\begin{aligned} L &= n(l_s + r_s)(1 + (t_0 - 32^\circ)\alpha_s) \\ &= n(l_b + r_b)(1 + (t_0 - 32^\circ)\alpha_b) \end{aligned}\right\} . \quad . \quad . \quad . \quad (2)$$

Since the temperature correction is relatively a very small quantity, we may put $l_s + r_s = l_b + r_b = l$, the length of the tape to which the temperature correction is applied.

We then have from (2)

$$(t_s - 32°) = \frac{(l_s + r_s) - (l_b + r_b)}{l(\alpha_b - \alpha_s)} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting this value of the temperature in (2), we obtain

$$L = n[l_s + r_s + \frac{\alpha_s}{\alpha_b - \alpha_s}((l_s + r_s) - (l_b + r_b))]. \quad . \quad (4)$$

If we put $l_s + r_s = S_s$ and $l_b + r_b = S_b$, we have

$$\left.\begin{aligned} L &= n\left[S_s + (S_s - S_b)\frac{\alpha_s}{\alpha_b - \alpha_s}\right] \\ &= n\left[S_b + (S_s - S_b)\frac{\alpha_b}{\alpha_b - \alpha_s}\right] \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

From either of the equations (5) we may compute the length of the line as corrected for temperature. If, however, it is desired to find the temperature correction separately, in order to combine it with the other corrections, we have

$$C_{st} = n(S_s - S_b)\frac{\alpha_s}{\alpha_b - \alpha_s}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

for the temperature correction to be applied to the measured length by the steel wire, or

$$C_{bt} = n(S_s - S_b)\frac{\alpha_b}{\alpha_b - \alpha_s}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

as the temperature correction to be applied to the measured length by the brass wire.

These formulæ all apply only to the *entire tape-lengths*. Any fractional length would have to be computed separately, or else a diminished weight given to their scale-readings in obtaining the mean values, r_a and r_b.

347. Correction for Alignment, both horizontal and vertical.—The relative elevations of the points of support are found by a levelling instrument, and the horizontal alignment done by a transit or by eye. An alignment by eye will be found sufficiently exact if points be established on line by transit every 500 or 1000 feet. The suspending nails and hooks afford considerable latitude for lateral adjustment when the tape is stretched taut; hence the horizontal deviation will be practically zero unless the stakes are very badly set, and the relative elevations of any two successive supports should be determined to less than 0.05 foot. If no care is taken to have more than two suspension points on grade, then each section of the tape will have a separate correction. Usually a single grade may as well extend over several sections, in which case the portion on a uniform grade may be reduced as a single section.

Let l_1, l_2, l_3, etc., be the successive lengths of uniform grades, and h_1, h_2, h_3, etc., the differences of elevation between the extremities of these uniform grades; then for a single grade we would have the correction

$$C = l - \sqrt{l^2 - h^2}$$

or

$$l^2 - 2Cl + C^2 = l^2 - h^2.$$

But since C is a very small quantity as compared with l, we may drop the C^2, whence we have $C = \frac{h^2}{2l}$ for a single grade.

The exact value of C, in ascending powers of h, is

$$C = \frac{h^2}{2l} + \frac{h^4}{8l^3} + \frac{h^6}{16l^5} + \text{etc.} \quad . \quad . \quad . \quad . \quad (1)$$

For the entire line, if all but the first term be neglected, the correction is

$$C_g = -\frac{1}{2}\left(\frac{h_1^2}{l_1} + \frac{h_2^2}{l_2} + \frac{h_3^2}{l_3} + \ldots \frac{h_n^2}{l_n}\right). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the l's are all equal, as when no two successive suspension points fall in the same grade, then we have

$$C_g = -\frac{1}{2l}(h_1^2 + h_2^2 + h_3^2 + \ldots h_n^2) = -\frac{\Sigma h^2}{2l}. \quad . \quad (3)$$

Since the relative elevations are determined, and not the angles of the grades, these formulæ are more readily applied than one involving the grade angles.

The error made in rejecting the second power of C in the above equations is given in the table on the following page, where l and h are taken in the same unit of length.*

If the grades are given in vertical angles, as they always are with the ordinary base apparatus, then we have for the correction to each section whose length is l, and whose grade is θ above or below the horizon,

$$C_g = -l(1 - \cos\theta) = -2l\sin^2\frac{\theta}{2}.$$

If θ be expressed in minutes of arc, and if the grade angle is less than about six degrees, or if the slope is less than one in ten, we may write

$$C_g = -2l\sin^2\frac{\theta}{2} = -\frac{1}{2}l\theta^2\sin^2 1' = -\frac{\sin^2 1'}{2}\theta^2 l$$

$$= -0.00000004231\ \theta^2 l;$$

* From Jäderin's Geodätische Längenmessung.

or by logarithms,

$$\log C_g = \text{const. } \log 2.626422 + 2 \log \theta + \log l.$$

TABLE OF ABSOLUTE ERRORS IN THE FORMULA $C_g = \frac{h^2}{2l}$

Length of Uniform Grade l	Absolute Error in the Same Units used for l and h. 0.00005	0.00015	0.00025	0.00035	0.00045
	h — Rise or Fall in Length l.				
1	0.14	0.19	....	....	....
2	.44	.31	....	....	....
3	.18	.49	....	....	....
4	.40	.55	....	....	....
5	.47	.64	....	....	....
6	.54	.71	0.81	....	....
7	.61	.80	.91	....	....
8	.67	.88	1.00	....	....
9	.71	.97	1.10	1.19	....
10	.70	1.05	1.19	1.20	....
11	.85	1.10	1.28	1.39	....
12	.91	1.20	1.30	1.48	....
13	.97	1.27	1.45	1.57	1.67
14	1.03	1.35	1.53	1.66	1.77
15	1.08	1.40	1.61	1.75	1.86
16	1.13	1.40	1.69	1.84	1.90
17	1.13	1.50	1.77	1.93	2.05
18	1.21	1.60	1.85	2.01	2.14
19	1.40	1.69	1.93	2.09	2.23
20	1.44	1.76	2.00	2.17	2.32
21	1.46	1.82	2.05	2.25	2.40
22	1.44	1.86	2.15	2.33	2.48
23	1.48	1.95	2.22	2.41	2.57
24	1.55	2.04	2.29	2.49	2.65
25	1.58	2.08	2.36	2.57	2.73
26	1.61	2.14	2.43	2.65	2.82
27	1.67	2.20	2.50	2.72	2.90
28	1.72	2.20	2.57	2.80	2.98
29	1.77	2.32	2.64	2.87	3.06
30	1.81	2.36	2.72	2.95	3.14

348. Correction for Sag.—From equation (4), p. 478, we have

$$C_s = -\frac{L}{24}\left(\frac{wd}{P}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the standard length be given with the pull P_0, and the distance between supports d_0, while in the field the pull P and distance d between supports be used, then the correction for sag is

$$C_s = \frac{Lw^2}{24}\left(\frac{d_0^2}{P_0^2} - \frac{d^2}{P^2}\right) = \frac{L}{24}\left(\frac{w}{P_0P}\right)^2(d_0^2P^2 - d^2P_0^2), \quad . \quad (5)$$

where L, d, and C_s are taken in the same unit of length, and w is the weight of a unit's length of tape in the same units used for P.

349. Correction for Pull.—From equation (1), p. 477, we may write at once

$$C_p = + \frac{(P - P_0)L}{SE}.$$

Here P is taken in pounds, L and C_p in inches, and S in square inches, since E is usually given in inch-pound units. If E has not been determined by experiment, it may be taken at 28000000. The cross-section S is best found by weighing the tape and computing its volume, counting 3.6 cubic inches to the pound. Knowing the length, the cross-section can then be found. If the stretch has been observed for different weights, and the value of E computed, the value of S is of no consequence, provided the same value be used for both observations.

350. Elimination of Corrections for Sag and Pull.—Since the correction for sag is negative and that for pull is positive, we may make them numerically equal, and so eliminate them both from the work. If this be done, the normal or standard length of the tape should be obtained for no sag and no pull, and its normal or standard temperature found such that at this temperature, and for no sag and no pull, its graduated length is its true length.

If T_0 is the temperature at which the tape is of standard length for the pull P_0 and the distance d_0 between supports, and if l is the length of the tape, then we have,

$$\text{Shortening from sag} = \frac{l}{24}\left(\frac{wd_0}{P_0}\right)^2,$$

$$\text{Lengthening from pull} = \frac{P_0 l}{SE},$$

$$\text{or net lengthening from sag and pull} = \frac{P_0 l}{SE} - \frac{l}{24}\left(\frac{wd_0}{P_0}\right)^2;$$

$$\text{Lengthening from } x \text{ degrees F.} = x\alpha l.$$

If, therefore, the effects of sag and pull were eliminated, the tape would be of standard length at a temperature x degrees above T_0, where

$$x = \frac{1}{\alpha}\left(\frac{P_0}{SE} - \frac{1}{24}\left(\frac{wd_0}{P_0}\right)^2\right), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where all dimensions are in inches and weights in pounds.

The standard temperature for no sag and no pull would be, therefore,

$$T_n = T_0 + x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We will call this the *normal temperature*.

In order that the corrections for sag and pull shall balance each other, we must have

$$\frac{Pl}{SE} = \frac{l}{24}\left(\frac{wd}{P}\right)^2,$$

or

$$P_n = \sqrt[3]{\frac{SE}{24}(wd)^2}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which we will call the ***normal tension.***

If the stretch in inches is known for one pound of pull for the given tape, we may call this e, and we will have

$$e = \frac{l}{SE} \quad \text{or} \quad SE = \frac{l}{e}.$$

Also, $lw = W =$ weight of entire tape between end graduations, or $w = \frac{W}{l}$.

And $\frac{l}{d} = n =$ number of sags in the tape.

Substituting these values in (3), we obtain

$$P_n = \sqrt[3]{\frac{W^2 l}{24 e n^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $W =$ weight of entire tape in pounds;
l = length of tape in inches;
e = elongation of tape for a one-pound pull;
n = number of sags in tape $= \frac{l}{d}$.

If the tape has no intermediate supports, then $n = 1$, and we have for the normal tension

$$P_n = \sqrt[3]{\frac{W^2 l}{24 e}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

EXAMPLE.—For the 300-foot steel tape, whose constants the author determined, we have $W = 2$ lbs., $l = 3600$ inches, $e = 0.066$ inch. If the supports are 30 feet apart, $n = 10$, whence, from eq. (4), $P_n = 4.48$ pounds.

If $n = 6$, or if the supports were placed 50 feet apart, we would find $P_n = 6.32$ pounds.

If $n = 3$, or if the supports are 100 feet apart, $P_n = 10.03$ pounds.

In the last case, the sag would be ten inches midway between supports.

351. To reduce a Broken Base to a Straight Line.—It is sometimes necessary or convenient to introduce one or more angles into a base-line. These would never deviate much from 180°. Let the difference between the angle and 180° be θ, and let the two measured sides be a and b, to find the side c. If θ be expressed in minutes of arc and if it is not more than about 3°, the following approximate formula will prove sufficiently exact:

$$\text{side } c = a + b - \frac{\sin^2 1'}{2} \cdot \frac{ab\theta^2}{a+b}$$

$$= a + b - 0.0000000423I \frac{ab\theta^2}{a+b}.$$

If θ is greater than from 3° to 5°, the triangle would have to be computed by the ordinary sine formula.

352. To reduce the Length of the Base to Sea-level.—In geodetic work, all distances are reduced to what they would be if the same lines were projected upon a sea-level surface by radii passing through the extremities of the lines. It is not necessary, however, to reduce all the lines of a triangulation system in this manner, since if the length of the base-line is so reduced the computed lengths of all the other lines of the system will be their lengths at sea-level. The angles that are measured are the *horizontal* angles, and are not affected by the differences of elevation of the various stations. It is necessary, therefore, to know the approximate elevation of the base above sea-level.

Let r = mean radius of earth;
a = elevation above sea-level;
B = length of measured base;
b = length of base at sea-level.

Then $r + a : r :: B : b$,

or
$$b = B\frac{r}{r+a}.$$

The correction to the measured length is always negative, and is

$$C = b - B = -B\left(1 - \frac{r}{r+a}\right) = -B\left(\frac{a}{r+a}\right).$$

Since a is very small as compared to r, we may write

$$C = -B\frac{a}{r}.$$

The mean radius* in feet is

$$\text{mean } r = \frac{20926062 + 20855121}{2} = 20890592 \text{ feet},$$

$$\log r \text{ (in feet)} = 7.3199507.$$

353. **Summary of Corrections.**—For the significance of the notation used in the following equations, see the preceding articles where they are derived. The corrections are all for

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* Rigidly, we should use the length of the *normal* for the given latitude, but the mean radius as above found is sufficient for most cases.

the entire line measured, or rather for that portion of it composed of entire tape-lengths, and are to be applied with the signs given to the measured length.

1. *Correction for Temperature.*

For a single standard with mercurial temperatures,

$$C_t = + \alpha(T_m - T_s)L. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For metallic thermometer-readings, as found from steel and brass standards, for instance, the correction to be applied to the length as found by the steel wire, or standard, is

$$C_{st} = n(S_s - S_b)\frac{\alpha_s}{\alpha_b - \alpha_s}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

2. *Correction for Grade.*

In terms of the difference of elevation of grade, points at a common distance, l, apart,

$$C_g = \frac{\Sigma h^2}{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In terms of the grade angles, expressed in minutes of arc,

$$C_g = - 0.00000004231\Sigma\theta^2 l. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

3. *Correction for Sag.*

For the standard length given for a pull P_s, and a distance between supports d_s, while P and d are used in the field-work,

$$C_s = \frac{L}{24}\left(\frac{w}{P_s P}\right)^2(d_s^2P^2 - d^2P_s^2). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For the standard length given for no pull and no sag,

$$C_s = -\frac{L}{24}\left(\frac{wd}{P}\right)^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

4. *Correction for Pull.*

$$C_p = +\frac{P - P_0}{SE} L ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$C_p = (P - P_0)en. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

5. *To reduce Standard Temperature to Normal Temperature.*

When the temperature of the tape (T_0) is known at which the graduated is the true length for the pull P_0 and distance between supports d_0, to find the corresponding temperature for no pull and no sag, this being called the ***normal temperature*** (T_n), we have, in degrees,

$$T_n = T_0 + \frac{1}{\alpha}\left[\frac{P_0}{SE} - \frac{1}{24}\left(\frac{wd_0}{P_0}\right)^2\right] . \quad . \quad . \quad . \quad . \quad (9)$$

6. *To eliminate Corrections for Sag and Pull.*

Make the pull $P_n = \sqrt[3]{\frac{SE}{24}(wd)^2}$; . . . . (10)

or

$$P_n = \sqrt[3]{\frac{W^2 l}{24en^2}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For no intermediate supports to tape,

$$P_n = \sqrt[3]{\frac{W^2 l}{24e}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

P_n is called the ***normal tension.***

7: *Correction for Broken Base.*

If a and b are the two measured sides which make an angle of $180° - \theta$, the correction to be added to $a + b$ to get the distance between their extremities, θ being less than $5°$, and expressed in minutes of arc, is

$$C_b = -0.0000000423 1 \frac{ab\theta^2}{a+b}.$$

8. *Correction to Sea-level.*

$$C_l = -L\frac{a}{r},$$

where L is the length of the measured base at an altitude a above sea-level.

$$\log r \text{ (in feet)} = 7.3199507.$$

354. **To compute any Portion of a Straight Base which cannot be directly measured.**—It sometimes is convenient

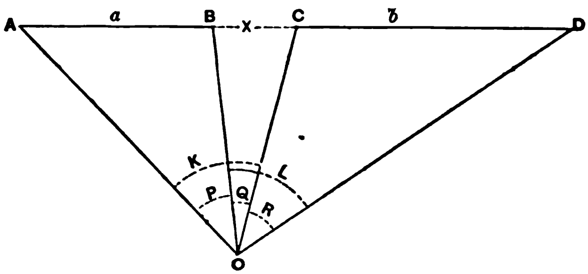

FIG. 131.

to take a base-line across a stream or other obstruction to direct measurement. In such a case a station may be chosen as O in Fig. 131, and the horizontal angles $AOB = P$, $BOC = Q$, and $COD = R$ measured. If the parts AB and CD lie in the same straight line, and $AB = a$ and $CD = b$ are known, then $BC = x$ may be found *by measuring only the angles at O.*

Thus in the triangles ABO and ACO we have

$$\frac{CO}{BO} = \frac{x + a}{a} \frac{\sin P}{\sin (P + Q)},$$

also from the triangles BDO and CDO we have

$$\frac{CO}{BO} = \frac{b}{x + b} \frac{\sin (Q + R)}{\sin R}.$$

Let $K = P + Q$ and $L = Q + R$, then by equating the above values of $\frac{CO}{BO}$ we have

$$(x + a)(x + b) = \frac{ab (\sin K \sin L)}{\sin P \sin R},$$

whence

$$x = -\frac{a + b}{2} \pm \sqrt{\frac{ab (\sin K \sin L)}{\sin P \sin R} + \left(\frac{a - b}{2}\right)^2}.$$

Evidently only the positive result is to be taken.

The points A, O, and D should be chosen so as to give good intersections at A and D.

355. Accuracy attainable by Steel-tape and Metallic-wire Measurements.—The following results have been attained by using the methods herein described:

1. In Sweden, Mr. Edw. Jäderin measured a primary base-line two kilometres in length three times, by means of steel and brass wires 25 metres long, in ordinary summer

weather, mostly clear, with a probable error of a single determination of 1 in 600,000, and a probable error of the mean result of 1 in 1,000,000, as compared with the true length of the line as obtained by a regular primary base apparatus.*

2. On the trigonometrical survey of the Missouri River, in 1885, Mr. O. B. Wheeler, U. S. Asst. Engineer, obtained the following results, using one steel-tape 300 feet long:

Glasgow Base.

First measurement..................	7923.237 feet.
Second "	7923.403 "
Mean........................	7923.320 $\pm$ 0.056 feet.

In this case the sun was shining more or less on both measurements. The probable error of a single result is 1 in 100,000, and of the mean of two measurements 1 in 140,000.

Benton Base.

First measurement..................	9870.443 feet.
Second "	9870.388 "
Mean........................	9870.415 $\pm$ 0.018 feet.

The probable error of a single measurement is 1 in 380,000, and of the mean, 1 in 533,000.

Trovers' Point Base.

First measurement..................	9711.915 feet.
Second "	9711.892 "
Mean........................	9711.904 $\pm$ 0.0078 feet.

* For title of Mr. Jäderin's pamphlet describing his methods and results, see foot-note, p. 474.

The probable error of a single measurement is 1 in 900,000, and of the mean it is 1 in 1,250,000.

Olney Base.

First measurement	10821.9658 feet.
Second "	10821.9665 "
Mean	10821.9662 ± 0.0002 feet.

This base had been measured by the U. S. Lake Survey Repsold base apparatus, with a probable error of about 1 in 1,000,000. This portion of it, about half the entire base, was remeasured with the tape in order to determine the absolute length of the tape. The work was done on both the tape-measurements in a drizzling rain, so that the temperatures were obtained with great accuracy. The mean temperatures of the two measurements differed, however, by several degrees, so that the two sets of graduations on the zinc strips were quite divergent, and it was only after the final reduction that the two results were known to be so nearly identical.*

3. The author has measured a number of bases about one half mile in length, in connection with students' practice surveys, by the methods given above, and in each case obtained a probable error of the mean of three or four measurements of less than one-millionth part of the length of the line. The work was always done on densely cloudy days, all the constants of tape and thermometers being well determined.

NOTE.—Prof. R. S. Woodward when assistant on the U. S. C. and G. Survey, in 1892, made five measurements of a base line 3,807 metres long, in four sections, using two steel tapes, making two measurements with each at night, and one measurement in the daytime in clear sunlight. These results gave a probable error in the mean of all of the results of $\frac{1}{2,000,000}$ part, not including the error in the length of the tape itself, and a probable error of $\frac{1}{1,280,000}$ part when all sources of error are taken into account. See a paper on *The Use of Long Steel Tapes for Measuring Base Lines*, *Trans. Am. Soc. C. E.*, Vol. XXX. (1893), p. 81.

* From the Report of the Missouri River Commission, 1886.

FIG. 132.

MEASUREMENT OF THE ANGLES.

356. **The Instruments** used in triangulation are designed especially for the accurate measurement of horizontal angles. This demands very accurate centring and fitting at the axis, and strict uniformity of graduation. It was formerly supposed that the larger the circle the more accurate the work which could be done. It is now known that there is no advantage in having the horizontal limb more than ten or twelve inches in diameter.

There are two general methods of reading fractional parts of the angle, smaller than the smallest graduated space on the limb. One is by verniers, the other by micrometer microscopes. Verniers may be successfully used to read angles to the nearest ten or twenty seconds of arc, but if a nearer approximation is desired microscopes should be employed.

Fig. 132 shows a high grade of vernier transit, capable also of reading vertical angles to 70°. Its horizontal limb is 8 inches in diameter and reads by verniers to ten seconds. It may be used as a repeating * instrument, and used either with or without a tripod. To mount such an instrument upon a station or post, a trivet, made of brass and shown in Fig. 135, is used. The pointed steel legs are driven into the station, the centre of the opening being over the station point. The arms have angular grooves cut in their upper surface. On this trivet may be set any three-legged instrument, so long as the radius of its base is not greater than the length of the trivet arms.

In Fig. 133 is shown a theodolite (not a transit since the telescope does not revolve on its horizontal axis) designed for the measurement of horizontal angles exclusively. Here micrometer microscopes are used. The horizontal limb is from eight to twelve inches in diameter. There is no vertical circle or arc, so that no vertical angles can be read. Since the relative heights of triangulation stations are usually determined

* See Art. 359 for an explanation of this term.

FIG. 133.

FIG. 134.

from their relative angular elevations or depressions, it is usually necessary to have a vertical limb. This instrument could not be used as a repeater.

In Fig. 134 is shown an altazimuth instrument, or an instrument designed for accurately measuring altitudes as well as the azimuths of points or lines. Both horizontal and vertical limbs are read by means of micrometer microscopes. Such an instrument is designed especially for astronomical observations for latitude and azimuth, but may also be used as a meridian or transit instrument for observing time as well as for measuring horizontal and vertical angles in triangulation. It is in fact the universal geodetic instrument, just as the complete engineer's transit is the universal instrument in ordinary surveying. In almost all cases where micrometers are used in reading the angles the limbs are graduated to five or ten minutes and the readings made to single seconds.

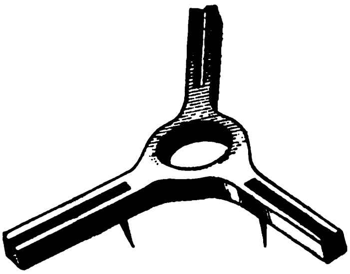

FIG. 135.

357. **The Filar Micrometer*** is used for the accurate measurement of small distances or angles, when the required exactness is greater than can be obtained by means of a vernier scale. It is usually combined with a microscope, the micrometer threads and scale lying in the plane of the image produced by the objective. This image is always larger than the object itself in microscopes, and therefore a given movement of the wires in the micrometer corresponds to a very much less distance on the object sighted at, according to the magnifying power of the objective.

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* From *filum* thread; *micros*, small, and *metros*, measure. The thread is in this case a spider's web, or scratches on glass.

The frame holding the movable wires has a screw with a very fine thread working in it, called the micrometer screw. This screw has a graduated cylindrical head, or disk, attached to it, there usually being sixty divisions in the circumference when used in angular measurements. The number of whole revolutions are recorded by noting how many teeth of a comb-scale are passed over, this scale being nearly in the plane of the wires and therefore in the focus of the eye-piece. The fractional parts of a revolution are read on the graduated screw-head outside. These micrometer attachments are shown on the two microscopes in Fig. 133 and on the five in Fig. 134.

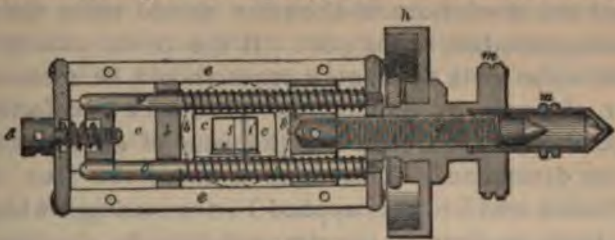

FIG. 136.

Fig. 136 is a sectional view of a filar micrometer. The graduated head *h* is attached to the milled head *m*, forming a nut into which the micrometer-screw *a* works. This screw is rigidly attached to the frame *b*, to which are fastened the movable wires *f'*. The comb-scale *s* and fixed wire *f* are attached to the frame *c*, which is adjusted to a zero-reading of the graduated head by the capstan-screw *d*. The lost motion on both of these frames is taken up by springs. The complete revolutions of the screw are counted on the comb-scale, and the fractional part of a revolution on the graduated head. The reading is made by bringing the double wires symmetrically over a graduation, the space between the wires being a little more than the width of the graduation, when the exact number of revolutions and sixtieths are read on the comb-scale and on the head.

If the limb is graduated to ten minutes and each revolution corresponds to one minute, then if the reading is taken on the *nearest* graduation, the number of revolutions need never exceed five. If, however, the reading be always taken to the last ten-minute mark counted on the limb, then ten revolutions may have to be read on the screw. The movement of the threads is as they appear to be, there being no inversion of image between wires and eye. The movement on the limb is, however, opposite from the apparent motion.

If the limb is graduated to ten minutes, and a single revolution of the screw corresponds to the space of one minute, then just ten revolutions of the screw should move the wires from one graduation to the next. If this is not exactly true, then the value of a ten-minute space should be measured a number of times, by running the wires back and forth, the mean result taken, and from this the value of *one revolution* of the screw determined. This value is called the "run of the screw," and a correction is applied to the readings, which are always made in degrees, minutes, and seconds, counting one revolution a minute and one division on the head a second of arc. This correction is called "correction for run," and should be determined for all parts of the screw used. If the value of one revolution is not exactly what it is designed to be, it can be adjusted by moving the objective of the microscope in or out a little, or the whole microscope up or down with reference to the limb, thereby changing the size of the image. Even when this adjustment is accurately made, there may be still a correction for run on account of the screw-threads not being of uniform value. In this case the value of each revolution of the screw is determined independently, these values tabulated, and the correction for run from this source determined for any given reading. Again, as the microscope revolves around the limb with the alidade, the plane of the graduations may not remain at a constant distance from the objective, in which case the size of the image would vary to a corresponding degree. To determine this, the values of ten-minute spaces are determined on various parts of the limb, and if these are not constant, then a table of corrections for run may be made out for different parts of the circle.

For reading on graduated straight lines the double threads give better results than either the single thread or the intersecting threads. The space between the threads should be a little greater than the width of the image of the graduation-line, so that a narrow strip of the limb's illuminated upper surface may appear on either side of the graduation and inside the wires. The setting is then made so as to make these illuminated lines of equal width. It is conceded that such an arrangement will give more exact readings than any other that has been used.

The magnifying power of the microscope is from thirty to fifty.

358. Programme of Observations.—There are two general methods of reading angles in triangulation work. One method consists in measuring each angle independently, usually by repeating it a number of times by successive additions on the limb, and then reading this multiplied angle, which is divided by the number of repetitions to give the true value of the angle. In the other method the readings are made on the several stations in order, as *A*, *B*, *C*, *D*, and *E*, in the figure, and the angles found by taking the difference between the successive readings. Each method has its advantages and disadvantages. If the instrument has an accurate fitting in the axis, clamps which can be set and loosened without disturbing the positions of the plates, is provided with verniers which have a coarse reading, as twenty or thirty seconds, and accurate work is desired, and if such an instrument

FIG. 137.

is mounted on a low, firm station, then the method by repetition would give superior results. If any of these conditions are not fulfilled, and especially if the instrument is provided with micrometer microscopes, whereby readings may be taken to the nearest second of arc, it is much more convenient, cheaper, and generally more accurate to read the stations continuously around the horizon, back and forth, until a sufficient number of readings have been obtained.

359. **The Repeating Method.**—This method was formerly used almost exclusively, but the other is the only one now used with the most accurate instruments. It was found that systematic errors were introduced in the method by repetition of a single angle, due largely to the clamping apparatus. If this method is used the repetitions should be made first towards the right and then towards the left; the number of repetitions making a set should be such as to make the multiplied angle a multiple of 360°, as nearly as possible, so as to eliminate errors of graduation on the limb. Thus, for an angle of 60° repeat it six times and then read. For the second set repeat six times in the opposite direction, and with telescope inverted. If triangulation work is to be done with the ordidary engineer's transit, which reads only to 30 seconds or one minute, this method may give very fair results, *provided there is no movement of circles from the use of the clamping apparatus and no lost motion in the axes.* The programme would be as follows:

PROGRAMME.

Telescope Normal.

1. Set on left station, and read both verniers.
2. Unclamp above and set on right station.
3. " below " " left "
4. " above " " right "
5. " below " " left "
6. " above " " right "

etc., etc.,

until the entire circle has been traversed, then read both verniers while pointing to right station. The total angle divided by the number of repetitions is the measure of the angle sought.

Telescope Reversed.

1.	Set on right station, and read both verniers.					
2.	Unclamp	above	and set on		left station.	
3.	"	below	"	"	right	"
4.	"	above	"	"	left	"
5.	"	below	"	"	right	"
6.	"	above	"	"	left	"
		etc.,			etc.,	

until the entire circle has been traversed by each vernier, when both verniers are read on the left station.

The repetition in opposite directions is designed to eliminate errors from clamp and axis movements, and the reversing of the telescope is designed to eliminate errors arising from the line of sight not being perpendicular to the horizontal axis, and from the horizontal axis not being perpendicular to the vertical axis of the instrument.*

As many such sets of readings may be made as desired, but there should always be an even number, or as many of one kind as of the other. It will be observed that two *pointings* are taken for each measurement of the angle, but comparatively few *readings* are made.

360. Method by Continuous Reading around the Horizon.—By this method the limb is clamped in any position, and

* In case the instrument used is a theodolite, and its telescope cannot be revolved on its horizontal axis, it should be lifted from the pivot bearings and turned over end for end, leaving the pivots in their former bearings. If this cannot be done conveniently, then the limb should be shifted by $\frac{360}{n}$ (see next page) each time, and this will result in mostly eliminating these same errors of collimation and inclination of horizontal axis

left undisturbed except between the different sets of readings. The pointings are made to the stations in succession around the horizon, and both verniers, or microscopes, read for each pointing. Thus, if the instrument were at o, Fig. 137, the pointings would be made to A, B, C, D, and E. If the telescope is now carried around to the right until the line of sight again falls on A, and a reading taken, the observer is said to close the horizon; that is, he has moved the telescope continuously around in one direction to the point of beginning. If the two readings here do not agree, the error is distributed among the angles in proportion to their number, irrespective of their size. It is questionable whether such an adjustment adds much to the accuracy of the angle values, and therefore it is common to read to the several stations back and forth without closing the horizon. Sum-angles can afterwards be read if desired. Thus, after the regular readings have been taken on the stations, the angle AOE, or AOC, and COE, may be read, and so one or more equations of condition obtained.

If the station is tall, there is always a twisting of its top in clear weather in the direction of the sun's movement. This twisting effect has been observed to be as much as $1''$ in a minute of time on a seventy-five-foot station. To eliminate this action the readings are taken both to the right and to the left. The reading of opposite verniers, or microscopes, eliminates errors of eccentricity, the inverting of the telescope eliminates errors of adjustment in the line of collimation and horizontal axis, and to eliminate periodic errors of graduation each angle is read on symmetrically distributed portions of the limb. To accomplish this the limb is shifted after each set of readings an amount equal to $\frac{180^\circ}{n}$,* where n is the number of sets of readings to be taken. The following is the

* For exception, see foot-note on previous page.

PROGRAMME.

1ST SET.	*Telescope normal.*	2D SET.	*Telescope inverted.*
	Read to right.		Read to right.
	Read to left.		Read to left.
	Telescope inverted.		*Telescope normal.*
	Read to right.		Read to right.
	Read to left.		Read to left.
	Shift the Limb.		Shift the Limb.

Evidently each set is complete in itself, and as many complete sets may be taken as desired, but no partial sets should be used. If the work is interrupted in the midst of one set of readings, the partial set of readings should be rejected, and when the work is resumed another set begun. In reducing the work, if one reading of any angle is so erroneous as to have to be rejected this should vitiate that entire set of readings of that angle.

If preferred, the telescope may be inverted between the right and left readings, and then two readings on each mark would constitute a complete set, when the limb could be shifted again. If this were done, the readings at o, Fig. 137, would be:

1ST SET.	Telescope Normal—Read	*ABCDE.*
	" Inverted "	*EDCBA.*
	Shift the Limb.	
2D SET.	Telescope Inverted—Read	*ABCDE.*
	" Normal "	*EDCBA.*
	Shift the Limb.	

361. Atmospheric Conditions.—In clear weather not even fair results can be obtained during the greater part of the day. From sunrise till about four o'clock in the afternoon in summer the air is so unsteady from the heated air-currents that any distant target is either invisible or else its image is so unsteady as to make a pointing to it very uncertain. From about four o'clock till dark in clear weather, and all day in densely cloudy weather with clear air, good work can be done. If heliotropes are used, the work is limited to clear weather. It has often been proposed to do such work at night, but the lack of a simple and efficient light of sufficient strength has usually prevented. The higher the line of sight above the ground the less it is affected by atmospheric disturbances.

362. Geodetic Night Signals.—Mr. C. O. Boutelle, of the U. S. Coast and Geodetic Survey, made a series of experiments in 1879 at Sugar Loaf Mountain, Maryland, for the purpose of testing the efficiency of certain night signals and the comparative values of day and night work. His report is given in Appendix No. 8 of the Report of the U. S. C. and G. Survey for 1880. It seems that either the common Argand or the "Electric" coal-oil lamp, assisted by a parabolic reflector or by a large lens, gives a light visible for over forty miles. His conclusions are:

1. That night observations are a little more accurate than those by day, but the difference is slight.

2. That the cost of the apparatus is less than that of good heliotropes.

3. That the apparatus can be manipulated by the same class of men as those ordinarily employed as heliotropers.

4. That the average time of observing in clear weather may be more than doubled by observing at night, and thus the time of occupation of a station proportionately shortened; "clear-cloudy" weather, when heliotropes cannot show, can be utilized at night.

363. Reduction to the Centre.—It sometimes happens that the instrument cannot be set directly over the geodetic point, as when a tower or steeple is used for such point. In this case two angles of each of the triangles meeting here may be measured and the third taken to be 180° minus their sum, or the instrument may be mounted near to the geodetic point and all the angles at this station measured from this position. These angles will then be very nearly the same as though measured from the true position, and may readily be reduced to what they would have been if the true station point had been occupied. Thus in Fig. 138 let C be the true station to which pointings were taken from other stations, and C' the position of the instrument for measuring the angles at this station. The line AB is a side of the system whose length has been found. From the measured angles at A and B the approximate value of the angle C is found and the lengths of the sides a and b computed. At C' the angle $AC'B$ is measured with the same exactness as though it were the angle C itself and the angle $CC'B = \alpha$ is measured by a single observation. The distance $CC' = r$ is also found. Since the exterior angle at the intersection D, as ADB, is equal to the sum of the opposite interior angles, we have

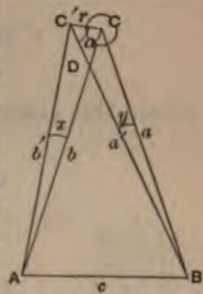

FIG. 138.

$$C + y = C' + x, \qquad \text{or} \qquad C = C' + (x - y). \quad . \quad . \quad (1)$$

In the triangle $AC'C$ we have the sides b and r and the angle $AC'C$ known, whence

$$\left.\begin{aligned} \sin x &= \frac{r \sin (C' + \alpha)}{b}; \\ \text{similarly} \qquad \sin y &= \frac{r \sin \alpha}{a}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since x and y are very small angles, their sines are proportional to their arcs, and we may write $\sin x = x \sin 1''$ where

x is expressed in seconds; similarly $\sin y = y \sin 1''$, and equations (2) become

$$\left.\begin{aligned} x &= \frac{r \sin (C' + \alpha)}{b \sin 1''}; \\ y &= \frac{r \sin \alpha}{a \sin 1''}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting these values in (1) we have

$$C = C' + \frac{r}{\sin 1''}\left(\frac{\sin C' + \alpha}{b} - \frac{\sin \alpha}{a}\right), \quad . \quad . \quad . \quad (4)$$

where the correction to C' is given in seconds of arc. The signs of the trigonometrical functions of the angle α must be carefully attended to, as it is measured continuously from B around to the left to 360°.

The following is another solution of the same problem: Measure the perpendiculars from C upon AC' and BC', Fig. 139, calling them m and n respectively. Then from equation (1) above we have

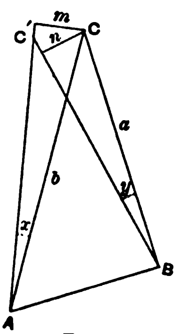

FIG. 139

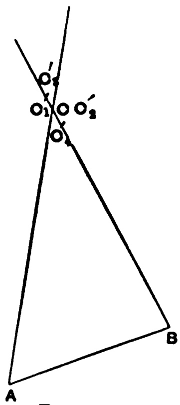

FIG. 140.

$$C = C' + (x - y).$$

But since the angles x and y are very small, their sines are equal to their arcs, and we have, in seconds of arc,

$$x = \frac{m}{b \sin 1''} \quad \text{and} \quad y = \frac{n}{a \sin 1''},$$

whence

$$C = C' + \frac{1}{\sin 1''}\left(\frac{m}{b} - \frac{n}{a}\right). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

There are four cases corresponding to the four positions of C', as shown in Fig. 140. For these several cases we have

$$\left.\begin{aligned} C &= C_1' + \frac{1}{\sin 1''}\left(\frac{m}{b} - \frac{n}{a}\right); \\ C &= C_2' - \frac{1}{\sin 1''}\left(\frac{m}{b} - \frac{n}{a}\right); \\ C &= C_3' + \frac{1}{\sin 1''}\left(\frac{m}{b} + \frac{n}{a}\right); \\ C &= C_4' - \frac{1}{\sin 1''}\left(\frac{m}{b} + \frac{n}{a}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

ADJUSTMENT OF THE MEASURED ANGLES.

364. Equations of Conditions.—When any continuous quantity, as an angle or a line, is measured, the observed value is always affected by certain small errors. Indeed, it would not be possible even to express exactly the value of a continuous quantity in terms of any unit, as degrees or feet and fractional parts of the same, even though this value could be exactly determined. If, therefore, the measured values of the three angles of a triangle be added together, the sum will not be exactly 180°. But we know that a rigid *condition* of all triangles is that the sum of the three angles is 180°. An equation which expresses a relation between any number of observed quantities which of geometrical necessity must exist is called an equation of condition, or a condition equation. Thus, in the above case, if A', B', and C' be the mean observed values of the angles, and A, B, and C their true values, we would have for our condition equation

$$A + B + C = 180°. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We would also have

$$A' + a' = A; \ B' + b' = B; \ C' + c' = C,$$

where a', b', and c' are small corrections to the measured values A', B', and C' which are to be determined.

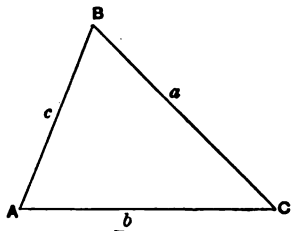

FIG. 141.

Let us suppose that the length of the side b has been exactly measured,* then when the true values of the angles are found we may compute the other two sides. If the sides b and c have both been measured, the length of the side c as computed from b must agree with its measured length, and so we might write the condition equation

$$c = \frac{b \sin (C' + c')}{\sin (B' + b')}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, if the side a had been measured and its exact length found, we would obtain the third condition equation,

$$a = \frac{b \sin (A' + a')}{\sin (B' + b')}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We now have three independent equations involving three unknown quantities, and can, therefore, find the quantities a', b', and c'. But if only one side had been measured, we should have had but one equation from which to determine three unknown quantities. Evidently there is an infinite number of

* This assumption is made in regard to the measured base-lines in a triangulation system, since its exactness is so much greater than can be obtained in the angle-measurements.

sets of values of a', b', and c', which would satisfy this equation. If we now impose the condition that the corrections shall be the *most probable* ones, then there is but one set of values that can be taken.

Equation (1) is called an *angle equation*, since only angles are involved; while equations (2) and (3) are called *side equations*, since the lengths of the sides are also involved.

365. Adjustment of a Triangle.—The finding and applying of the most probable corrections to the measured values of the angles of a system of triangulation is called *adjusting the system*. In the case of a single triangle with one known side and three measured angles, we have seen that there is but one equation of condition. If the three angles have been equally well observed, then it is most probable* that they are all equally in error, and hence this condition of highest probability gives us the probability equation

$$a' = b' = c' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which enables the corrections to be determined.

Thus, let $A' + B' + C' - 180° = a' + b' + c' = V$,

then from (4) we have

$$a' = b' = c' = \frac{V}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where V is the error of closure of the triangle.

* That is, this relation is more probable than are any other *single* relation that can be assigned, but of course it is not more probable than *all* other cases combined.

ADJUSTMENT OF A QUADRILATERAL.

366. The Geometrical Conditions.—In the quadrilateral in Fig. 142 there are eight observed angles, A_1, B_2, B_3, C_4, etc. The geometrical conditions which must here be fulfilled are:

(*a*) The sum of all the angles of any triangle must be 180° plus the spherical excess* and the opposite angles at the intersection of the diagonals must be equal.

(*b*) The computed length of any side, as *DC*, when obtained from any other side, as *AB*, through two independent sets of triangles, as *ABC*, *BDC*, and *ABD*, *ADC*, shall be the same in both cases.

The probability condition is that the set of corrections applied to the several angles shall be more probable than any other one of the infinite number of sets of corrections which would satisfy the other condition.

The condition given in (*a*) gives rise to the angle equations, and that given in (*b*) gives one side equation.

There are evidently eight unknown corrections to be determined.

367. The Angle-equation Adjustment.—In the quadrilateral *ABCD* we have four triangles in which all the angles have been observed, two sets of opposite angles where the other two angles of the corresponding triangles have been observed, and the quadrilateral itself in which all the angles have

* It is not necessary to take account of the spherical excess in computing a single triangle or quadrilateral; but if azimuth is to be carried over a series of triangles it is necessary that all the angles be spherical angles. In this place spherical excess will be omitted; but if it is desirable to introduce it, it is inserted in equations (1), (2), and (3), the right members then becoming $l_1 + e_1$, $l_2 + e_2$, and $l_3 + e_3$, where e_1 is the residual excess of the angle *AOB* over that of the angle *DOC* (being negative in this case), e_2 is the excess of angle *BOC* over that of the angle *AOD*, and e_3 is the spherical excess for the entire quadrilateral. The spherical excess may be taken as 1″ for each 75 square miles of area, and this is to be divided equally amongst the angles of the figure. The formula for spherical excess is E (in seconds) $= \dfrac{206000\,A}{r^2}$, where A is area in square miles, and r is radius of the earth in miles.

been observed; making, in all, seven geometric conditions to be fulfilled. Only three of these conditions are independent, however, since where any three independent conditions are fulfilled the remaining four are fulfilled also. Thus, a great variety of conditioned equations could be formed, but we will

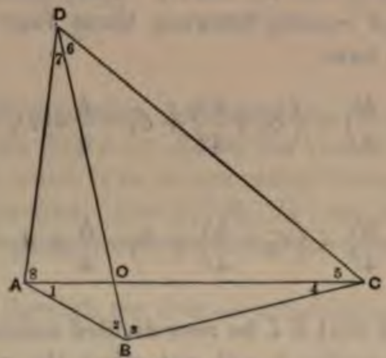

FIG. 142.

choose the three which give the simplest equations, viz.: that the opposite central angles shall be equal, and that the sum of all the angles of the quadrilateral shall be 360°. These give rise to the following equations:

If A_1, B_2, B_3, C_4, etc., be the observed angles, and l_1, l_2, and l_3 the residuals in the several combinations, due to erroneous determinations, then we have:

$$180° - (A_1 + B_2) - \{180 - (C_5 + D_6)\} = l_1,$$

$$\text{or} \quad -A_1 - B_2 \quad + C_5 + D_6 \quad = l_1. \quad (1)$$

$$\text{Similarly} \quad -B_3 - C_4 \quad + D_7 + A_8 \quad = l_2, \quad (2)$$

$$\text{and} \quad A_1 + B_2 + B_3 + C_4 + C_5 + D_6 + D_7 + A_8 - 360° = l_3. \quad (3)$$

If the angles have all been equally well observed,—that is, if their mean observed values have equal credence,—then they are said to have equal weight, and any residual arising from any combination of angles should be distributed uniformly among the angles forming such combination.* Thus l_1 arises from the angles A_1, B_2, C_5, and D_6. This residual should therefore, be divided equally between these four angles. When this is done we have

$$-\left(A_1+\frac{l_1}{4}\right)-\left(B_2+\frac{l_1}{4}\right)+C_5-\frac{l_1}{4}+D_6-\frac{l_1}{4}=0. \quad . \quad (4)$$

Similarly

$$-\left(B_3+\frac{l_2}{4}\right)-\left(C_4+\frac{l_2}{4}\right)+D_7-\frac{l_2}{4}+A_8-\frac{l_2}{4}=0. \quad . \quad (5)$$

It is evident that if l_3 be now divided uniformly among the eight observed angles, it will not affect the two adjustments already made; neither have the adjustments already made affected the third condition, expressed by eq. (3), since equal amounts have been added and subtracted. Hence these adjustments may be made in sequence as well as simultaneously, and we shall have for the total corrections for angle-equations

$$\begin{aligned} A_1&-\left(\frac{l_3}{8}-\frac{l_1}{4}\right)+B_2-\left(\frac{l_3}{8}-\frac{l_1}{4}\right)+B_3-\left(\frac{l_3}{8}-\frac{l_2}{4}\right)+C_4 \\ &-\left(\frac{l_3}{8}-\frac{l_2}{4}\right)+C_5-\left(\frac{l_3}{8}+\frac{l_1}{4}\right)+D_6-\left(\frac{l_3}{8}+\frac{l_1}{4}\right)+D_7 \\ &-\left(\frac{l_3}{8}+\frac{l_2}{4}\right)+A_8-\left(\frac{l_3}{8}+\frac{l_2}{4}\right)-360°=0 \quad . \quad . \quad . \quad . \quad . \quad (6) \end{aligned}$$

* The errors in the mean observed values of the angles are supposed to result from the small incidental errors and approximations made in pointing,

Or if v_1, v_2, v_3, etc., be the total corrections to the several mean observed angles for angle-equations, we have

$$\left.\begin{aligned} v_1 = v_2 &= -\frac{l_3 - 2l_1}{8}, \qquad v_5 = v_6 = -\frac{l_3 + 2l_1}{8}; \\ v_3 = v_4 &= -\frac{l_3 - 2l_2}{8}, \qquad v_7 = v_8 = -\frac{l_3 + 2l_2}{8}; \end{aligned}\right\} . \qquad (7)$$

368. The Side-equation Adjustment.—In the quadrilateral shown in the figure, let AB be the known side, and CD the required side, which is to be computed through two independent sets of triangles. Let A_1', B_2', B_3', etc., be the several angles corrected for angle conditions by the corrections found in eq. (7).

As computed through the first set of triangles, we have

$$DC = \frac{BC \sin B_3'}{\sin D_6'} = \frac{AB \sin A_1' \sin B_3'}{\sin C_4' \sin D_6'}. \quad . \quad . \quad (8)$$

Similarly $$DC = \frac{AD \sin A_8'}{\sin C_5'} = \frac{AB \sin B_2' \sin A_8'}{\sin C_5' \sin D_7'}. \quad . \quad . \quad (9)$$

Whence $$\frac{\sin A_1' \sin B_3'}{\sin C_4' \sin D_6'} = \frac{\sin B_2' \sin A_8'}{\sin C_5' \sin D_7'};$$

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reading, etc.; in other words, they are supposed to be *errors of observation* and not *instrumental errors*, these latter having been eliminated by the method of making the observations. Since the sources of the errors of observation are the same for small as for large angles, it follows that they should be credited with equal portions of the aggregate error of any combination of angles, regardless of the size of the angles themselves.

or
$$\frac{\sin A_1' \sin B_3' \sin C_5' \sin D_7'}{\sin B_2' \sin C_4' \sin D_6' \sin A_8'} = 1, \quad . \quad . \quad (10)$$

which is called the side-equation.

It is evident that in any case where the angles have all been observed, even after they have been adjusted for the angle-conditions, this equation will not hold true, the value of the left member being a little more or less than one. When put into the logarithmic form for computation, therefore, we will have

$$\log \sin A_1' + \log \sin B_3' + \log \sin C_5' + \log \sin D_7' \\ -\log \sin B_2' - \log \sin C_4' - \log \sin D_6' - \log \sin A_8' = l_4, \quad (11)$$

where l_4 is the logarithmic residual due to erroneous observations.

We must now distribute this residual l_4 among the log sines according to the most probable manner of the occurrence of the errors which caused it. For a given small error, as 1″, in any angle, the effect on the log sine is measured by the logarithmic tabular difference for 1″ for that angle. This tabular difference varies for different angles, being large for angles near zero or 180°, and small for angles near 90°.

Let v_1', v_2', v_3', etc., be the corrections to be made to the angles A_1', B_2', B_3', etc., for the side-equation (11), and let d_1, d_2, d_3, etc., be the corresponding logarithmic tabular differences for 1″.

Now, the influences on l_4 of the small angular errors were in direct proportion to the tabular differences of the corresponding log sines; therefore the corrections should be in proportion to the corresponding tabular differences. In other words, the corrections are weighted in proportion to their tabular differences.* We therefore have the numerical relation:

$$v_1' : d_1 :: v_2' : d_2 :: v_3' : d_3, \text{ etc.},$$

or, paying attention to signs,

$$\frac{v_1'}{d_1} = -\frac{v_2'}{d_2} = \frac{v_3'}{d_3} = -\frac{v_4'}{d_4} = \text{etc.} \quad . \quad . \quad . \qquad (12)$$

But since the log-sine correction is the angular correction multiplied by the tabular difference, and since the sum of these would equal l_4, we would have

$$v_1'd_1 - v_2'd_2 + v_3'd_3 - v_4'd_4 + v_5'd_5 - v_6'd_6 + v_7'd_7 - v_8'd_8 = -l_4 \quad . \quad . \qquad (13)$$

From equations (12) and (13) we are to find the side-equation corrections v_1', v_2', v_3', etc.

Dividing eq. (13) by eq. (12), term by term, we have

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2 + d_7^2 + d_8^2$$
$$= -\frac{l_4 d_1}{v_1'} = +\frac{l_4 d_2}{v_2'} = -\frac{l_4 d_3}{v_3'} = +\text{ etc.}$$

---

* To illustrate this principle more fully, let us suppose that for a change of 1″ in the angles A_1 and A_2 the corresponding changes in the log sines are 1 for A_1 and 2 for A_2; then for a given error of 1 in log sin A_1 + log sin $A_2 = l$ there are two chances that it came from A_2 to one chance that it came from A_1, when these angles were equally well observed. If the error is to be divided between the angles A_1 and A_2, therefore, we should make the correction to A_2 twice as great as the correction to A_1, or $v_1' : v_2' :: d_1 : d_2$, whence $\frac{v_1'}{d_1} = \frac{v_2'}{d_2}$. The same reasoning would hold evidently for any number of angles, hence eq. (12).

† By eq. (11) it will be seen that the corrections to the angles having odd subscripts must be of opposite sign from those having even subscripts.

Whence we have, for the values of these corrections,

$$\frac{v_1'}{d_1} = -\frac{v_2'}{d_2} = \frac{v_3'}{d_3} = -\frac{v_4'}{d_4} = \frac{v_5'}{d_5} = -\frac{v_6'}{d_6} = \frac{v_7'}{d_7}$$

$$= -\frac{v_8'}{d_8} = -\frac{l_4}{\Sigma(d^2)}. \quad (14)$$

We have now found a set of corrections, v_1, v_2, v_3, etc. (eq. 7), for the angle-equations, and a set of corrections, v_1', v_2', v_3', etc. (eq. 14), for the side-equation; but they were determined independently and not simultaneously, and therefore, when successively applied, each set of corrections will disturb the former adjustment somewhat. Thus, if the corrections in eq. (7) be first applied, and then those of eq. (14), using the partially corrected angles in finding l_4 by eq. (11), we would find eq. (10) would be satisfied, but l_1, l_2, and l_3, in equations (1), (2), and (3), would now not be zero when the newly adjusted angles were used. Another set of corrections v_1'', v_2'', v_3'', etc., might now be found by eq. (7) for the adjusted angles A_1'', B_2'', B_3'', etc., and so on by successive approximations, using the corrections of equations (7) and (14) alternately, until both sets of conditions were satisfied within the desired limits. It will usually be found, however, that the adjustment for side-equation does not materially disturb that for angle-equations. If the angles were all the same size, so that the corrections to the log sines would have equal weight, the first adjustment would remain undisturbed. In this case, the corrections for side-equation would all be numerically equal, the odd and even subscripts having opposite signs. If the observed angles range between 30° and 60°, as they would in a fairly symmetrical quadrilateral, then the errors of this approximation would be quite inappreciable.

369. Rigorous Adjustment for Angle- and Side-equations.—Let the angle-equation adjustments be applied as given by eq. (7). Then, using these adjusted angles, let the corrections to the angles for side-equation be so expressed that they shall not be inconsistent with the angle-equation conditions, whatever their values. This may be done by letting

$$\left.\begin{array}{ll} v_1' = x_0 + x_1, & v_5' = x_0 + x_3; \\ v_2' = x_0 - x_1, & v_6' = x_0 - x_3; \\ v_3' = -x_0 + x_2, & v_7' = -x_0 + x_4; \\ v_4' = -x_0 - x_2, & v_8' = -x_0 - x_4. \end{array}\right\} \quad . \quad . \quad (15)$$

Then, analogous to eq. (13), we may write

$$\begin{aligned} &d_1(x_0 + x_1) - d_2(x_0 - x_1) - d_3(x_0 - x_2) + d_4(x_0 + x_2) \\ &\quad + d_5(x_0 + x_3) - d_6(x_0 - x_3) - d_7(x_0 - x_4) + d_8(x_0 + x_4) = -l_4; \quad . \quad (16) \end{aligned}$$

or

$$\begin{aligned} &(d_1 - d_2 - d_3 + d_4 + d_5 - d_6 - d_7 + d_8)x_0 + (d_1 + d_2)x_1 \\ &\qquad + (d_3 + d_4)x_2 + (d_5 + d_6)x_3 + (d_7 + d_8)x_4 = -l_4, \quad . \quad (17) \end{aligned}$$

wherein l_4 is given by eq. (11), and the d's are the tabular differences for one second for the several log sines as before.

If, for simplicity, we write for the coefficients of x_0, x_1, x_2, x_3, and x_4, respectively, C_0, C_1, C_2, C_3, and C_4, then (17) becomes

$$C_0x_0 + C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 = -l_4. \quad . \quad . \quad (18)$$

It now remains to find the values of x_0, x_1, x_2, x_3, and x_4, such that their combinations which make up the angle-corrections as given in eqs. (15) shall be the most probable.

To make (18) symmetrical with (15), we may put it in the following form:

$$\left(\frac{C_0}{4}x_0 + C_1x_1\right) + \left(\frac{C_0}{4}x_0 + C_2x_2\right) + \left(\frac{C_0}{4}x_0 + C_3x_3\right)$$
$$+ \left(\frac{C_0}{4}x_0 + C_4x_4\right) = -l_4. \quad . \quad (19)$$

In the argument preceding the derivation of eq. (12) it was found that the measured angles required to be corrected by a series of quantities (v_1', v_2', etc.), which quantities were found to be to each other as the tabular differences of the angles themselves; and eq. (13), which is a summation of log sine corrections, shows that when eq. 12 is true it is equivalent to saying that the most probable set of angle corrections (v_1', v_2', etc.) is that set which are respectively proportional to their numerical coefficients (d_1, d_2, etc.). This is, in fact, a general law of the theory of probabilities; and hence we say, in eq. (19), that the most probable corrections (x_0, x_1, etc.) are those which are proportional to their several numerical coefficients, or, we may write at once, as a condition of the greatest probability:

$$x_0 : x_1 :: \frac{C_0}{4} : C_1 ; \; x_0 : x_2 :: \frac{C_0}{4} : \frac{C_2}{x_2}, \text{ etc.};$$

or,
$$\frac{4x_0}{C_0} = \frac{x_1}{C_1}; \; \frac{4x_0}{C_0} = \frac{x_2}{C_2}, \text{ etc.}$$

Therefore the condition of the highest probability gives

$$\frac{4x_0}{C_0} = \frac{x_1}{C_1} = \frac{x_2}{C_2} = \frac{x_3}{C_3} = \frac{x_4}{C_4}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Dividing (18) by (20), term by term, we have

$$\frac{C_0^2}{4} + C_1^2 + C_2^2 + C_3^2 + C_4^2 = -\frac{C_0 l_4}{4x_0} = -\frac{C_1 l_4}{x_1}$$

$$= -\frac{C_2 l_4}{x_2} = -\frac{C_3 l_4}{x_3} = -\frac{C_4 l_4}{x_4};$$

whence

$$\frac{4x_0}{C_0} = \frac{x_1}{C_1} = \frac{x_2}{C_2} = \frac{x_3}{C_3} = \frac{x_4}{C_4} = -\frac{l_4}{\frac{C_0^2}{4} + \Sigma(C^2)^*}. \quad . \quad (21)$$

From equation (21) the side-equation corrections can be computed, which will not disturb the angle-equation adjustment, and which are the most probable corrections to the several angle-values.

The second or rigid method will be found much more satisfactory than the method by approximations. The complete adjustment consists in applying to the mean measured values, the corrections from angle-equations given by equation (7), and then applying to these corrected angles the corrections found by equation (21).

NOTE.—The results obtained in the above adjustments are identical with those found by the method of least squares, and the fundamental principle by which they are obtained is also the same as that of least squares, viz.: that the arithmetic

* Note that $\Sigma(C^2)$ does not include C_0^2.

mean of properly weighted observations is the most probable result, and is identical with that obtained by making the sum of the squares of the corrections a minimum. For least-square solutions of this problem, see Clarke's "Geodesy," pp. 263–6, and Wright's "Adjustment of Observations," pp. 303–8.

EXAMPLE.

The following is the numerical computation of the quadrilateral shown in the figure. *AB* is the known side, and *CD* is to be found. The mean observed values of the angles are given in the second column. The corrections for angle-equations are given in the third column, and are the same for all three methods of solution given above. The spherical excess is here applied only to the quadrilateral as a whole, or to l_3, thus distributing it equally among the several angles. This is a common way of doing it, although if the excess is considerable, and the several triangles very unequal in size, as is the case here, it should be applied to the several triangles according to their size, as stated in the foot-note, p. 514.

In columns 7 and 8, the corrections for side-equation are worked out by the two methods given to show the relative results. Thus, from eq. (14) we find the values of v_1', v_2', etc., for the first approximation. Applying these to the first corrected values in column 4, and again taking out the values of l_1, l_2, and l_3, for angle-equation conditions, we find they are not zero, but very small. It would probably be sufficient to work out a new set of angle-corrections by eq. (7), and then consider the quadrilateral adjusted. In this example, the final values thus found would then differ from the final values by the rigid adjustment by not more than 0".2 for any angle.

If we compute *CD* from *AB*, assuming the latter to be 25,000 feet in length, we obtain 88,670.9 ft. in passing through the triangles *ABC* and *BCD*, while if we pass through the triangles *ABD* and *ADC* we obtain 88671.1 ft., a discrepancy of 0.2 ft., and giving a mean value of 88671.0 ft. The discrepancy of 0.2 ft. in the two results by the rigid solution results from not computing the corrections beyond tenths of a second.

If simply a check on the final corrected values is desired, it may be obtained by adding them, when their sum should equal 360° + spherical excess, or by taking out the log sines and seeing if l_4' in eq. (11) is zero. In this case it is not zero, but 9, resulting from not carrying out the corrections beyond tenths of seconds, as mentioned above.

COMPUTATION OF ANGLE-CORRECTIONS FOR A QUADRILATERAL ADJUSTMENT.

Angles.	Observed Values.	Corrections for Angle-equations.	First Corrected Values.	Log. Sines.	Tabular Differences.	Corrections for Side-equation. First Approx. First Method.	Corrections for Side-equation. By Second Method, $\pm x_0, \pm x_1$, etc.	Final Corrected Angles. Rigid Solution (Third Method).	Check. Log. Sines of the Corrected Angles.
A_1	30° 12′ 52″.4	− 0″.6	51″.8	9.7017725	+ 36.2	+ 0″.4	+ 0″.6	30° 12′ 52″.4	9.7017747
B_2	88 55 19 .3	− 0 .3	19 .0	9.9999232	+ 0.4	0 .0	+ 0 .6	88 55 19 .6	9.9999229
C_5	39 48 06 .4	− 0 .8	05 .6	9.8062686	+ 25.2	+ 0 .3	+ 0. 5	39 48 06 .1	9.8062698
D_7	18 49 38 .3	− 1 .2	37 .1	9.5088140	+ 61.8	+ 0 .6	+ 0 .4	18 49 37 .5	9.5088166
				.0167785					9.0167840
B_3	47 31 20 .4	− 0 .7	19 .7	9.8677851	+ 19.3	− 0 .2	− 0 .4	47 31 19 .3	9.8677843
C_4	13 20 30 .1	− 0 .3	29 .8	9.3631540	+ 88.8	− 0 .9	− 0 .8	13 20 29 .0	9.3631467
D_6	37 56 06 8	− 0 .8	06 .0	9.7887106	+ 27.0	− 0 .3	− 0 .3	37 56 05 .7	9.7887098
A_8	83 26 13 .1	− 1 .2	11 .9	9.9971443	+ 2.5	0 .0	− 0 .6	83 26 11 .3	9.9971441
Sums ...	360 00 06 .8	− 5 .9	00 .9	.0167940	261.2	− 0 .1	0 .0	360 00 00 .9	9.0167849

Spher. ex. = 0.9
$l_3 = +5''.9$
$l_2 = +2.0$
$l_1 = +0.4$

Hence from eq. (7) we find v_1, v_2, v_3, etc.

$l_4 = -155$ $l'_4 = -9$

FIRST METHOD.

$-\frac{l_4}{\Sigma(d^2)} = +\frac{155}{14756} = +0.01$, whence, from eq. (14) we find v_1', v_2', etc., and then from eqs. (1), (2), and (3), find l_1', l_2', and l_3' to be: $l_1' = -0''.2$; $l_2' = +1''.3$; and $l_3' = -0''.1$.

SECOND METHOD.

$C_0 = 44.2$	$\frac{1}{4}C_0^2 = 488$	$\frac{4x_0}{44.2} = -\frac{l_4}{\frac{C_0^2}{4} + \Sigma(C^2)}$	$x_0 = +0.1$	Whence from eq. (15) find v_1', v_2', etc.
$C_1 = 55.5$	$C_1^2 = 3080$	$= \frac{155}{18384}$	$x_1 = +0.5$	
$C_2 = 89.2$	$C_2^2 = 7957$	$= 0.0083$	$x_2 = +0.7$	
$C_3 = 52.2$	$C_3^2 = 2725$		$x_3 = +0.4$	
$C_4 = 64.3$	$C_4^2 = 4134$		$x_4 = +0.5$	

ADJUSTMENT OF LARGER SYSTEMS.

370. Used only in Primary Triangulation.—The simultaneous adjustment of all the angles in an extended system of triangles with one measured base which is taken as exact, is a very complicated problem. The methods of least squares must here be applied, so that a discussion of this problem belongs rather to a treatise on geodesy than to one on surveying. The adjustments of a triangle and of a quadrilateral will be found sufficient for all secondary work, or such as is intended to serve only for topographical or geographical purposes. Especially is this true if the stations be so selected that the observed lines will form a series of quadrilaterals. The adjustment of these quadrilaterals by the rigid method given above gives nearly as good results as could be obtained by reducing the work as a single system. For a discussion of the least-square methods of adjustment of an extended system of triangles the student is referred to "Primary Triangulation of the U. S. Lake Survey," being Professional Papers, Corps of Engineers U. S. A., No. 24; Report of the U. S. Coast and Geodetic Survey for 1875; Clarke's Geodesy; and especially to Wright's "Adjustment of Observations."

The facility and accuracy with which base-lines may now be measured by means of long steel tapes will result in actually measuring many more lines than has heretofore been done, and so errors from angular measurements will not be allowed to accumulate to any great extent. It is not improbable that geodetic methods will be materially influenced by this new method of accurate measurement.

371. **Computing the Sides of the Triangles.**—After the angles of the system are adjusted, the sides of the triangles are computed by the ordinary sine ratio for plane triangles. If the system consist of simple triangles, then one side is known and the other two sides computed from it. In this case there is no check on the computation except what the computer

carries along with him, or what may be obtained from a duplicate computation.

If the system be made up of a series of quadrilaterals, then the line which is common to two successive quadrilaterals is computed through two sets of triangles from the previous known side. Thus if the quadrilateral of Fig. 142 be one of a series, the lines in common being AB and CD, then AB is computed in duplicate from the previous quadrilateral, and the mean of the two results taken. In the triangle ABD compute AD, and then in the triangle ADC compute DC; in the triangle ABC compute BC, and then in the triangle BCD compute DC again. There are thus obtained two independent values of DC, as computed from AB. If the adjustment had been exact these values would have agreed exactly, but the adjusted angles were computed only to the nearest second, or tenth of a second; hence the two values of DC will agree only to a corresponding exactness. If the system be composed of quadrilaterals and the adjustment be made to the nearest second, then the two values of DC would probably differ in the fifth or sixth significant figure. If the adjustment be made to the nearest tenth of a second, and a seven-place logarithmic table be used, then the two values of DC should begin to differ in the sixth or seventh place. Of course the adjusted values are not the *true* values of the angles, but simply the *most probable* values. If the angles were not accurately measured the adjusted values may still be considerably in error, but any such errors would not prevent the two values of CD from agreeing, since this agreement is one of the conditions which the adjustment is made to satisfy. The disagreement between the two computed values of CD comes only from the *inexactness* of the computed corrections to the angles, an angle, like a length, being an incommensurable quantity, and hence some degree of approximation is necessary in its expression. If the true computed values of CD differ by more than the amounts above signified, then it is probable that an error has been made in the computation.

LATITUDE AND AZIMUTH.

372. Conditions.—In the methods here given for obtaining latitude, azimuth, and time, the instrument used may either be an ordinary field transit mounted on its tripod, or a more elaborate altazimuth instrument, such as shown in Figs. 132 and 134. The accuracy sought is only such as is sufficient for topographical or geographical purposes. Both the field methods and the office reductions are of the simplest character; but all large errors are eliminated, so that the results will be found as accurate as it is possible to obtain with anything less than the regular field astronomical instruments. This higher grade of work falls within the sphere of the astronomer rather than of the surveyor.

373. Latitude and Azimuth by Observations on Circumpolar Stars at Culmination and Elongation.—When latitude and azimuth are to be found to a small fraction of a minute, or as accurately as can be read on the instrument used, if this be an ordinary field transit, the most convenient method is by means of observations on circumpolar stars. The observation for latitude is made on such a star when it is at its upper or lower culmination, since it is then not changing its altitude, and the observation for azimuth is made at elongation, since then the star is not changing its azimuth. At these times a number of readings may be taken on the star, thus eliminating instrumental constants by reversals, since a half hour may be utilized for this work without the star sensibly changing its position so far as the use it is serving is concerned. Two close circumpolar stars have been chosen whose right ascensions differ by about five hours and thirty minutes. They therefore always give a culmination and an elongation about thirty minutes apart. This is very convenient, since it allows observations to be made for latitude and azimuth at one setting with a sufficient intervening interval to complete one set of observations before commencing the next.

The two stars selected are Polaris (α Ursæ Minoris), which is of the second magnitude, and 51 Cephei, which is of the fifth magnitude. Their relative positions are shown in Fig. 143.

FIG. 143.

The position of 51 Cephei may be described with reference to the line joining "the pointers," in the constellation of the Great Bear, with Polaris. Thus, 51 Cephei is to the *right* of this line, when looking *towards* the pole-star along the line, at a distance of about *three times* the sun's disk from the line, and of about *five times* the sun's disk from Polaris, in the direction of the pointers.

In case 51 Cephei is not visible to the naked eye, as it may not be on moonlight nights, or with a slightly hazy atmosphere, it may be found, when near elongation, by the telescope, as follows:

Having carefully levelled the instrument, turn upon Polaris. When 51 Cephei is near its eastern elongation Polaris is near its upper culmination, and when near its western elongation Polaris is near its lower culmination. To find 51 Cephei at eastern elongation, therefore, after taking a pointing on Polaris, lower the telescope (diminish the vertical angle) by about one degree (if the time is about twenty minutes before elongation), and then turn off towards the east about two and a half degrees. This will bring the cross wires approximately upon the star.

To find it at western elongation, simply reverse these angles; that is, increase the vertical angle one degree, and turn off to the west two and one half degrees.

The following table gives the times of the elongations and culminations of these two stars for 1885 for latitude 40°, which may be used for observing azimuth and latitude. The times given are for the *nights following* the dates named in the first column.

TIMES OF ELONGATION AND CULMINATION, 1885.
LATITUDE, 40°.

DATE.	POLARIS (α Urs. Min.).				51 CEPHEI.			
	Elongation.	Time.	Culmination.	Time.	Elongation.	Time.	Culmination.	Time.
Jan. 1	W	$12^h 24^m.6$ A.M.	U	$6^h 29^m.9$ P.M.	W	$5^h 48^m.3$ A.M.	U	$11^h 58^m.6$ P.M.
Feb. 1	"	10 22 .2 P.M.	L	4 25 .6 A.M.	"	3 46 .4 "	"	9 56 .7 "
Mar. 1	"	8 31 .8 "	"	2 35 .1 "	"	1 56 .1 "	"	8 6 .4 "
April 1	"	*6 29 .7 "	"	12 33 .1 "	"	11 54 .0 P.M.	"	*6 4 .3 "
May 1	E	*4 36 .6 A.M.	"	10 35 .2 P.M.	"	9 55 .9 "	L	4 4 .2 A.M.
June 1	"	2 37 .0 "	"	8 33 .7 "	"	*7 53 .9 "	"	2 2 .2 "
July 1	"	12 39 .0 "	"	*6 36 .2 "	E	*6 12 .6 A.M.	"	12 4 .2 "
Aug. 1	"	10 38 .1 P.M.	U	4 32 .8 A.M.	"	4 10 .8 "	"	10 2 .4 P.M.
Sept. 1	"	8 36 .6 "	"	2 31 .3 "	"	2 9 .1 "	"	8 0 .8 "
Oct. 1	"	*6 38 .9 "	"	12 33 .6 "	"	12 11 .4 "	"	*6 3 .1 "
Nov. 1	W	4 26 .4 A.M.	"	10 31 .7 P.M.	"	10 9 .8 P.M.	U	3 59 .5 A.M.
Dec. 1	"	2 28 .2 "	"	8 33 .5 "	"	8 12 .0 "	"	2 1 .8 "

* Probably not visible to the naked eye.

From the above table it is evident that both an elongation and a culmination of one of these stars can always be obtained.

For other days than those given in the table, either interpolate, or find by allowing $3^m.94$ for one day, remembering that each succeeding day the elongation occurs *earlier* by this amount.

For other years than 1885, take from the table the time corresponding to the given month and day, and add for Polaris $0^m.3$, and for 51 Cephei $0^m.5$ for each year after 1885; also,

Add 1^m if the year is the second after leap-year.
Add 2^m if the year is the third after leap-year.
Add 3^m if the year is leap-year before March 1.
Subtract 1^m if the year is leap-year after March 1.

For the first year after leap-year there is no correction except the periodic ones of $0^m.3$ and $0^m.5$ per annum.

For other latitudes between 30° and 50° north latitude correct the times of elongation as follows:

For each degree south of 40°,
Add to the western or subtract from the eastern time of elongation $\begin{cases} 0^m.14 \text{ for Polaris.} \\ 0^m.29 \text{ for 51 Cephei.} \end{cases}$

For each degree north of 40°,
Subtract from the western or add to the eastern time of elongation $\begin{cases} 0^m.18 \text{ for Polaris.} \\ 0^m.39 \text{ for 51 Cephei.} \end{cases}$

The following table gives the pole distances of Polaris and 51 Cephei for Jan. 1 of each third year from 1885 to 1930:

POLE DISTANCE (90° – DECLINATION).

STAR.	1885.	1888.	1891.	1894.	1897.	1900.	1903.	1906.
Polaris....	1°18′16″	1°17′19″	1°16′23″	1°15′26″	1°14′30″	1°13′33″	1°12′37″	1°11′41″
51 Cephei..	2 46 35	2 46 47	2 47 00	2 47 13	2 47 26	2 47 40	2 47 54	2 48 09

STAR.	1909.	1912.	1915.	1918.	1921.	1924.	1927.	1930.
Polaris....	1°10′45″	1° 9′49″	1° 8′53″	1° 7′58″	1° 7′ 2″	1° 6′ 7″	1° 5′12″	1° 4′16″
51 Cephei..	2 48 24	2 48 39	2 48 55	2 49 12	2 49 28	2 49 45	2 50 01	2 50 18

Interpolate for intermediate dates.

To observe for latitude no knowledge of the geographical position is needed.

374. **The Observation for Latitude** consists simply in observing the altitude of a circumpolar star at upper or lower culmination and correcting this altitude for the pole distance of the star and for refraction.

Let
$$\phi = \text{latitude};$$
$$d = \text{polar distance};$$
$$r = \text{refraction};$$
$$h = \text{altitude};$$

then
$$\phi = h \mp d - r; \quad \ldots \ldots \ldots \quad (1)$$

the minus sign being used for upper, and the plus sign for lower, culmination observations. The value of r is taken from the following table of mean refractions computed for barometer 30 inches, and temperature 50° F.

TABLE OF MEAN REFRACTIONS.

Altitude.	Refraction.	Altitude.	Refraction.
10°	5′ 19″	20°	2′ 39″
11	4 51	25	2 04
12	4 28	30	1 41
13	4 07	35	1 23
14	3 50	40	1 09
15	3 34	45	0 58
16	3 20	50	0 49
17	3 08	60	0 34
18	2 58	70	0 21
19	2 48	80	0 10

The index error of the vertical circle is eliminated by reading with the telescope direct and reversed, providing the vertical circle is complete. If the vertical limb is but an arc of 180° or less, the index error cannot be eliminated in this way. In this case the second method is recommended.

375. First Method.—Mount the instrument firmly, preferably on a post, and adjust carefully the plate-bubbles, especially the one parallel to the plane of the vertical circle. About five or ten minutes before the star comes to its culmination read the altitude of the star with telescope direct. Revolve the telescope on its horizontal axis and also on its vertical axis, *relevel the instrument* if the bubbles are not in the middle, but do not readjust the bubbles, and bring the telescope upon the star. Make two readings in this position. Revolve the telescope and instrument again about their axes, relevel, and read again in first position. This gives two direct and two reversed readings taken in such a way as to eliminate the error from collimation, the index error of vertical circle, and also the error of adjustment of the plate-bubbles. The result, when corrected for refraction and the pole distance of the star, should be the latitude of the place within the limits of accuracy and exactness of the vertical circle-readings.

376. Second Method.—An "artificial horizon," formed by the free surface of mercury in an open vessel, may be used in conjunction either with the transit or a sextant. If the former is used two pointings are made—one to the star and the other to its image in the mercury surface. The angle measured is then twice the apparent altitude of the star. The position of the vessel of mercury will be on a line as much below the horizontal as the star is above it. The instrument is first set up and then the artificial horizon put in place. The surface of the mercury must be free from dust. If the mercury is not clean it may be strained through a chamois-skin or skimmed by a piece of cardboard. Any open vessel three or more inches in diameter may be used for holding the mercury. It should be placed on a solid support and protected from the wind.

The observations with a transit would then consist in taking a reading on the star just before culmination, two readings on the image, and then one on the star. The index error of the vernier on the vertical circle will then be eliminated, since both plus and minus angles have been read, and their sum taken for twice the altitude of the star. This method is adapted to transits with incomplete vertical limbs.

The Sextant may also be used with the artificial horizon and will give more accurate results than can be obtained with the ordinary field transits. The double altitude angle is then measured at once by bringing the direct and reflected images of the star into coincidence. In both cases the observed angle is $2h$, and the latitude is found from equation (1), as before. If there is much wind the mercury basin may be partially covered, leaving only a narrow slit in the vertical plane through instrument and star, or the regular covered mercurial horizon may be used. This is covered by two pieces of plate-glass set at right angles to each other like the roof of a house. If the opposite faces of these glasses are not parallel planes, an error is introduced. This is eliminated by reversing the horizon apparatus on half the observations. It is best, however, to avoid the use of glass covers, if possible.

If tin-foil be added to the mercury an amalgam is formed, whose surface remains a perfect mirror, which is not readily disturbed by wind. As much tin-foil should be used as the mercury will unite with. Observations may then be made in windy weather without the aid of a glass cover.

377. Correction for Observations not on the Meridian.—If the star is more than five or ten minutes of time from the meridian, it is necessary to apply a correction to the observed altitude to give the altitude at culmination. The following approximate rule gives these corrections for the two circumpolar stars here used, with an error of less than 1″ of arc when the observation is taken not more than 18 minutes of time from the star's meridian passage, and the error is less than 10″ of arc when the observation is made 32 minutes of time from the meridian.

Rule for reducing circummeridian altitudes to the altitude at culmination.

For Polaris: Multiply the square of the time from meridian passage, in minutes, by 0.0444, and the product is the correction in seconds of arc.

For 51 *Cephei:* Multiply the square of time from meridian passage, in minutes, by 0.1017, and the product is the correction in seconds of arc.

The correction is to be added to the observed altitude for upper culmination, and subtracted for lower culmination.

By using these corrections an observation for latitude may be made at any time for a period of about one hour, near the time of culmination.

378. **The Observation for Azimuth** is made on one of the two stars here chosen when it is at or near its eastern or western elongation, for the same reason that latitude observations are taken at culmination. The azimuth of a star at elongation is found from the formula,

$$\text{sine of azimuth} = \frac{\text{sine of polar distance}}{\text{cosine of latitude}}. \quad . \quad . \quad (1)$$

This formula is so simple that it is hardly necessary to give a table of values of azimuths for various latitudes. Such a table is given for Polaris, however, on p. 33. The pole distances are given on p. 531, and the latitude is found by observation. It is not necessary to know the azimuth of the star at elonga-

[redacted]

tion before making the observations. This can be computed afterwards from the observed latitude.

The observation for azimuth consists simply in measuring the horizontal angle between the star and some conveniently located station, marked by an artificial light. The operation is in no sense different from the measurement of the horizontal angle between two stations at different elevations. The great source of error is in the horizontal axis of the telescope. If this is not truly horizontal then the line of sight does not describe a vertical plane, and since the two objects observed have very different elevations, the angle measured will not be that subtended by vertical planes passing through the objects and the axis of the instrument. To eliminate this error the telescope is reversed, and readings taken in both positions. The following programme is recommended:

PROGRAMME FOR OBSERVING FOR AZIMUTH ON A CIRCUMPOLAR STAR AT ELONGATION.

Instrument.	Time of Observation.	Reading on
Direct..........	10 min. before elongation.........	Mark.
Reversed.......	7 " "	"
"	4 " "	Star.
"	2 " "	"
Direct..........	2 min. after "	"
"	4 " "	"
"	7 " "	Mark.
Reversed.......	10 " "	"

The instrument should not be relevelled nor the bubbles adjusted after the observations have begun. If the instrument should be disturbed of course the series is spoiled. If the change of level is gradual, it and all other errors will be eliminated except those of graduation. Of course both verniers are to be read each time.*

Having found the latitude, the azimuth of the star at elongation is found from equation (1) above. This is then added to or subtracted from the horizontal angle between mark and star, as the case may be, to give the azimuth of the mark from the north point. If the azimuth is to be referred to the south point, which it generally is, we must add or subtract 180°.

379. Corrections for Observations near Elongation.—As in the case of observations for latitude, we may have an approximate rule for reducing an observed azimuth when near elongation to what it would have been if taken at elongation. The limits of accuracy are also about the same, but the factors are slightly different.

Rule for reducing azimuth observations on Polaris and 51 *Cephei near elongation to their true values at elongation, for latitude* 40°.

For Polaris, multiply the square of the time from elongation in minutes by 0.058, and the product will be the correction in seconds of arc.

For 51 *Cephei*, multiply the square of the time from elongation in minutes by 0.124, and the product will be the correction in seconds of arc.

The formula for reduction, when near elongation, is

$$c = 112.5\, t^2 \sin 1'' \tan A,$$

where c = correction to observed azimuth in seconds of arc;
t = time from elongation in seconds of time;
A = azimuth of star at elongation.

$$\log 112.5 \sin 1'' = 6.7367274.$$

From this formula and that of equation (1) we may compute the coefficients for the above approximate rules for any latitude.

* If a mercury surface be used and alternate readings be taken on the star and on the image, all errors from inclined horizontal axis are eliminated, and extremely accurate work can be done with an ordinary transit.

Thus, for latitude 30° we have azimuth of Polaris, 1885, 1° 30′.4, whence the coefficient of reduction for elongation of Polaris in latitude 30° is found to be 0.052, and for latitude 50° it is 0.069.

For 51 Cephei, this coefficient for latitude 30° is 0.110, and for latitude 50°, 0.148.

From the above data the corrections for an observation of a circumpolar star near elongation may be computed.

If azimuth be reckoned from the south point, as is common in topographical and other geodetic work, and if it increase in the direction S.W.N.E., then a star at western elongation has an azimuth of less than 180°, and at eastern elongation its azimuth is more than 180°.

The corrections to reduce to elongation, as above computed, should be added to the computed azimuth of the star at western elongation, and subtracted when at eastern elongation.

380. The Target.—This may be a sort of box, in which a light may be placed. A narrow vertical slit should be cut, subtending an angle, at the instrument, from one to two seconds of arc. This should be set as far from the instrument as convenient, as from a quarter of a mile to one mile. The width of slit desired may be computed for any given angular width and distance by remembering that the arc of one second is three-tenths of an inch for a mile radius. The target should be sufficiently distant to enable it to be seen with the stellar focus without appreciable parallax, as the instrument should not be refocused on the target. This target may be set on any convenient azimuth from the observation-station, as upon one triangulation station when the observations are taken at another, thus obtaining directly the azimuth of this line.

381. Illumination of Cross-wires.—Various methods are used to illuminate the wires, the crudest of which is, perhaps, to hold a bull's-eye lantern so as to throw light down the telescope-tube through the objective, taking care not to obstruct the line of sight.

A very good reflector may be made from a piece of new tin, cut and bent as in Fig. 144. The straight strip is bent about the object end of the telescope tube, leaving the annular elliptic piece projecting over in front. This is then bent to any desired angle, preferably about forty-five degrees, and turned so that an attendant can reflect light down the tube by illuminating the disk from a convenient position. This position should be so chosen that the lantern may throw the light *from* the observer, rather than *towards* him. If the reflecting side of the disk be whitened, the effect is very good. The opening should be about three-fourths or seven-eighths inch in its shorter diameter, the longer diameter being such as to make its vertical projection equal to the shorter one. There is, of course, no necessity of limiting or of making true the outer edges of the disk.

FIG. 144.

381a. Azimuth from Polaris at any Hour.—In the "Manual of Instructions," issued by the Commissioner of the General Land Office in 1890, there appeared a new set of tables designed to enable observations for azimuth to the nearest minute to be made at any hour by an observation on Polaris. These tables are condensed into Table XII.

By the use of this table an observation for azimuth, of sufficient accuracy for ordinary purposes, can be made at any time when Polaris is visible.

Considering the two pages as composing one table, the two middle columns give the time of upper culmination of Polaris for any day of the year 1893. For other years add $0^m.3$ for each year after 1893; also, add 1^m if the year is the second after leap-year; add 2^m if the year is the third after leap-year; add 3^m if the year is leap-year before March 1; subtract 1^m if the year is leap-year after March 1.

For the first year after leap-year there is no correction except the periodic one of 0.3^m per annum.

The table is arranged for giving the azimuth of Polaris at any point in its path, the argument being the time which has elapsed since its last upper culmination. The upper culmination sought is therefore the last one preceding the time chosen for the observation.

Suppose this time to be August 9, 1895, at 9 P.M. By referring to the two central columns of the table, we see at a glance that in August Polaris culminates some fifteen or sixteen hours after mean noon, or at about 3 or 4 o'clock the next morning. The culmination sought, therefore, is the one following mean noon on the 8th. We therefore wish to find the time of culmination of Polaris after mean noon on August 8, 1895.

We have, from the table, the time of the star's upper culmination:

For August 1, 1893................	$16^h\ 35^m.1$
Tab. dif. for 7 days................	$-27\ .5$
For August 8, 1893................	$16^h\ 07^m.6$
Correction to 1895................	$+\ 2\ .6$
Time up. culm. August 8, 1895......	$16^h\ 10^m.2$

after noon, which is 4 o'clock and 10.2 min. A.M. of August 9.

The time chosen for the observation is 9 P.M., or $16^h\ 49^m.8$ after the star's last upper culmination. This is called the hour angle of the star. Evidently it has passed its lower culmination, and is now moving upward on the eastern half of its orbit. Since its position in its orbit with reference to the meridian is the significant thing, we can find this by subtracting $16^h\ 49^m.8$ from its period of revolution, which is a siderial day, or $23^h\ 56^m$. Making this subtraction, we find the star's position to be $7^h\ 06^m.2$ from its upper culmination, on the east side. Entering the table with the argument $7^h\ 06^m.2$ for the year 1895, and for latitude 40°, we find we must interpolate between 1° 32′ and 1° 34′, which gives us the azimuth 1° 33′, which is the true azimuth of Polaris at the time of observation.

Furthermore, the table shows us that to change the azimuth at this time by 2′ requires a time of $8^{min.}$ or $9^{min.}$ to make a change of 1′ in azimuth.

When near the culminating points the star moves much faster in azimuth, and it here requires but $2^{min.}$ to produce a change of azimuth of 1′. It is evident, therefore, that if the local time is known to within one or two minutes, the method will always give the azimuth to the nearest minute of arc.

It must be remembered, also, that it is the *local* time which must be used as the time of the observation, and not the "standard" time, now universally used in America.

By the peculiar and ingenious arrangement of this table,* all the data necessary to make an observation for azimuth at any hour of any day or year, until 1900, are presented on two opposite pages. Never before has this matter been so simplified. It is usually very inconvenient to await the time of elongation of Polaris, and at times both the elongations occur in the daylight hours. By means of this table, and where an accuracy of one minute of arc is sufficient, the observation can be taken at pleasure, simply noting the time, and the azimuth of the star may be taken out later for that particular time.

TIME AND LONGITUDE.

382. **Fundamental Relations.**—In all astronomical computations the observer is supposed to be situated at the centre of the celestial sphere and the stars appear projected upon its surface. Their positions with respect to the observer may be fixed by two angular coördinates. The most common plane of reference for these coördinates is that of the celestial equator, and the coördinates referring to it are known as Right Ascension and Declination—corresponding to Longitude and Latitude on the earth's surface.

Right ascension is counted on the equator from west tow-

* Prepared originally by Mr. J. B. Shinn, of the United States General Land Office, Washington, D. C.

35

ards east. As a zero of right ascension the vernal equinox is taken.

Declination is counted on a great circle perpendicular to the equator, and is called positive when the star is north and negative when south.

In Fig. 145

P is the pole;
Z is the zenith of the observer;
S is the star;

Then R. A. star = VPS = arc VE;
Dec. star = SE.

These coördinates are fixed, varying only by slow changes due to the shifting of the reference-plane.

Another system of coördinates is often used in fixing the place of a star, namely: Hour-angle and Declination. Hour-angle is the angle at the pole between the meridian and the great circle passing through the star and the pole perpendicular to the equator. Hour-angle will of course be constantly changing each instant. In Fig. 145 hour-angle = ZPS.

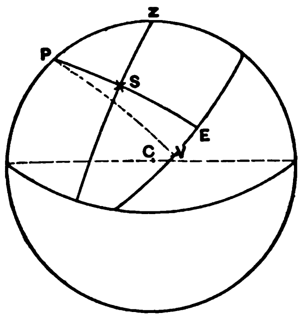


FIG. 145.

383. Time.—The motion of the earth on its axis is perfectly uniform. We obtain, therefore, a uniform measure of time by employing the successive transits of a point in the equator across the meridian of any place. The point naturally chosen is the vernal equinox.

A *Sidereal Day* is the interval of time between two successive upper transits of the vernal equinox over the same meridian.

The *Sidereal Time* at any instant is the hour-angle of the vernal equinox at that instant reckoned from the meridian westward from 0^h to 24^h. Thus, when the vernal equinox is on the meridian, the hour-angle is $0^h\ 0^m\ 0^s$ and the sidereal time is $0^h\ 0^m\ 0^s$. When the vernal equinox is 1^h west of the meridian the sidereal time is $1^h\ 0^m\ 0^s$.

We have in Fig. 145

$$\begin{aligned} &\text{Hour-angle of ver. eq.} = ZPV = \theta = \text{sidereal time;} \\ &\text{Right asc. of star} \quad = VPS = \alpha; \\ &\text{Hour-angle of star} \quad = ZPS = H; \end{aligned}$$

Whence $\theta - \alpha = H.$ . . . . (1)

From this equation, knowing the sidereal time and the R. A. of the star, the hour-angle may always be computed.

When $H = 0$, i.e., when the star is on the meridian, $\theta = \alpha$, or, in other words, the R. A. of any star is equal to the true local sidereal time when the star is on the meridian. By noting the exact time of transit of any star whose R. A. is known, the local sidereal time will be at once known.

An Apparent Solar Day is the interval of time between two successive upper transits of the true sun across the same meridian.

Apparent or True Solar Time is the hour-angle of the true sun.

Owing to the annual revolution of the earth, the sun's right ascension is constantly increasing. It follows, therefore, that a solar day is longer than a sidereal day. In one year the sun moves through 24^{hrs} of right ascension. There will be, therefore, in one tropical year (which is the interval between two successive passages of the sun through the vernal equinox) exactly one more sidereal day than solar days ; or, in other words, in a tropical year the vernal equinox will cross the meridian of any given place once more than the sun will.

The solar days will, however, be unequal for two reasons :

1st. The sun in its apparent motion round the earth does not move in the equator, but in the ecliptic.

2d. Its motion in the ecliptic is not uniform.

On account of these inequalities the true solar day cannot be used as a convenient measure of time. But a mean solar day has been introduced, which is the mean of all the true solar days of the year and which is a uniform measure of time.

Suppose a fictitious sun to start out from perigee with the true sun, to move uniformly in the ecliptic, returning to perigee at the same moment as the true sun. Now, suppose a second fictitious sun moving in the equator in such a way as to make the circuit of the equator in the same time that the first fictitious sun makes the circuit of the ecliptic, the two fictitious suns starting together from the vernal equinox and returning to it at the same moment. The second fictitious sun will move uniformly in the equator and will be therefore a uniform measure of time. This second fictitious sun is known as the Mean Sun.

A Mean Solar Day is therefore the interval between the upper transits of the mean sun over the meridian of any place.

Mean Solar Time at any meridian is the hour-angle of the mean sun at that meridian counted from the meridian west from 0^h to 24^{hrs}.

The Equation of Time is the quantity to be added to or subtracted from apparent solar time to obtain mean time.

The equation of time is given in the American Ephemeris for Washington mean and apparent noon of each day. If the value is required for any other time it can be interpolated between the values there given.

384. To convert a Sidereal into a Mean-time Interval, and vice versa.—According to Bessel, the tropical year contains 365.24222 mean solar days, and since the number of sidereal days will be greater by one than the number of mean solar days, we have

$$365.24222 \text{ mean sol. days} = 366.24222 \text{ sid. days};$$
$$1 \text{ mean sol. day} = 1.00273791 \text{ sid. days}.$$

Let I_m = mean solar interval;
I_s = sidereal interval;
$k = 1.00273791.$

Thus

$$I_s = I_m k = I_m + I_m(k-1) = I_m + 0.0027379 I_m;$$
$$I_m = \frac{I_s}{k} = I_s - I_s\left(1 - \frac{1}{k}\right) = I_s - 0.0027304 I_s.$$

By the use of these formulæ the process of converting a sidereal interval into a mean-time interval, and *vice versa*, is made very easy. It is rendered more easy by the use of Tables II. and III. of the Appendix to the American Ephemeris and Nautical Almanac, where the quantity $I_m(k-1)$ is given with the argument I_m, and $I_s\left(1 - \frac{1}{k}\right)$ with the argument I_s.

Example.—Given the sidereal interval $I_s = 15^h\ 40^m\ 50^s.50$, find the corresponding mean-time interval.

	$I_s = 15^h\ 40^m\ 50^s.50$
Table II. gives for $15^h\ 40^m$	2 33.996
" " " $50^s.50$	0.138
	$I_m = 15\ \ 38\ \ 16.37$

385. To change Mean Time into Sidereal.—Referring to Fig. 145, suppose S to represent the mean sun.

Then ZPS = hour-angle of mean sun = mean-time = T;
VPE = R. A. of mean sun = α_s;
θ = sidereal time.
From equation (1), p. 543,
$$\theta = \alpha_s + T.$$

The right ascension of the mean sun is given in the American Ephemeris both for Greenwich and Washington mean noon of each date. It is called ordinarily the ***sidereal time of mean noon***, which is of course the right ascension of the mean sun at noon, since at mean noon the mean sun is on the meridian and its right ascension is equal to the sidereal time. Since the sun's right ascension increases 360° or 24^{hrs} in one year, it will change at the rate of $3^m\ 56^s.555$ in one day, or $9^s.8565$ in one hour.

Suppose θ_0' = sid. time of mean noon at Greenwich;
θ_0 = " " " " " " the place for which T is known;
L = longitude west of Greenwich.

Then $\theta_0 = \theta_0' + 9^s.8565L$,

where L is expressed in hours and decimals of an hour. In this way the sidereal time of mean noon may be obtained for the meridian of observation.

Substituting for α_s its equivalent, and reducing the mean-time interval to sidereal,

$$t = \theta_0 + T + T(k - 1).$$

Example.—Longitude of St. Louis, $6^h\ 0^m\ 49^s.16 = 6^h.0136$. Mean time, 1886, June 10, $10^h\ 25^m\ 25^s.5$. Required corresponding sidereal time.

From Amer. Ephem., p. 93:

$$
\begin{array}{lrrr}
\theta_0' \text{ (for Greenwich)} & = 5^h & 15^m & 3^s.30 \\
6.0136 \times 9.8565 & = & 0 & 59.27 \\
\hline
\theta_0 & = 5 & 16 & 2.57 \\
T & = 10 & 25 & 25.50 \\
T(k-1), \text{Table III.}, & = & 1 & 42.74 \\
\hline
\theta & = 15 & 43 & 10.81
\end{array}
$$

It should be remarked that the quantity $59^s.27$ will be a constant correction, to be added to the sid. time of mean noon at Greenwich to obtain the sid. time of mean noon at St. Louis.

386. To change from Sidereal to Mean Time.—This process is simply the reverse of that for changing from mean to sidereal time. Using the same notation as before, we shall have

$$T = \theta - \theta_0 - (\theta - \theta_0)\left(1 - \frac{1}{k}\right).$$

Subtracting from the given sidereal time (θ) the sidereal time of mean noon (θ_0), we have the sidereal interval elapsed since mean noon, and this needs simply to be changed into a mean-time interval.

EXAMPLE.—Given 1886, June 10, $15^h\ 43^m\ 10^s.81$ sidereal time, to find the corresponding mean time.

$$
\begin{array}{rrrr}
\theta = & 15 & 43 & 10.81 \\
\text{(as before)} \qquad \theta_0 = & 5 & 16 & 2.57 \\
\hline
\theta - \theta_0 = & 10 & 27 & 8.24 \\
(\theta - \theta_0)\left(1 - \frac{1}{k}\right) \text{ (Table II.)} = & & 1 & 42.74 \\
\hline
T = & 10 & 25 & 25.50
\end{array}
$$

387. The Observation for Time, as here described,* consists in observing the passage, or transit, of a star across the meridian. The direction of the meridian is supposed to have been determined by an azimuth observation. If the instrument be mounted over a station the azimuth from which to some other visible point is known, the telescope can be put in the plane of the meridian. An observation of the passage of a star across the meridian will then give the local time, when the *mean local time of transit* of that star has been computed. In order to eliminate the instrumental errors at least two stars should be observed, at about the same altitude. If the instrument has no prismatic eye-piece, then only south stars can be observed with the ordinary field-transits; that is, only stars having a south declination, if the observer is in about 40° north latitude. Stars near the pole should not be chosen, since they move so slowly that a small error in the instrument would make a very large error in the time of passage.

388. Selection of Stars.—The stars should be chosen in pairs, each pair being at about the same altitude, or declination. It is supposed that the American Ephemeris is to be used. The "sidereal time of transit, or right ascension of the mean sun," is its angle reckoned easterly on the equatorial from the vernal equinox. This is given in the Ephemeris for every day of the year. Similarly, the right ascension of many fixed stars is given for every ten days of the year, under the head of "Fixed Stars, Apparent Places for the Upper Transit at Washington." These latter change by a few seconds a year, from the fact that the origin of coördinates, the vernal equinox itself, changes by a small amount annually. If, therefore, the hour-angle, or right ascension, of both the mean sun and a fixed

* It is assumed that the engineer or surveyor has only the ordinary field-transit, without prismatic eye-piece, so that he can only read altitudes less than 60°. The accuracy to be attained is about to the nearest second of time.

star be found for any day of the year, the difference will be the *sidereal interval* intervening between their meridian passages, the one having the greater hour-angle crossing the meridian much later than the other. When this interval is changed to *mean time* the result is the mean or clock time intervening between their meridian passages. If a fixed star is chosen whose right ascension is eight hours greater than that of the mean sun for any day in the year, then this star will come to the meridian eight hours (sidereal time) after noon, or at $7^h\ 58^m\ 41^s.364$ after noon of the civil day indicated in the Nautical Almanac. If, therefore, one wishes to make his observations for time from 8 to 10 o'clock P.M. he should select stars whose hour-angles, or right ascensions, are from 8 to 10 hours greater than that of the mean sun for the given date.

In the following table such lists are made out for the first day of each month for the year 1888. The mean time of transit is given for the meridian of Washington to the nearest minute, as well as its mean place for the year. None of these values will vary more than three or four minutes from year to year, and therefore the table may be used for any place and for any time. The table merely enables the observer to *select the stars* to be observed. After these are chosen their *local mean time of transit* must be worked out with accuracy from the Nautical Almanac.* For any other day of the month we have only to remember that the star comes to the meridian $3^m\ 56^s$ earlier (mean time) each succeeding day, so that for n days after the first of the month we subtract $3.93\ n$ minutes from the mean time of transit given in the table, and this will give the approximate mean time of transit for that date. If we take n days before

* Even this trouble may be avoided by using *Clarke's Transit Tables* (Spon, London). Price to American purchasers less than one dollar. They are published annually in advance, and give the Greenwich mean time of transit of the sun and many fixed stars for every day in the year. They are computed for popular use from the Nautical Almanac.

LIST OF SOUTHERN TIME-STARS FOR EACH MONTH.

Star.	Mag.	Declination.	Right Ascension.	Approx. Mean Time of Transit.	Star.	Mag.	Declination.	Right Ascension.	Approx. Mean Time of Transit.
JANUARY 1.					FEBRUARY 1.				
ε Eridani	3	− 9° 50′	$3^h\ 28^m$	$8^h\ 44^m$	δ Orionis	2	− 0° 23′	$5^h\ 26^m$	$8^h\ 40^m$
γ Eridani	3	− 13 49	3 53	9 09	α Leporis	3	− 17 54	5 28	8 42
o^1 Eridani	4	− 7 08	4 06	9 23	ε Orionis	2	− 1 16	5 31	8 45
β Eridani	3	− 5 14	5 02	10 18	α Columbæ	2	− 34 08	5 36	8 50
β Orionis	1	− 8 20	5 09	10 25	κ Orionis	3	− 9 42	5 42	8 57
τ Orionis	4	− 6 58	5 12	10 28	α Canis Maj	1	− 16 34	6 40	9 54
MARCH 1.					APRIL 1.				
ε Canis Maj	2	− 28° 49′	$6^h\ 54^m$	$8^h\ 14^m$	θ Hydræ	4	+ 2° 47′	$9^h\ 08^m$	$8^h\ 26^m$
δ Canis Maj	2	− 26 13	7 04	8 24	α Hydræ	2	− 8 10	9 22	8 40
15 Argus	3	− 23 59	8 03	9 23	μ Hydræ	4	− 16 16	10 21	9 38
30 Monoc	4	− 3 32	8 20	9 39	α Antliæ	4	− 30 30	10 22	9 39
σ Hydræ	5	+ 3 44	8 33	9 53	δ Crateris	3	− 14 10	11 14	10 32
ε Hydræ	3	+ 6 50	8 41	10 00	ξ Hydræ	4	− 31 14	11 27	10 45
MAY 1.					JUNE 1.				
ε Corvi	3	− 22° 00′	$12^h\ 04^m$	$9^h\ 23^m$	α Virginis	1	− 10° 35′	$13^h\ 19^m$	$8^h\ 36^m$
γ Corvi	2	− 16 55	12 10	9 29	ζ Virginis	3	− 0 01	13 29	8 46
η Virginis	3	− 0 03	12 14	9 33	π Hydræ	4	− 26 08	14 00	9 17
δ^2 Corvi	2	− 15 53	12 24	9 43	κ Virginis	4	− 9 45	14 07	9 24
β Corvi	2	− 22 47	12 28	9 48	λ Virginis	5	− 12 51	14 13	9 30
γ Virginis	3	− 0 50	12 36	9 55	α^2 Libræ	2	− 15 35	14 45	10 01
θ Virginis	4	− 4 56	13 04	10 23	γ Scorpii	3	− 24 50	14 57	10 14

LIST OF SOUTHERN TIME-STARS FOR EACH MONTH—*Continued.*

Star.	Mag.	Declination.	Right Ascension.	Approx. Mean Time of Transit.	Star.	Mag.	Declination.	Right Ascension.	Approx. Mean Time of Transit.
July 1.					**August 1.**				
β Libræ	2	− 8° 58′	15ʰ 11ᵐ	8ʰ 30ᵐ	θ Ophiuchi	3	− 24° 53′	17ʰ 15ᵐ	8ʰ 32ᵐ
δ Scorpii	2	− 22 18	15 54	9 13	δ Ophiuchi	5	− 24 04	17 20	8 36
β¹ Scorpii	2	− 19 30	15 59	9 18	γ² Sagitt.	3	− 30 25	17 59	9 15
δ Ophiuchi	3	− 3 24	16 08	9 28	μ¹ Sagitt.	4	− 21 05	18 07	9 24
α Scorpii	1	− 26 11	16 23	9 42	η Serpentis	3	− 2 56	18 15	9 32
ζ Ophiuchi	3	− 10 20	16 31	9 50	λ Sagitt.	3	− 25 29	18 21	9 38
η Ophiuchi	3	− 15 35	17 04	10 23	ι Aquilæ	4	− 8 19	18 29	9 46
September 1.					**October 1.**				
σ Sagitt.	2	− 26° 26′	18ʰ 48ᵐ	8ʰ 03ᵐ	φ Capri	4	− 25° 40′	20ʰ 39ᵐ	7ʰ 56ᵐ
d Sagitt.	5	− 19 09	19 11	8 25	μ Aquarii	5	− 9 24	20 47	8 03
κ Aquilæ	5	− 7 16	19 31	8 45	ζ Capri	4	− 22 54	21 20	8 36
ι Sagitt.	5	− 28 01	19 56	9 10	β Aquarii	3	− 6 04	21 26	8 42
θ Aquilæ	3	− 1 09	20 06	9 20	θ Aquarii	4	− 8 20	22 11	9 27
α² Capri	3	− 12 53	20 12	9 26	γ Aquarii	3	− 1 57	22 16	9 32
November 1.					**December 1.**				
λ Aquarii	4	− 8° 11′	22ʰ 47ᵐ	8ʰ 01ᵐ	ι Ceti	3	− 9° 27′	0ʰ 14ᵐ	8ʰ 29ᵐ
α Pis. Aus.	1	− 30 13	22 51	8 05	β Ceti	2	− 18 36	0 38	8 54
φ Aquarii	4	− 6 39	23 09	8 22	θ¹ Ceti	3	− 8 46	1 18	9 34
ι¹ Aquarii	5	− 18 54	23 38	8 52	ζ Ceti	3	− 10 53	1 46	10 01
δ Sculp.	4	− 28 45	23 43	8 57	δ Ceti	4	− 0 09	2 34	10 49
33 Piscium	5	− 6 20	24 00	9 14	γ Ceti	3	+ 2 46	2 37	10 53

a date in the table, add 2.93 *n* minutes to the corresponding time of transit to find the approximate time of transit for the given date. This table is therefore a mere matter of convenience to assist in selecting the stars to be used. They are nearly all southern stars, since these only can be observed with the ordinary field-transit.

389. Finding the Mean Time of Transit.—As explained above, the mean or clock time of transit is simply the sidereal interval between the mean sun and star for the given place and date, reduced to mean time. To find this interval we find the right ascension of both mean sun and star, and take their difference. But the right ascension or sidereal time of the mean sun or mean noon is given for the meridian of Greenwich, whereas by the time the sun has reached the given American meridian its right ascension or sidereal time has increased somewhat, the hourly increase being $9^s.8565$. To find the "sidereal time of mean noon" for the given place, therefore, we take the value for the given date for Greenwich and add to it $9^s.8565$ for every hour of longitude the place is west of Greenwich. This then gives the "local sidereal time of mean noon." The right ascension of the star, or the sidereal time of its meridian passage, is then found. This changes only by a few seconds in a year, and is given for every ten days in the Washington Ephemeris. This, therefore, needs no correction to reduce it to its local value for any place. The difference between the "local sidereal time of mean noon" and the sidereal time of the star is the *sidereal interval* of time elapsing between local mean noon and the transit of the star. When this sidereal interval is changed to a mean-time interval, which is effected by means of a table at the back of the Nautical Almanac, the result is the local mean time of transit of the star.

EXAMPLE.—Compute the local mean time of transit of ϵ Eridani at St. Louis on Jan. 16, 1888.

Sidereal time of mean noon at Greenwich	$= 19^h\ 41^m\ 27^s.80$
Correction for longitude 6.05^h west	$= +\quad 59.63$
Local sidereal time of mean noon	$= 19\ 42\ 27.43$
Right ascension ϵ Eridani Jan. 16	$= 3\ 27\ 39.21$
Sidereal interval after mean noon	$= 7\ 45\ 11.78$
Correction to reduce to mean time	$= -\ 1\ 16.21$
Local mean time of transit	$= 7^h\ 43^m\ 55^s.57$

390. Finding the Altitude.—The relation between latitude, declination, and altitude is shown by Fig. 146, which represents a meridian section of the celestial sphere. Let PP' be the line through the earth's axis; QQ' the plane of the equator; Z the zenith, and HH' the horizon. Then $H'P = ZQ = \phi$ is the latitude of the place, and $QS = \delta$ and $QS'' = -\delta''$ are the declinations of S and S'' respectively. The altitude of the star S is $H'S$, or measured from the south point it would be HS. The altitude of the star S'' is HS''.

FIG. 146.

We have therefore for altitude of S

$$h = HZ - ZQ + QS = 90° - \phi + \delta.$$

Also for altitude of S'',

$$h'' = HZ - ZQ - QS'' = 90° - \phi - \delta''$$

But since south declination is considered as negative, we have, in general, for altitude from the south point, of a star in the meridian,

$$A = 90° - \phi + \delta.$$

The latitude is supposed to be known and the declination is given in the table, whence the altitude of any star in the list is readily found.

391. Making the Observations.—The meridian is supposed to be established. This may be done either by having two points in it fixed, one of which is occupied by the instrument and the other by a target, or an azimuth may be known to any other station or target. In either case the instrument is put into the meridian by means of *both verniers*, either making the mean of the two read zero on the meridian post, or by making the mean of their readings on the azimuth station differ from their mean reading in the meridian by an amount equal to the azimuth of the given line.

Or, the setting may be approximately on the meridian and the angle measured so that the true deviation of the instrument from the meridian is observed for each star observation. The error in time, from a given small error in azimuth, is then found from the differential equation*

$$dt = \frac{\sin(\phi - \delta)}{\cos\delta}\, da,$$

where dt is the error in *hour-angle in seconds of arc* when da is the deviation from the meridian in seconds of arc, ϕ being the latitude of the place, and δ the declination of the star.

---

* One of the fundamental equations that may be written from an inspection of Fig. 11, p. 49, is

$$\cos\delta \sin t = -\cos h \sin a,$$

where h is the altitude and t, δ, and a as above. Differentiating with reference to t and a, we have

$$dt = -\frac{\cos h \cos a}{\cos\delta \cos t}\, da.$$

For observations very near the meridian both $\cos a$ and $\cos t$ become unity, and then we have

$$dt = -\frac{\cos h}{\cos\delta}\, da = \frac{\sin(\phi - \delta)}{\cos\delta}\, da.$$

Having found the time correction in seconds of arc, the correction in seconds of time is found by dividing by fifteen.

If the declination is south, or negative, the equation becomes

$$dt = \frac{\sin(\phi + \delta)}{\cos \delta} da.$$

The error from this cause diminishes as the altitude of the star increases, and is zero for a zenith observation.

The stars are chosen in pairs, the two stars of a pair having about the same altitude or declination. Thus, from the January group we might select o' Eridani and β Orionis as one pair, and β Eridani and τ Orionis for another. The stars are of course observed in the order of their coming to the meridian, irrespective of the way they are paired, but they are paired in the reduction.

The visual angle of the field of view of the ordinary engineer's field-transit is something over one degree. The star will therefore be visible in the telescope for something over two minutes before it comes to the vertical wire, it being here assumed that there is but one vertical thread. Let an attendant hold the watch or chronometer and note the time to the nearest second when the star is on the wire, as noted by the observer. If this time be compared with that of the computed mean time of transit, the error of the chronometer is obtained, so far as this observation gives it.

The instrument must be reversed on the second star of each pair. This is to eliminate the instrumental errors. The horizontal angle to the station-mark (whether this be on the meridian or not) should also be read for every reading on a star, or at least before and after the star-readings.

The following programme would be adapted to observations on the four stars selected above:

PROGRAMME.

1. Set on azimuth station and read horizontal angle (both verniers).

2. Set in the meridian and read both verniers.

3. Set the approximate altitude of o^1 Eridani.

4. Note time of passage of o^1 Eridani.

5. Set on azimuth station and read both verniers.

6. Set in the meridian and read verniers.

7. Note time of passage of β Eridani.

8. *Revolve the telescope* 180° *on its horizontal axis, relevel*, and read on the azimuth station.

9. Set in the meridian and read verniers.

10. Note time of passage of β Orionis.

11. Note time of passage of τ Orionis.

12. Read both verniers again in the meridian before the instrument is disturbed.

13. Read to azimuth station.

We have thus obtained four measurements of the horizontal angle, and read with the telescope normal and inverted on each pair of stars. Especial care must be taken to see that the plate-bubble set perpendicular to the telescope is exactly in the centre when readings are taken to the stars. The mean chronometer error for the two stars of a pair is its true error, provided it has no rate. If the chronometer has a known rate, that is, if it is known to be gaining or losing at a certain rate, then its error must be found for some particular time, as that of the first observation. Its rate must then be applied to the observed time of transit of the other stars for the intervening intervals before comparing results. If local time alone is desired, the result is obtained as soon as a pair of stars has been observed and their mean result found.

392. Longitude.—If geographical position or longitude is sought, it remains to compare the chronometer with the standard or meridian time for that region. This standard time is now transmitted daily from fixed observatories to almost all railroad stations in the United States. The time thus transmitted is probably never in error more than a few tenths of a second. It is usually sent out from 10 A.M. to noon daily. If the rate of the station clock is known, and also that of the watch used in the time observation, then a comparison of these subsequent to the observation would give the difference between local time and the hourly meridian time used, which difference changed to longitude would be the longitude of the place east or west of that standard meridian. If the station clock cannot be relied on as to its rate, then the watch used in the observation must have a constant known rate. In this case the observer compares his watch on the following day with the time signal as it is transmitted over the railroad company's wires, and so obtains his longitude.

Local time can be observed in this way by means of an ordinary transit to the nearest second of time, and the longitude obtained to the same accuracy if the rate of the chronometer used is constant and accurately known. It is probable, however, that several seconds error may be made if a watch is used, since probably no watch has a rate which is constant within one second in twelve hours. Therefore if longitude is desired a portable chronometer should be used whose rate is well known.*

393. **Computing the Geodetic Positions.**— After the angles of the system are adjusted, and the sides of the triangles computed, we have the plane angles and linear distances from point to point in the system. It now remains to compute the

* This method has been extensively used for obtaining approximate geodetic positions for the U. S. Geological Survey in the West, comparisons being made daily with the Washington University time signals which are transmitted to the railways in that region.

latitudes and longitudes of the several stations, and the azimuths of the lines.

The following formulæ, though not exact, are quite sufficient when the sides of the triangles do not exceed ten or fifteen miles in length: *

NOTATION.

Let L' = latitude of the known point;
L = latitude of the unknown point;
M' = longitude of the known point;
M = longitude of the unknown point;
Z' = azimuth of the unknown point from the known, counting from the south point in the direction S.W.N.E.;
Z = azimuth of the known point from the unknown, or the back azimuth;
K = length in metres of line joining the two points;
e = eccentricity of the earth's meridian section;
N = length of the normal, or radius of curvature of a section perpendicular to the meridian of the middle latitude, in metres.
R = radius of curvature of the meridian in metres.

Then we have in terms of the length and azimuth of a given line, in seconds of arc, when the distances are given in metres,

$$\left.\begin{aligned} L' - L &= -dL = BK\cos Z' + CK^2\sin^2 Z' + Dh^2; \\ M' - M &= +dM = AK\frac{\sin Z'}{\cos L''}; \\ Z' - 180^\circ - Z &= -dZ = dM\sin L_m; \end{aligned}\right\} \quad (1)$$

* For a summarized derivation of these formulæ for computing the L M Z's from triangulation data, together with extended tables of factors used, see Report U. S. Coast and Geodetic Survey, 1884, Appendix No. 7. The derivation of the formulæ is further amplified in Appendix D of this book.

where $A^* = \dfrac{1}{N' \text{ arc } 1''}$; $B = \dfrac{1}{R' \text{ arc } 1''}$; $C = \dfrac{\tan L'}{2R'N' \text{ arc } 1''}$;

$D = \dfrac{\frac{3}{2}e^2 \sin L' \cos L' \sin 1''}{(1 - e^2 \sin^2 L')\dagger}$; $L_m = \text{mean latitude} = \dfrac{L + L'}{2}$;

and h = value of first term in right member $= BK \cos Z'$. *Careful attention must be paid to the signs of the Z functions.*

TABLE OF *L M Z* COEFFICIENTS.

Latitude.	Log. A + 10.	Log. B + 10.	Log. C + 10.	Log. D + 10.
30°	8.5093588	8.5115729	1.16692	2.3298
31	3363	5054	.18416	.3382
32	3134	4368	.20108	.3460
33	2901	3669	.21772	.3532
34	2665	2959	.23409	.3597
35	8.5092425	8.5112239	1.25024	2.3656
36	2182	1510	.26617	.3709
37	1936	0772	.28193	.3756
38	1687	8.5110027	.29753	.3797
39	1437	8.5109275	.31299	.3833
40	8.5091184	8.5108517	1.32833	2.3863
41	0930	7755	.34358	.3888
42	0675	6989	.35875	.3907
43	0419	6220	.37386	.3921
44	8.5090162	5449	.38894	.3930
45	8.5089904	8.5104677	1.40400	2.3933
46	9647	3905	.41906	.3932
47	9390	3134	.43414	.3924
48	9133	2364	.44926	.3912
49	8878	1598	.46443	.3894
50	8.5088623	8.5100835	1.47968	2.3871

* A is to be evaluated for L.

† This denominator is given to the $\frac{3}{2}$ power in the Coast Survey formulæ. The rigid development used in Appendix D. shows it to be as given above The error, however, is small.

Logarithmic values of the coefficients *A*, *B*, *C*, and *D* are given in the above table for each degree of latitude from 30° to 50°. By the aid of this table the *LMZ*'s are readily found. These tabular values are computed from the constants of Clarke's spheroid. In this we have

Equatorial semidiameter = 6 378 206 metres.
Polar semidiameter = 6 356 584 "

Whence the ratio of the semidiameters is $\frac{294.98}{293.98}$.

Clarke's value of the metre has been taken, which is

$$1 \text{ metre} = 39.37000 \text{ inches.}$$

The difference of azimuth of the two ends of a line is due to the convergence of the meridians passing through its extremities, this convergence, as seen from the last of equations (1), being equal to the difference of longitude into the sine of the mean latitude.

When the sides of a system of triangulation have been computed, and the azimuths of the lines are desired from the several stations, the successive differences of latitude and longitude are first computed, and from these the azimuths of the lines, using equations (1). If the longitude is unknown, the longitude of the first station may be assumed without affecting the accuracy of the computed relative positions. The last of equations (1) gives the difference between the forward and back azimuth of the line joining the two stations. This difference being applied, with the proper sign, gives the azimuth of the first station as seen from the second. But when the azimuth of one line from a station is known, the azimuths of all other lines from that station are found from the adjusted plane-angles at that station, provided the spherical excess had been deducted or allowed for, in the adjustment. If no account has been taken of spherical excess, the error in azimuth accumulates in working eastward or westward, and soon becomes appreciable.

For any other station the azimuth correction is again found for the line joining this station with a station where azimuths have been computed, which when applied gives the azimuth of this line as taken at the forward station, whence the azimuths of all the lines from this station are known, and so on.

394. Example.—In Fig. 142, p. 495, let the azimuth of the line CA, from C, be 80°; latitude of C be 40°; the length of the line CD be 25000 metres (over 15 miles); required the geodetic position of D, and the azimuth of the line DC from D.

COMPUTATION OF LMZ.

Z	C to A	80°	00′	00″.0
$\angle$	$ACD = C_3$ (see p. 495)	39	48	06 .1
Z'	C to D	119	48	06 .1
dZ	....	−	9	49 .5
180°	....	180		
Z	D to C	299	38	16 .6

L'	40°	00′	00″,000	C	M'	90°	00′	00″,000
dL	+	6	41 .817	25000 metres.	dM	+	15	16 ,019
L	40	06	41 .847	D	M	90	15	16 ,019

		B	8.5108517	C	1.32833	D	2.3863
		K	4.3979400	K^2	8.79588	h^2	5.2103
		$\cos Z'$	9.6963560	$\sin^2 Z'$	9.87679		
1st term	− 402″.853	h	2.6051477		0.00100		7.5966
2d and 3d terms	+ 1 .006				1.0023		0.00395
$-dL$	− 401 .847	A	8.5091156				
$L_m = 40° 03' 22''$		K	4.3979400				
		$\sin Z'$	9.9383948	dM	2.96190		
		$\cos L'$(a.c.)	0.1164540	$\sin L_m$	9.80857		
			2.9619044		2.77047		
		dM	+ 916″.019	$-dZ$	+ 589″.48		

GEODETIC LEVELLING.

395. Geodetic Levelling is of two kinds: (*A*) *Trigonometrical Levelling* and (*B*) *Precise Spirit-levelling*. In trigonometrical levelling the relative elevations of the triangulation-stations are determined by reading the vertical angles between the stations. When these are corrected for curvature of the earth's surface and for refraction it enables the actual difference of elevation to be found. In precise spirit-levelling a special type of the ordinary spirit or engineer's level is used, and great care taken in the running of a line of levels from the sea-coast inland, connecting directly or indirectly with the triangulation stations and base-lines. Both these methods will be described.

(*A*) TRIGONOMETRICAL LEVELLING.

396. Refraction.—If rays of light passed through the atmosphere in straight lines, then in trigonometrical levelling we should

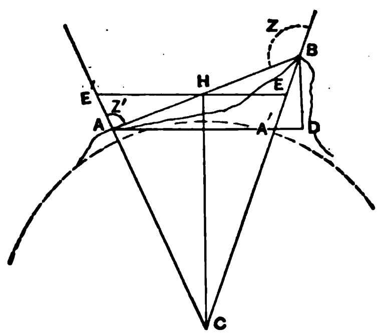


FIG. 147.

have to correct only for the curvature of a level surface at the locality. It is found, however, that rays of light near the surface of the earth usually are curved downwards—that is, their paths are convex upwards. This curve is quite variable, sometimes being actually convex downwards in some localities. It has its greatest curvature about daybreak, diminishes rapidly till 8 A.M., and is nearly constant from 10 A.M. till 4 P.M., when it begins to increase again. The curve may be considered a circle having a variable radius, the mean value of which is about seven times the radius of the earth.

397. Formulæ for Reciprocal Observations.—In Fig. 147 the dotted curve represents a sea-level surface.

Let $H =$ height of station B above sea-level;
$H' =$ height of station A above sea-level;
$C =$ angle subtended by the radii through A and B;
$Z =$ true zenith distance of A as seen from B;
$Z' =$ true zenith distance of B as seen from A;
$\delta =$ true altitude of A as seen from $B = 90° - Z$;
$\delta' =$ true altitude of B as seen from $A = 90° - Z'$;
$h =$ apparent altitude of A as seen from $B = \delta +$ refraction;
$h' =$ apparent altitude of B as seen from $A = \delta' +$ refraction.
$d =$ distance at sea-level between A and B;
$r =$ radius of the earth;
$m =$ coefficient of refraction.

In the figure join the points A and B by a straight line. This would be the line of sight from A to B if there were no refraction. Through A and B draw the radii meeting at C, extending them beyond the surface.* Take the middle point of the line AB, as H, and draw HC. Take AA' perpendicular to HC, and EE' through H and perpendicular to HC. Extend AA' to meet a perpendicular to it from B. Then do we have

$$A'C = AC; \quad E'E = AD; \quad \text{and} \quad HC = r + \frac{H + H'}{2}.$$

* In reality these are the *normals* at A and B, but will here be taken as the radii.

Neither of these three relations is quite exact, because HC does not quite bisect the angle C. The figure is greatly exaggerated as compared to any possible case in practice, for the angle C would never be more than 1° in such work. The error in practice is inappreciable.

From the geometrical relations shown in the figure we have

$$H - H' = A'B = DB \sec \frac{C}{2}. \quad . \quad . \quad . \quad . \qquad (1)$$

But since C is never more than 1°, and usually much less, we may say

$$H - H' = A'B = DB = AD \tan BAD. \quad . \quad . \quad . \qquad (2)$$

But $AD = E'E =$ distance between the stations reduced to their mean elevation above sea-level $= d'$; also

$$BAD = \tfrac{1}{2}(Z - Z');$$

$$\therefore H - H' = d' \tan \tfrac{1}{2}(Z - Z'). \quad . \quad . \quad . \quad . \qquad (3)$$

But since $d =$ distance between stations at sea-level, we have

$$d' : d :: r + \frac{H + H'}{2} : r,$$

or

$$d' = d\left(1 + \frac{H + H'}{2r}\right); \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

whence we have, for reciprocal observations at A and B,

$$H - H' = d \tan \tfrac{1}{2}(Z - Z')\left(1 + \frac{H + H'}{2r}\right). \quad . \quad . \qquad (5)$$

or, in terms of δ and δ',

$$H - H' = d \tan \tfrac{1}{2}(\delta' - \delta)\left(1 + \frac{H + H'}{2r}\right), \quad . \quad . \quad . \quad (6)$$

where attention must be paid to the signs of δ and δ'.

The effect of refraction is to increase δ and δ' by equal amounts (presumably), whence their difference remains unaffected. Equations (5) and (6) are therefore the true equations to use for reciprocal observations at two stations. Since the refraction is so largely dependent on the state of the atmosphere, the observations should be made simultaneously for the best results. This is seldom practicable, however, and therefore it is highly probable that a material error is made in assuming that the refraction is the same at the two stations when the observations are made at different times.

398. Formulæ for Observations at One Station only.— If the vertical angle be read at only one of the two stations, then the refraction becomes a function in the problem. Since the curve of the refracted ray is assumed to be circular (it probably is not when stations have widely different elevations), the amount of angular curvature on a given line is directly proportional to the length of the line or to the angle C. The difference at A or B between the directions of the right line AB and the ray of light passing between them is one half the total angular curvature of the ray; that is, it is the angle between the tangent to the curved ray at A and the cord AB. The ratio between this refraction angle at A or B and the angle C is a constant for any given refraction curve; that is, this ratio does not change for different distances between stations. This ratio is called the coefficient of refraction, and is here denoted by m. The true angle BAD is equal to $\delta' + \frac{C}{2}$, but since the observed altitude is increased by the amount of the refraction, we have for the apparent altitude of B, as seen from A,

$$h' = \delta' + mC;$$

whence $$BAD = h' + \frac{C}{2} - mC. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Using this value of the angle BAD in equation (2), we obtain

$$H - H' = d' \tan\left(h' + \frac{C}{2} - mC\right)$$
$$= d \tan\left(h' + \frac{C}{2} - mC\right)\left(1 + \frac{H + H'}{2r}\right), \quad . \quad . \quad (8)$$

where h' is positive above and negative below the horizon.

Equation (8) is used where the vertical angle is read from one station only.

Since the total angular curvature of the ray of light between A and B is $2mC$, and the curvature of the earth is C, we may write

$$C : 2mC :: r' : r, \qquad \text{or} \qquad r' = \frac{r}{2m}, \quad . \quad . \quad . \quad (9)$$

where r' is the radius of the curve of the refracted ray.

Since the curvature of the ray is of the same kind as that of the earth, but less in amount, the total correction for curvature and refraction is for an angle equal to $\frac{C}{2} - mC = \frac{C}{2}(1 - 2m)$.

Also, since C is always a small angle, we may put

$$C \text{ (in seconds of arc)} = \frac{d}{r \sin 1''}.$$

If the mean radius is used, we have, in feet,

$$\log r = 7.32020, \text{ and } \log \sin 1'' = 4.6855749,$$

whence in seconds of arc and distance in feet we have

$$\left.\begin{aligned} \log C &= \log d - 2.00577 \\ \text{or} \qquad C &= \frac{d}{101.34} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or the curvature is approximately equal to 1″ for 100 feet in distance.

The following table gives computed values of the combined mean corrections for curvature and refraction for short distances, either for horizontal or inclined sights. Both the distance d and the correction c_n are in feet, except for the last column, where the distance is given in miles. For a more extended table for long distances, see page 453.

CORRECTION FOR EARTH'S CURVATURE AND REFRACTION.

d	c_n	d	c_n	d	c_n	d	c_n	d	c_n	miles	c_n
300	.002	1300	.035	2300	.108	3300	223	4300	.379	1	.571
400	.003	1400	.040	2400	.118	3400	.237	4400	.397	2	2.285
500	.005	1500	.046	2500	.128	3500	.251	4500	.415	3	5.142
600	.007	1600	.052	2600	.139	3600	.266	4600	.434	4	9.141
700	.010	1700	.059	2700	.149	3700	.281	4700	.453	5	14.282
800	.013	1800	.066	2800	.161	3800	.296	4800	.472	6	20.567
900	.017	1900	.074	2900	.172	3900	.312	4900	.492	7	27.994
1000	.020	2000	.082	3000	.184	4000	.328	5000	.512	8	36.563
1100	.025	2100	.090	3100	.197	4100	.345	5100	.533	9	46.275
1200	.030	2200	.099	3200	.210	4200	.362	5200	.554	10	57.130

399. Formulæ for an observed Angle of Depression to a Sea Horizon.—In Fig. 148 let A be the point of observation and S the point on the sea-level surface where the tangent from A falls. Then we have

$$\begin{aligned} H = AD &= {}^{*}AS \tan ASD \\ &= r \tan C \tan \frac{C}{2}. \quad . \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

* Let the student prove this relation

Since the angle C is always very small, we may let the arc equal its tangent, whence

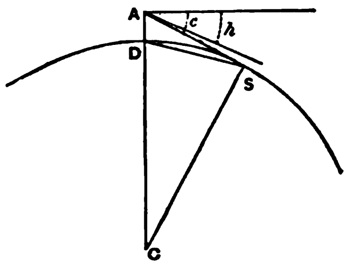


FIG. 148.

$$H = \frac{r}{2} \tan^2 C. \quad . \quad (12)$$

If the observed angle of depression be $h = C - mC$,

then
$$C = \frac{h}{1 - m},$$

and
$$H = \frac{r}{2} \tan^2 \left(\frac{h}{1 - m}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or
$$H = \frac{r}{2}\left(\frac{h}{1 - m}\right)^2 \tan^2 1'', \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where h is expressed in seconds of arc.

$$\text{Log} \frac{r}{2} \tan^2 1'' = 6.39032 \text{ for distances in feet.}$$

400. To find the Value of m we have

$$Z = 90^\circ - h + mC,$$
$$Z' = 90^\circ - h' + mC;$$

whence
$$Z + Z' = 180^\circ + C = 180^\circ - h - h' + 2mC,$$

or
$$1 - 2m = -\frac{h + h'}{C},$$

or
$$m = \tfrac{1}{2}\left(1 + \frac{h + h'}{C}\right). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where h and h' are the observed altitudes above the horizon. It is evident that every pair of reciprocal observations at two stations will give a value for m. The mean values of m, as found from observations on the United States Coast Survey in New England, were:

Between primary stations, ...	$m = 0.071$
For small elevations, ...	$m = 0.075$
For a sea horizon, ...	$m = 0.078$

On the New York State Survey the value from 137 observations was $m = 0.073$.*

In this work also the term $\dfrac{H + H'}{2r}$ in equations (4) to (8) never affected the result by more than $\frac{1}{4000}$ part of its value.

PRECISE SPIRIT-LEVELLING.†

401. **Precise Levelling** differs from ordinary spirit-levelling both in the character of the instruments used and in the methods of observation and reduction. It is differential levelling over long lines, the elevations usually being referred to mean sea-level. In order that the elevations of inland points, a thousand miles or more from the coast, may be determined with accuracy, the greatest care is required to prevent the accumulation of errors. In order that triangulation distances may be reduced to sea-level, the elevations of the bases at least must be found. It is impossible to carry elevations accurately from one triangulation-station to another by means of the vertical angles, on account of the great variations in the refraction. Barometric determinations of heights are also subject to great uncertainties unless observations be

* See pages 435 and 436 for a case of excessive refraction profitably utilized.

† See Appendix F for description of methods used on the Miss. River Survey.

made for long periods. The only accurate method of finding the elevations of points in the interior above sea-level is by first finding what mean sea-level is at a given point by means of automatic tide-gauge records for several years, and then running a line of precise spirit-levels from this gauge inland and connecting with the points whose elevations are required. Most European countries have inaugurated such systems of geodetic levelling, this work being considered an integral part of the trigonometrical survey of those countries. In the United States this grade of work was begun on the U. S. Lake Survey in 1875, by carrying a duplicate line of levels from Albany, N. Y., and connecting with each of the Great Lakes. The Mississippi River Commission has carried such a line from Biloxi, on the Gulf of Mexico, to Savannah, Ill., along the Mississippi River, and thence across to Chicago, connecting there with the Lake Survey Elevations.* The U. S. Coast and Geodetic Survey is carrying a line of precise levels from Sandy Hook, N. J., across the continent, passing through St. Louis, their line here crossing that of the Mississippi River Commission. On all these lines permanent bench-marks are left at intervals of from one to five miles, whose elevations above mean sea-level are determined and published.

402. **The Instruments** used in precise levelling differ in many respects from the ordinary wye levels used in America. The levelling instrument prescribed by the International Geodetic Commission held in Berlin in 1864 is shown in Fig. 149.

These instruments are made by Kern & Co., of Aarau, Switzerland, and this illustration is almost an exact representation of the instruments used on the U. S. Lake and Mississippi River Surveys.† The bubble is enclosed in a wooden case (metal case in the cut), and rests on top of the pivots or rings; it is carried in the hand when the instrument is transported. A mirror is provided which enables the observer to read the

* The author had charge of about 600 miles of this work.

bubble without moving his eye from the eye-piece. There is a thumb-screw with a very fine thread under one wye which is used for the final levelling of the telescope when pointed on the rod. There are three levelling-screws, and a circular or box level for convenience in setting. The telescope bubble is very

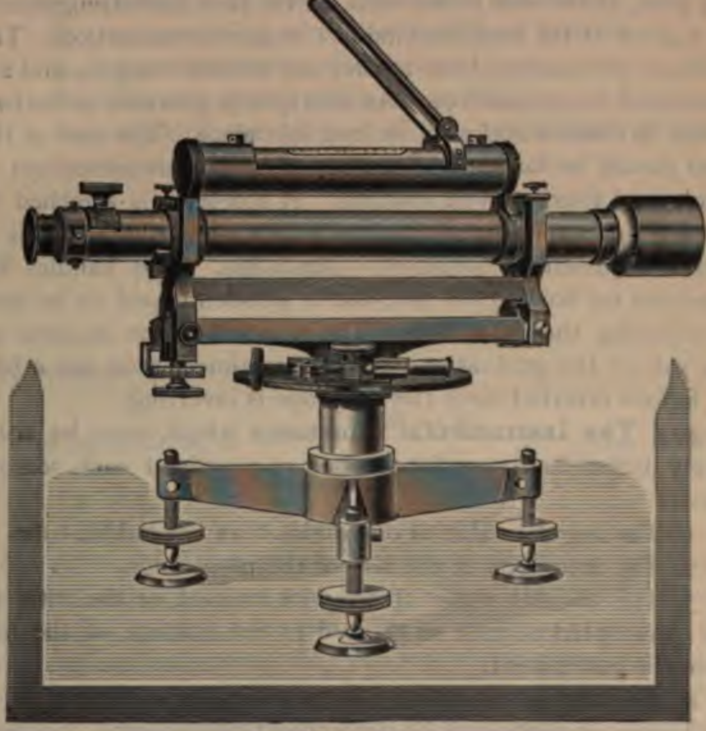


FIG. 149.

delicate, one division on the scale corresponding to about three seconds of arc. The bubble-tube is chambered also, thus allowing the length of the bubble to be adjusted to different temperatures. The magnifying power is about 45 diameters. There are three horizontal wires provided, set at such a distance apart that the wire interval is about one hundredth of the distance to the rod. The tripod legs are covered with white cloth to diminish the disturbing effects of the sun upon them. The level itself is always kept in the shade while at work.

The levelling-rod is made in one piece, three metres long, of dry pine, about four inches wide on the face, and strengthened by a piece at the back, making a T-shaped cross-section. The rods are self-reading, that is, they are without targets, and are graduated to centimetres. An iron spur is provided at bottom which fits into a socket in an iron foot-plate. The end of the spur should be flat and the bottom of the socket turned out to a spherical form, convex upwards. A box-level is attached to the rod to enable the rodman to hold it vertically, and this in turn is adjusted by means of a plumb-line. Two handles are provided for holding the rod, and a wooden tripod to be used in adjusting the rod-bubble. The decimetres are marked on one side of the graduations and the centimetres on the other, all figures inverted since the telescope is inverting.

403. The Instrumental Constants which must be accurately determined once for all, but re-examined each season, are—

1. The angular value of one division on the bubble-tube.
2. The inequality in the size of the pivot-rings.
3. The angular value of the wire-interval, or the ratio of the intercepted portion on the rod to the distance of the rod from the instrument.
4. The absolute lengths of the levelling-rods.

These constants may be determined as follows:

The value of one division of the bubble may be readily found by sighting the telescope on the rod, which is set at a known distance from the instrument, and running the bubble from end to end of its tube, taking rod-readings for each position of the bubble. The bubble-graduations are supposed to be numbered from the centre towards the ends.

Let E_1 = *mean* of all the eye-end readings of the bubble when it was run to the eye-end of its tube;
E_2 = same for bubble at object-end of tube;
O_1 = *mean* of all the object-end readings when bubble was at eye-end of tube;
O_2 = same for bubble at object-end of tube;
R_1 = *mean* reading of rod for bubble at eye-end,
R_2 = same for bubble at object-end;
D = distance from instrument to rod;
v = value of one division of the bubble (sine of the angle) at a unit's distance.

Then
$$v = \frac{R_2 - R_1}{D\left(\frac{E_1 - O_1}{2} - \frac{E_2 - O_2}{2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In seconds of arc we would have

$$v \text{ (in seconds)} = \frac{R_2 - R_1}{D \sin 1'' \left(\frac{E_1 - O_1}{2} - \frac{E_2 - O_2}{2}\right)}. \quad . \quad (2)$$

If a table is to be prepared for corrections to the rod-readings for various distances and deviations of the bubble from the centre of its tube, then the value as given by equation (1) is most convenient to use. The value of one division of a level bubble should be constant, but it is often affected by its rigid fastenings, which change their form from changes in temperature.

The inequality in the size of the rings is found by reversing the bubble on the rings, and also reversing the telescope in the wyes. The bubble is reversed only in order to eliminate its error of adjustment. The following will illustrate the method of making and reducing the observations:

		Bubble-readings.		
		North.		South.
Tel. eye-end north.	Lev. direct.	4.3		5.5
" " "	" reversed.	4.7		5.2
		9.0	(− 1.7)	10.7
			− 0.42	
Tel. eye-end south	Lev. direct.	6.2		3.7
" " "	" reversed.	6.6		3.3
		12.8	(+ 5.8)	7.0
			+ 1.45	
Tel. eye-end north	Lev. direct.	4.4		5.5
" " "	" reversed.	4.8		5.2
		9.2	(− 1.5)	10 7
			− 0.38	

Mean reading north = − 0.40
" " south = + 1.45
North minus south = − 1.85

That is to say, the bubble moves 1.85 divisions *towards the object-end* when the telescope is reversed in the wyes. This is evidently *twice* the inequality of the pivot-rings; and since the axis of a cone is inclined to one of its elements by one half the angle at the apex, so the line of sight is inclined to the tops of the rings by one fourth of 1.85 divisions, or 0.46 divisions of the bubble. It is also evident that the eye-end ring is the smaller, and that therefore when the top surfaces of the rings are horizontal the line of sight inclines downward from the instrument. The correction is therefore positive. This is called the *pivot-correction*, and changes only with an unequal wear in the pivot-rings.

The angular value of the wire-interval is found by measuring a base on level ground of about 300 feet from an initial point *f* * in front of the objective. Focus the telescope on a very distant object, and measure the distance from the inside of the objective to the cross-wires, this being the value

* See art. 200 for the significance of this term, as well as for the theory of the problem.

of f for that instrument. Measure the space intercepted on the rod between the extreme cross-wires.

If d = length of base, counting from the initial point;
s = length of the intercepted portion of the rod;
$r = \frac{f}{i}$ = constant ratio of distance to intercept;

then $r = \frac{d}{s}$;

and for any other intercept s' on the rod we have

$$d' = rs' + f + c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When r, f, and c are found, a table can be prepared giving distances in terms of the wire-intervals.

The errors in the absolute lengths of the rods affect only the final differences of elevation between bench-marks. This correction is usually inappreciable for moderate heights.

404. The Daily Adjustments.—The adjustments which are examined at the beginning and close of each day's work are as follows:

1. The collimation, that is, the amount by which the line of sight, as determined by the mean reading of the three wires, deviates from the line joining the centres of the rings.

2. The bubble-adjustment—that is, the inclination of the axis of the bubble to the top surface of the rings.

3. The rod-level. This is examined only at the beginning of each day's work, and made sufficiently perfect.

The first two adjustments are very important, since it is by means of these (in conjunction with the pivot-correction, determined once for the season) that the relation of the bubble to the line of sight is found. It is not customary in this work to try to reduce these errors to zero, but to make them reasonably small, and then *determine their values* and correct for them. It is evident that if the back and fore sights be kept exactly equal between bench-marks, then the errors in the instrumental adjustments are fully eliminated; and in any case these errors can only affect the *excess* in length of the sum of the one over that of the other. It is to this excess in length of back-sights over fore-sights, or *vice versa*, that the instrumental constants are applied; but in order to apply them their values must be accurately determined. The curvature of a level surface would also enter into this excess, but it is usually so small a residual distance, that the correction for curvature is quite insignificant. There are, however, three instrumental corrections to be applied for the amount of the excess, namely, the corrections for collimation, inclination of bubble, and inequality of pivots, designated respectively by c, i, and p. Since three horizontal wires are read on the rod, the wire-intervals can be used in place of the distances, for they are linear functions practically, and so a record is kept of the continued sum of the lengths of the back and fore sights, and from these the final difference is found.

The collimation-correction is taken out for a distance of one unit (the metre has been universally used in this kind of levelling), and then the correction for any given case found by multiplying by the residual distance.

Let R_1 = rod-reading for telescope normal;
R_2 = " " " " inverted;
d = distance of rod from instrument.

Then
$$c = \frac{R_2 - R_1}{2d}. \qquad (1)$$

The correction for the inclination of the bubble to the tops of the rings is found by reversing the bubble on the telescope and reading it in both positions. In such observations the initial and final readings are taken with the bubble in the same position, thus giving an odd number of observations. Usually two direct and one reversed reading are taken. The correction is found in terms of divisions on the bubble, the correction in elevation being taken from the table prepared for that purpose.

Let E_1 = *mean* of the eye-end* readings for level direct;
E_2 = " " " " " " reversed;
O_1 = " " object " " " direct;
O_2 = " " " " " " reversed;

then
$$i = \frac{1}{2}\left(\frac{E_2 - O_2}{2} - \frac{E_1 - O_1}{2}\right). \quad . \quad . \quad . \quad . \quad (2)$$

The pivot correction has already been found, and is supposed to remain constant for the season.

If E be the excess of the sum of the back-sights over that of the fore-sights, then the final correction for this excess is

$$C = E[c + v(i + p)], \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where v is taken from eq. (1), p. 551. Evidently, if the fore-sights are in excess, the correction is of the opposite sign.

405. **Field Methods.**—The great accuracy attained in precise levelling is due quite as much to the methods used and precautions taken in making the observations as to the instrumental means employed. Aside from errors of observation and instrumental errors, we have two other general classes of

* By eye-end is always meant the end towards the eye-end of the telescope, whether in a direct or a reversed position.

errors, which can be avoided only by proper care being used in doing the work. These two classes are *errors from unstable supports* and *atmospheric errors.*

Any settling of the rod between the fore and back readings upon it will result in the final elevation being too high, while any settling of the instrument between the back and fore readings from it will also result in too high a final elevation. Such errors are therefore cumulative, and the only way in which they can be eliminated is to duplicate the work over the same ground *in the opposite direction.* As a general precaution, the duplicate line should always be run in the opposite direction. This will result in larger *discrepancies* than if both are run in the same direction, but the mean is nearer the truth.

Atmospheric errors may come from wind, heated air-currents causing the object sighted to tremble or "dance," or from variable refraction. For moderate winds the instrument may be shielded by a screen or tent, but if its velocity is more than eight or ten miles an hour, work must be abandoned. To avoid the evil effects of an unsteady atmosphere the length of the sights is shortened; but when a reading cannot be well taken at a distance of about 150 feet, or 50 metres, it would be better to stop, since the errors arising from the number of stations occupied would make the work poor. At about 8 o'clock A.M. and 4 P.M. very large changes in the refraction have been observed on lines over ground which is passing from sun to shade, or *vice versa*, when the image was apparently very steady. In clear weather not more than three or four hours a day can be utilized for the best work, and sometimes, with hot days and cool nights, it is impossible to get an hour when good work can be done.

In making the observations the bubble is brought exactly to the centre of its tube, the observer being able to do this by means of the thumb-screw under one wye, and the mirror which reflects the image of the bubble to the observer at the eye-piece. If there is no mirror to the bubble, then it is brought approximately to the centre, and the recorder reads it while the observer is reading the three horizontal wires. In any case the bubble-reading is recorded in the note-book, and if it was not in the middle a correction is made for the eccentric position by means of a table prepared for the purpose. The mean of the three wire-readings is taken as the reading of that rod, the observer estimating the tenths of the centimetre spaces, thus reading each wire to the nearest millimetre. The wires should be about equally spaced so that the mean of the three wires coincides very nearly with the middle wire. The differences between the middle and extreme wire-readings are also taken out to give the distance, as well as to check the readings themselves by noting the relation of the two intervals. If they are not about equal, then one or more of the three readings is erroneous. This is a most important check, and constitutes an essential feature of the method.

It has been found economical to have two rodmen to each instrument, so that no time shall be lost between the back and fore sight readings from an instrument-station. Since but a small portion of the day can generally be utilized, it is desirable to make very rapid progress when the weather is favorable. When two rodmen are used, and the air is so steady that 100-metre sights can be taken,* it is not difficult for an experienced observer to move at the rate of a mile an hour.

On the U. S. Coast and Geodetic Survey a much more laborious method of observing than the one above outlined has been followed. There a special kind of target-rod has been employed, the target being set approximately and clamped. The thumb-screw under the wye is used as a micrometer-screw, and two readings are taken on it one when

* This is about the limiting length of sight for first-class work, even under the most favorable conditions.

the bubble is in the middle and the other when the centre wire bisects the target, the bubble now not being in the middle, since the target's position was only approximate. The bubble is then reversed, and two more readings of the screw taken. The telescope is now revolved in the wyes, and readings taken again with bubble direct and reversed. Thus there are four independent readings taken on the rod, each necessitating two micrometer-readings. The reduction is also very complicated, each sight being corrected for curvature and refraction as well as for instrumental constants. The duplicate line is carried along with the first one by having two sets of turning-points for each instrument-station.* The instrument, however, is set but once, so that the lines are not wholly independent. The alternate sections are run in opposite directions, thus partly obviating the objection to running both lines in the same direction. The method first described was used on the U. S. Lake and Mississippi River surveys, and is also the method used on most of the European surveys of this character.

The instrument is always shaded from the sun, both while standing and while being carried between stations. It is absolutely necessary to do this in order to keep the adjustments approximately constant, and the bubble from continually moving.

406. Limits of Error.—On the U. S. Coast and Geodetic Survey the limit of discrepancy between duplicate lines is $5^{mm}\sqrt{2K}$ where K is the distance in kilometres. On the U. S. Lake Survey the limit was $10^{mm}\sqrt{K}$, and on the Mississippi River Survey it was $5^{mm}\sqrt{K}$. These limits are respectively 0.029 0.041, and 0.021 feet into the square root of the distance in miles. If any discrepancies occurred greater than these the stretch had to be run again.

The "probable error" of the mean of several observations on the same quantity is a function of the discrepancies of the

* Now abandoned, 1890.

several results from the mean. If v_1, v_2, v_3, etc., be the several residuals obtained by subtracting the several results from the mean, and if $\Sigma[vv]$ be the sum of the squares of these residuals, and if m be the number of observations, then the *probable error of the mean* is $R = \pm .6745\sqrt{\frac{\Sigma[vv]}{m(m-1)}}$.

This is the function which is universally adopted for measuring the *relative accuracy* of different sets of observations.

If there be but two observations this formula reduces to

$$R = \pm \tfrac{1}{3}V,$$

where V is the discrepancy between two results.

The European International Geodetic Association have fixed on the following limits of probable error per kilometre in the mean or adopted result: $\pm 3^{mm}$ per km. is tolerable; $\pm 5^{mm}$ per km. is too large; $\pm 2^{mm}$ per km. is fair; and $\pm 1^{mm}$ per km. is a very high degree of precision. On the U. S. coast and geodetic line from Sandy Hook to St. Louis, a distance of 1109 miles, the probable error per kilometre was $\pm 1.2^{mm}$.* For the 670 miles of this work on the Mississippi River Survey, of which the author had charge, the probable error of the mean for the entire distance was 23.5^{mm} (less than one inch), and the probable error per kilometre was $\pm 0.7^{mm}$.† Of course very little can be predicated on these results as to the *actual* errors of the work, since the number of observations on each value was usually but two; but they may fairly be used for the purpose of comparing the relative accuracy of different lines where this function has been computed from similar data.

407. Adjustment of Polygonal Systems in Levelling.—If

* Report U. S. Coast and Geodetic Survey, 1882, p. 522.

† Reports of the Miss. Riv. Commission for the years 1882, 1883, and 1884.

a line of levels closes upon itself the summation of all the differences of elevation between successive benches should be zero. If it is not, the residual error must be distributed among the several sides, or stretches, composing the polygon, according to some law, so that the final corrections which are applied to the several sides shall be independent of all personal considerations. These corrections should also be the most probable corrections. There are two general criterions on which to found a theory of probabilities. One may be called *a priori*, and the other *a posteriori*. By the former we would say that the errors made are some function of the distance run, as that they are directly proportional to this distance, or to the square root of this distance, etc.; while by the latter, or *a posteriori* method, we would say the errors made on the several lines are a function of the discrepancies found between the duplicate measurements on those lines, or to the computed "probable error per kilometre," as found from these discrepancies. Both methods are largely used in the adjustment of observations. These laws of distribution are equivalent to establishing a method of weighting the several sides of the system, a larger weight implying that a larger share of the total error is to be given to that side. When any system of weights is fixed upon, then the corrections may be computed by the methods of least squares so as to comply with the condition that the corrections shall be the *most probable* ones for that system of weighting. The most probable set of corrections is that set the sum of whose squares is a minimum. If the system includes more than a few polygons, this method of reduction is exceedingly laborious, while the increased accuracy is very small over that from a much simpler method.

Fig. 150 represents the Bavarian network of geodetic levels, there being four polygons. Every side has been levelled, and the difference of elevation of its extremities found. These elevations must now be adjusted so that the differences of elevation on each polygon shall sum up zero. When these sums are taken the following residuals are found: I., $+20.2^{mm}$; II., $+39.3^{mm}$; III., -25.2^{mm}; and IV., $+108.0^{mm}$. It was supposed that an error of one decimetre had been made in the fourth polygon, but in the absence of any knowledge in the case this error must be distributed with the rest.

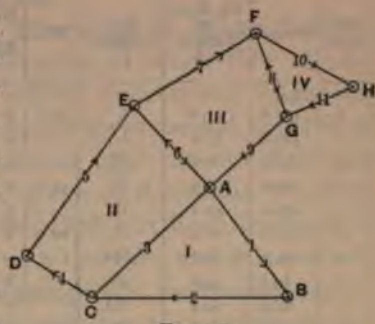


FIG. 150.

The method which the author would recommend is a modification of Bauernfeind's, in that the errors are to be made proportional to the square roots of the lengths of the sides instead of the lengths of the sides directly. Since the errors in levelling are compensating in their nature they would be expected to increase with the square root of the length of the line, and it is the author's experience that the error is much nearer proportional to the square root of the distance than to the distance itself.

Instead of treating the four polygons as one system and solving by least squares, the polygon *having the largest error of closure* is first adjusted by distributing the error among its sides in proportion to the square roots of the lengths of those sides. Then the polygon having the next largest error is adjusted, using the new value for the adjusted side, if it is contiguous to the former one, and distributing the remaining error among the remaining sides of the figure, leaving the previously adjusted side undisturbed. The adjustment proceeds in this manner until all the polygons are adjusted. The Bavarian system is worked out on this plan in the following tabulated form:

ADJUSTMENT OF THE BAVARIAN SYSTEM OF LEVEL POLYGONS.

No. Side.	Length.	Sq. Root of Length $= \lambda$.	No. Polygon.	$\Sigma\lambda$.	Difference of Elevation.	Error of Closure	Corrected Error of Closure	Correction.	Corrected Difference of Elevation.
	km.				m.	mm.			
1	125.8	11.2	I.	24.6	+ 35.8723	+ 20.2	+ 31.3	− 14.3	+ 35.8580
2	179.0	13.4	I.		− 217.5062			− 17.0	− 217.5232
3	147.3	12.1	II.		± 181.6541	+ 39.3	+ 39.3	− 11.1	± 181.6652
4	60.6	7.8	II.	43.1	+ 32.0958			− 7.1	+ 32.0887
5	174.0	13.2	II.		+ 179.5981			− 12.0	+ 179.5861
6	101.1	10.0	II.	20.9	∓ 30.0005	− 25.2	+ 19.9	− 9.1	∓ 30.0096
7	134.9	11.6	III.		− 38.6644			− 11.0	− 38.6754
8	80.1	9.0	IV.		∓ 48.8053			− 36.0	± 48.7693
9	87.0	9.3	III.		+ 57.4440			− 8.9	+ 57.4351
10	96.8	9.8	IV.	27.0	− 100.1619	+ 108.0	+ 108.0	− 39.2	− 100.2011
11	67.9	8.2	IV.		+ 51.4646			− 32.8	+ 51.4318

Beginning with polygon IV., we find its error of closure to be $+ 108.0^{mm}$, this being distributed among the three sides so that $\frac{90}{270}$ goes to side 8, $\frac{98}{270}$ to side 10, and $\frac{82}{270}$ to side 11. The corrected values for these sides are now found. Next take the polygon having the next largest error of closure, which is number II., and distribute its error in like manner. This leaves polygons I. and III. to be adjusted, one side of the former and two of the latter being already adjusted. The corrected errors of closure for these polygons are 31.3^{mm} and 19.9^{mm} respectively, the former to be distributed between the sides 1 and 2 and the latter between the sides 7 and 9. The resulting corrected values cause all the polygons to sum up zero.

The sum of the squares of the corrections here found is 50.02 square centimetres, whereas if the differences of elevation had been weighted in proportion to the lengths of the sides and the system adjusted rigidly by least squares the sum of the squares of the corrections would have been 49.65 square centimetres, showing that the method here used is practically as good as the rigid method which is commonly used. It has been found in practice to give, in general, about the same sized corrections as the rigid system.

408. Determination of the Elevation of Mean Tide.—To determine accurately the elevation of mean tide at any point on the coast requires continuous observations by means of an automatic self-registering gauge for a period of several years. The methods of making these observations with cuts of the instruments employed are given in Appendix No. 8 of the U. S. Coast Survey Report for 1876. A float, inclosed in a perforated box, rises and falls with the tide, and this motion, properly reduced in scale by appropriate mechanism, is recorded by a pencil on a continuous roll of paper which is moved over a drum at a uniform rate by means of clockwork. An outer staff-gauge is read one or more times a day by the attendant, who records the height of the water and the time of the observation on the continuous roll. This outer staff is connected with fixed bench-marks in the locality by very careful levelling, and this connection is repeated at intervals to test the stability of the gauge.

To find from this automatic record the height of mean tide, ordinates are measured from the datum-line of the sheet to the tide-curve for each hour of the day throughout the entire period. This period should be a certain number of entire lunar months. The mean of all the hourly readings for the period may be taken as mean tide. It may be found advisable to reject all readings in stormy weather, in which case the entire lunation should be rejected.

CHAPTER XV.

PROJECTION OF MAPS, MAP-LETTERING, AND TOPOGRAPHICAL SYMBOLS.

I. PROJECTION OF MAPS.

409. THE particular method that should be employed in representing portions of the earth's surface on a plane sheet or map depends, *first*, on the extent of the region to be represented; *second*, on the use to be made of the map or chart; and *third*, on the degree of accuracy desired.

Thus, a given kind of projection may suffice for a small region, but the approximation may become too inaccurate when extended over a large area. It is quite impossible to represent a spherical surface on a plane without sacrificing something in the accuracy of the relative distances, courses, or areas; and the use to which the chart is to be put must determine which of these conditions should be fulfilled at the expense of the others. A great many methods have been proposed and used for accomplishing various ends, some of which will be described.

410. Rectangular Projection.—In this method the meridians are all drawn as straight parallel lines; and the parallels are also straight, and at right angles with the meridians. A central meridian is drawn, and divided into minutes of latitude according to the value of these at that latitude as given in Table XI. Through these points of division draw the parallels of latitude as right lines perpendicular to the central meridian. On the central parallel lay off the minutes of longitude, according to their value for the given latitude, by Table XI.; and through these points of division draw the other meridians parallel with the first.

The largest error here is in assuming the meridians to be parallel. For the latitude of 40°, two meridians a mile apart will converge at the rate of about a foot per mile. A knowledge of this fact will enable the draughtsman to decide when this method is sufficiently accurate for his purpose. Thus, for an area of ten miles square, the distortion at the extreme corners in longitude, with reference to the centre of the map as an origin of coördinates, will be about twenty-five feet. At the equator this method is strictly correct.

In this kind of projection, whether plotted from polar or rectangular coördinates, or from latitudes and longitudes, all straight lines of the survey, whether determined by triangulation or run out by a transit on the ground, will be straight on the map; that is, the fore and back azimuth of a line is the same, or, in other words, a straight line on the drawing gives a constant angle with all the meridians.

This is the method to use on field-sheets, where the survey has all been referred to a single meridian.

411. Trapezoidal Projection.—Here the meridians are made to converge properly, but both they and the parallels are straight lines. A central meridian is first drawn, and graduated to degrees or minutes; and through these points parallels are drawn, as before. Two of these parallels are selected; one about one fourth the height of the map from the bottom, and the other the same distance from the top. These parallels are then subdivided, according to their respective latitudes, from Table XI.; and through the corresponding points of division the remaining meridians are drawn as straight lines. The map is thus divided into a series of trapezoids. The parallels are perpendicular to but one of the meridians. The principal distortion comes from the parallels being drawn as straight lines, and amounts to about thirty-two feet in ten miles in latitude 40°, and is nearly proportional to the square of the distance east or west from the central meridian.

The work should be plotted from computed latitudes and longitudes. The method is adapted to a scheme which has a system of triangulation for its basis, the geodetic position of the stations having been determined. These conditions would be fulfilled in a State topographical or geological survey for the separate sheets, each sheet covering an area of not more than twenty-five miles square.

412. The Simple Conic Projection.—In this projection, points on a spherical surface are first projected upon the surface of a tangent cone, and then this conical surface is developed into the plane of the map. The apex of the cone is taken in the extended axis of the earth, at such an altitude that the cone becomes tangent to the earth's surface at the middle parallel of the map. When this conical surface is developed into a plane, the meridians are straight lines converging to the apex of the cone, and the parallels are arcs of concentric circles about the apex as the common centre.

The sheet is laid out as follows: Draw a central meridian, and graduate it to degrees or minutes, according to their true values as given in Table XI. Through these points of division draw parallel circular arcs, using the apex of the cone as the common centre. For values of the length of the side of the tangent cone, which is the length of the central parallel above, see Table XI. The rectangular coördinates of points in these curves are also given in the same table.

On the middle parallel of the map the degrees or minutes of longitude are laid off, and through these are drawn the remaining meridians as straight lines radiating from the apex of the tangent cone.

It will be seen that the latitudes are correctly laid off, and the degrees of longitude will be sufficiently accurate for a map covering an area of several hundred miles square. The meridians and parallels are at right angles.

In this projection the degrees of longitude on all parallels, except the middle one, are too great; and therefore the area represented on the map is greater than the corresponding area on the sphere.

The chart should be plotted from computed latitudes and longitudes.

413. De l'Isle's Conic Projection.—This is very similar to the above, except that two parallels, one at one fourth, and one at three fourths the height of the map, are properly graduated, and the meridians drawn as straight lines through these points of division. The parallels are drawn as concentric circles, as in the simple conic projection. This is therefore but a combination of the second and third methods, and is more accurate than either of them. The cone here is no longer tangent, but intersects the sphere in the two graduated parallels. In this case the region between the parallels of intersection is shown too small, and that outside these lines is shown too large; so that the area of the whole map will correspond very closely to the corresponding area on the sphere. When these parallels are so selected that these areas will be to each other exactly as the scale of the drawing, then it is called "Murdoch's projection."

414. Bonne's Projection.—This differs from the simple conic in this—that all the parallels are properly graduated, and the meridians drawn to connect the corresponding points of division in the parallels. These latter are, however, still concentric circles. The meridians are at right angles to the parallels only in the middle portion of the map. The same scale applies to all parts of the chart. There is a slight distortion at the extreme corners, from the parallels being arcs of concentric circles. The proportionate equality of areas is preserved. A rhumb-line appears as a curve; but when once drawn, its length may be properly scaled.

It will be noted that the last three methods involve the use of but one tangent or intersecting cone.

415. The Polyconic Projection.—For very large areas it is preferable to have each parallel the development of the base of a cone tangent in the plane of the given parallel. This projection differs from Bonne's only in the fact that the parallels are no longer concentric arcs, but each is drawn with a radius equal to the side of the cone which is tangent at that latitude. These, of course, decrease towards the pole; and therefore the parallels diverge from each other towards the edge of the chart. The result of this is, that a degree of latitude at the side of the map is not equal to a degree on the central meridian; or, in other words, the same scale cannot be applied to all parts of the map. These defects appear, however, only on maps representing very large areas. The whole of North America could be represented by this method without any material distortion.

This method of projection is exclusively used on the United States Coast and Geodetic Survey, and for all other maps and charts of large areas in this country. Extensive tables are published by the War and Navy Departments, and also by the Coast Survey, to facilitate the projection of maps by the polyconic system. Table VIII. gives in a condensed form the rectangular coördinates of the points of intersection of the parallels and meridians referred to the intersection of the several parallels with the central meridian as the several origins.

416. Formulæ used in the Projection of Maps.*—The fundamental relations on which the method of polyconic projection rests are given in the following formulæ:

* See Appendix D for the derivation of equations (1) and (2).

Normal, being the radius of curvature of a section perpendicular to the meridian at a given point........$N = \dfrac{R_e}{(1 - e^2 \sin^2 L)^{\frac{1}{2}}}$, (1)

where R_e is the equational radius,
e is the eccentricity,
and L is the latitude.

Radius of the meridian............$R_m = N^3 \dfrac{(1 - e^2)}{R_e^2}$. . (2)

Radius of the parallel.. ....$R_p = N \cos L$. . . (3)

Degree of the meridian............$D_m = \dfrac{\pi}{180} R_m$ . . . (4)

$= 3600 R_m \sin 1''$.

Degree of the parallel.............$D_p = \dfrac{\pi}{180} R_p$ . . . (5)

$= 3600 R_p \sin 1''$.

Radius of the developed parallel, or side of tangent cone............$r = N \cot L$.. . . (6)

If n be any arc of a parallel, in degrees, or any difference of longitude from the central meridian of the drawing, and if θ be the corresponding angle, in degrees, at the vertex of the tangent cone, subtended by the developed parallel, then since the angular value of arcs of given lengths are inversely as their radii, we have

$$\frac{\theta}{n} = \frac{R_p}{r} = \sin L,$$

or $$\theta = n \sin L. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

Since the developed parallels are circular arcs, the rectangular coördinates of any point an angular distance of θ from the central meridian is,

$$\left.\begin{array}{lll} \text{Meridian distance,} & d_m & = x = r \sin \theta. \\ \text{Divergence of parallels,} & d_p & = y = r \text{ vers } \theta. \\ & & = x \tan \tfrac{1}{2}\theta. \end{array}\right\} \quad . \ . \quad (9)$$

For arcs of small extent, the parallel may be considered coincident with its chord; but the angle between the axis of x and the chord is $\frac{1}{2}\theta$. If, then, the length of the arc, which is nD_p, be represented by the chord, we may write

$$\left.\begin{array}{lll} & d_m = \text{meridian distance} & = x = nD_p \cos \tfrac{1}{2}\theta, \\ \text{and} & d_p = \text{divergence of parallels} & = y = nD_p \sin \tfrac{1}{2}\theta. \end{array}\right\} \quad (10)$$

If, now, d_{m1} = meridian distance for 1 degree of longitude,
and d_{mn} = meridian distance for n degrees of longitude,

we have
$$\frac{d_{mn}}{d_{m1}} = \frac{nD_p \cos \frac{1}{2}\theta_n}{D_p \cos \frac{1}{2}\theta_1}.$$

But $\theta = n \sin L$, so that $\theta_1 = 1° \times \sin L = 38'$ for latitude 40°. Therefore

$$\cos \tfrac{1}{2}\theta_1 = \cos 19' = 1, \text{ nearly;}$$

so that
$$\frac{d_{mn}}{d_{m1}} = n \cos \tfrac{1}{2}(n \sin L), \text{ nearly.} \quad . \ . \ . \ . \quad (11)$$

For $L = 30°$, we have $\sin L = \frac{1}{2}$. Therefore, for latitude 30°,

$$\frac{d_{mn}}{d_{m1}} = n \cos \tfrac{1}{4}n = n \cos (0.25n), \text{ nearly.}$$

If we have obtained the meridian distance, d_m, for 1 degree of longitude, and wish to obtain it for n degrees in latitude 30°, we have but to multiply the distance for 1 degree by $n \cos (0.25n)$.

417. In Table VIII. the meridian distances are given, at various latitudes, for a difference of longitude of one degree. To find the meridian distance for any number of degrees or parts of degrees, multiply the distance for one degree by the factor there given for the given latitude. The factor given in the table for latitude 30° is $n \cos (0.288n)$, in place of $n \cos (0.25n)$, as obtained above. The difference is a correction which has been introduced to compensate the error made in assuming that the chord was equal in length to its arc. The corrected factors enable the table to be used without material error up to 25 degrees longitude either side of the central meridian.

To obtain the divergence of the parallels for differences of longitude more or less than one degree, multiply the divergence for one degree by the square of the number of degrees. It is evident that this rule is based on the assumption that the arc of the developed parallel is a parabola, and so it may be considered for a distance of 25 degrees either side of the central meridian between the latitudes 30° and 50° without material error.

If the whole of the United States were projected by this table, using the factors given, to a scale of 1 to 1,500,000, thus giving a map some 8 by 10 feet, the maximum deviation of the meridians and parallels from their true positions (which would be at the upper corners) would be but about 0.02 inch.

Thus, for a map of this size, covering 20 degrees of latitude and 50 degrees of longitude, the geodetic lines would have their true position within the width of a fine pencil line, by the use of Table VIII. Fig. 151 will illustrate the use of the table in projecting a map by the polyconic method.

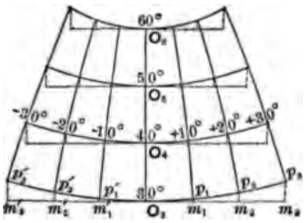


FIG. 151.

The map covers 30 degrees in latitude (30° to 60°) and 60 degrees in longitude. The straight line O_3O_6 is first drawn through the centre of the map, and graduated according to the lengths of one degree of latitude, as given in the second column of Table VIII. Through these points of division the lines $m'_3 m_3$, are drawn in pencil at right angles to the central meridian. On these lines the points m_1, m_2, etc., are laid off by the aid of the first part of Table VIII. This table gives the meridian distances when n is less than one degree, as well as when n is greater. From the points m_1, m_2, etc., the divergence of the parallels is laid off above the lines Om, by the aid of the second portion of Table VIII., thus obtaining the positions of the points p_1, p_2, etc. The points p mark the intersection of the meridians and parallels; and these may be drawn as straight lines between these points, provided a sufficient number of such points have been located. The map is then to be plotted upon the chart from computed latitudes and longitudes.

418. Summary.—We have seen that there are, in general, two ways of plotting a map or chart, and two corresponding uses to which it may be put:

First. We may plot by a system of plane coördinates, either polar (azimuth and distance) or rectangular (latitudes and departures). This gives a map from which distance, azimuth (referred to the meridian of the map), and areas are correctly determined.

Second. We may plot the map by computed latitudes and longitudes, and determine from it the relative position of points in terms of their latitude and longitude.

The first system is adapted to small field sheets and detail charts for which the notes were taken by referring all points to a single point and meridian. For this purpose the system of rectangular projection should be selected, as long as the area of the chart is not more than about one hundred square miles. If it be larger than this, the trapezoidal system should be used. In case this is done, the work is still plotted as before, provided it has all been referred to a given meridian in the field work, and then converging meridians are drawn as described above. From such a chart, not only the azimuth (referred to the central meridian) and distance may be determined, but the correct longitude and nearly correct latitude are given.

In the case of topographical charts, based on a system of triangulation, each sheet is referred to a meridian passing through a triangulation-station on that sheet, or near to it, and the rectangular system used.

In the case of a survey of a long and narrow belt, as for a river, railroad, or canal, if the survey was based on a system of triangulation, the convergence of meridians has been looked after in the computation of the geodetic positions of these stations, and each sheet is plotted by the rectangular system, being referred to the meridian through the adjacent triangulation-station. When many of these are combined into a single map on a small scale, then they must be plotted on the condensed map by latitudes and longitudes, these being taken from the small rectangular projections, and plotted on the reduced chart in polyconic projection; the meridians and parallels having been laid out as shown above.

In case the belt extends mostly east and west, and is not based on a triangulation scheme, then observations for azimuth should be made as often as every fifty miles. It will not do to run on a given azimuth for this distance, however; for there has been a change in the direction of the parallel (or meridian) in this distance, in latitude 40°, of about 40 minutes. According to the accuracy with which the work is done, therefore, when running wholly by back azimuths, the setting of the instrument must be changed. Thus, if in going 1 degree (53 miles), east or west, in latitude 40°, the meridian has shifted 40′, then in going 13 miles east or west the meridian has changed 10′; and this is surely a sufficiently large correction to make it worth while to apply it.

When running west, this correction is applied in the direction of the hands of a watch, and when running east, in the opposite direction; that is, having run west 13 miles by back azimuth, then the pointing which appears north is really 10′ west of north, and the telescope must be shifted 10′ around to the right.

If the azimuth be corrected in this way, a survey can be carried by back azimuth an indefinite distance, and still have the entire survey referred to the true meridian.

419. **The Angle of Convergence of Meridians** is the angle θ in the equations given in the above formula. Then

$$\theta = n \sin L,*$$

where n is the angular change in degrees of longitude, and L is the latitude of the place.

For $L = 30°$, $\sin L = \frac{1}{2}$; or, in latitude 30° a change of longitude of one degree changes the direction of the meridian by 30 minutes.

For $L = 40°$, $\sin L = 0.643$; or, a change of longitude of one degree changes the direction of the meridian by 0.643 of 60 minutes, or 38.6 minutes, being approximately 40 minutes.

For $L = 50°$, $\sin L = 0.766$; or, in going east or west one

* From Eq. (G), p. 643, when $\cos \frac{1}{2} \Delta L$ is taken as unity.

degree, the meridian changes 0.766×60 minutes $= 46$ minutes, or approximately 50 minutes.

Therefore we may have the approximate rule, that a change of longitude of one degree changes the azimuth by as many minutes as equals the degrees of latitude of the place. This rule gives results very near the truth between plus and minus 40° latitude, that is, over an equatorial belt 80 degrees in width.

II. MAP-LETTERING AND TOPOGRAPHICAL SYMBOLS.

420. Map-Lettering.—The best-drawn map may have its appearance ruined by the poor skill or bad taste displayed in the lettering. The letters should be simple, neat, and dignified in appearance, and have their size properly proportioned to the subject. The map is lettered before the topographical symbols are drawn. When a map is drawn for popular display, some ornamentation may be allowed in the title; but even then, the lettering on the map itself should be plain and simple. When the map is for official or professional use, even the title should be made plain.

On Plate IV. are given several sets of alphabets which are well adapted to map work. Of course the size should vary according to the scale of the map and the subject, as shown on Plate V. It is a good rule to make all words connected with water in italics. The small letters in stump writing will be found very useful, and these should be practised thoroughly. The italic capitals go with these small letters also.

In place of the system of letters above described, and which has heretofore been almost exclusively used for mapping purposes, a new system, called "round writing," may be used. A text-book on this method, by F. Soennecken, is published by Messrs. Kueffel & Esser, New York. This work is done with blunt pens, all lines being made with a single stroke.

APPENDICES.

APPENDIX A.

THE JUDICIAL FUNCTIONS OF SURVEYORS.

BY JUSTICE COOLEY OF THE MICHIGAN SUPREME COURT.

WHEN a man has had a training in one of the exact sciences, where every problem within its purview is supposed to be susceptible of accurate solution, he is likely to be not a little impatient when he is told that, under some circumstances, he must recognize inaccuracies, and govern his action by facts which lead him away from the results which theoretically he ought to reach. Observation warrants us in saying that this remark may frequently be made of surveyors.

In the State of Michigan all our lands are supposed to have been surveyed once or more, and permanent monuments fixed to determine the boundaries of those who should become proprietors. The United States, as original owner, caused them all to be surveyed once by sworn officers, and as the plan of subdivision was simple, and was uniform over a large extent of territory, there should have been, with due care, few or no mistakes; and long rows of monuments should have been perfect guides to the place of any one that chanced to be missing. The truth unfortunately is that the lines were very carelessly run, the monuments inaccurately placed; and, as the recorded witnesses to these were many times wanting in permanency, it is often the case that when the monument was not correctly placed it is impossible to determine by the record, with the aid of anything on the ground, where it was located. The incorrect record of course becomes worse than useless when the witnesses it refers to have disappeared.

It is, perhaps, generally supposed that our town plats were more accurately surveyed, as indeed they should have been, for in general there can have been no difficulty in making them sufficiently perfect for all practical purposes. Many of them, however, were laid out in the woods; some of them by proprietors themselves, without either chain or compass, and some by imperfectly trained surveyors, who, when land was cheap, did not appreciate the importance of having correct lines to determine boundaries when land should become dear. The fact probably is that town surveys are quite as inaccurate as those made under authority of the general government.

It is now upwards of fifty years since a major part of the public surveys in what is now the State of Michigan were made under authority of the United States. Of the lands south of Lansing, it is now forty years since the major part were sold and the work of improvement begun. A generation has passed away since they were converted into cultivated farms, and few if any of the original corner and quarter stakes now remain.

The corner and quarter stakes were often nothing but green sticks driven into the ground. Stones might be put around or over these if they were handy, but often they were not, and the witness trees must be relied upon after the stake was gone. Too often the first settlers were careless in fixing their lines with accuracy while monuments remained, and an irregular brush fence, or something equally untrustworthy, may have been relied upon to keep in mind where the blazed line once was. A fire running through this might sweep it away, and if nothing were substituted in its place, the adjoining proprietors might in a few years be found disputing over their lines, and perhaps rushing into litigation, as soon as they had occasion to cultivate the land along the boundary.

If now the disputing parties call in a surveyor, it is not likely that any one summoned would doubt or question that his duty was to find, if possible, the place of the original stakes which determined the boundary line between the proprietors. However erroneous may have been the original survey, the monuments that were set must nevertheless govern, even though the effect be to make one half-quarter section ninety acres and the one adjoining but seventy; for parties buy or are supposed to buy in reference to those monuments, and are entitled to what is within their lines, and no more, be it more or less. *McIver v. Walker*, 4 Wheaton's Reports, 444; *Land Co. v. Saunders*, 103 U. S. Reports, 316; *Cottingham v. Parr*, 93 Ill. Reports, 233; *Bunton v. Cardwell*, 53 Texas Reports, 408; *Watson v. Jones*, 85 Penn. Reports, 117.

While the witness trees remain there can generally be no difficulty in determining the locality of the stakes. When the witness trees are gone, so that there is no longer record evidence of the monuments, it is remarkable how many there are who mistake altogether the duty that now devolves upon the surveyor. It is by no means uncommon that we find men whose theoretical education is supposed to make them experts who think that when the monuments are gone, the only thing to be done is to place new monuments where the old ones should have been, and where they would have been if placed correctly. This is a serious mistake. The problem is now the same that it was before: to ascertain, by the best lights of which the case admits, where the original lines were. The mistake above alluded to is supposed to have found expression in our legislation; though it is possible that the real intent of the act to which we shall refer is not what is commonly supposed.

An act passed in 1869, Compiled Laws, § 593, amending the laws respecting the duties and powers of county surveyors, after providing for the case of corners which can be identified by the original field-notes or other unquestionable testimony, directs as follows:

"*Second.* Extinct interior section-corners must be re-established at the intersection of two right lines joining the nearest known points on the original section lines east and west and north and south of it.

"*Third.* Any extinct quarter-section corner, except on fractional lines, must be re-established equidistant and in a right line between the section corners; in all other cases at its proportionate distance between the nearest original corners on the same line."

The corners thus determined, the surveyors are required to perpetuate by noting bearing trees when timber is near.

To estimate properly this legislation, we must start with the admitted and unquestionable fact that each purchaser from government bought such land as was within the original boundaries, and unquestionably owned it up to the time when the monuments became extinct. If the monument was set for an interior-section corner, but did not happen to be "at the intersection of two right lines joining the nearest known points on the original section lines east and west and north and south of it," it nevertheless determined the extent of his possessions, and he gained or lost according as the mistake did or did not favor him.

It will probably be admitted that no man loses title to his land or any part thereof merely because the evidences become lost or uncertain. It may become more difficult for him to establish it as against an adverse claimant, but theoretically the right remains; and it remains as a potential fact so long as he can present better evidence than any other person. And it may often happen that, notwithstanding the loss of all trace of a section corner or quarter stake, there will still be evidence from which any surveyor will be able to determine with almost absolute certainty where the original boundary was between the government subdivisions.

There are two senses in which the word extinct may be used in this connection: one the sense of physical disappearance; the other the sense of loss of all reliable evidence. If the statute speaks of extinct corners in the former sense, it is plain that a serious mistake was made in supposing that surveyors could be clothed with authority to establish new corners by an arbitrary rule in such cases. As well might the statute declare that if a man lose his deed he shall lose his land altogether.

But if by extinct corner is meant one in respect to the actual location of which all reliable evidence is lost, then the following remarks are pertinent:

1. There would undoubtedly be a presumption in such a case that the corner was correctly fixed by the government surveyor where the field-notes indicated it to be.

2. But this is only a presumption, and may be overcome by any satisfactory evidence showing that in fact it was placed elsewhere.

3. No statute can confer upon a county surveyor the power to "establish" corners, and thereby bind the parties concerned. Nor is this a question merely of conflict between State and Federal law; it is a question of property right. The original surveys must govern, and the laws under which they were made must govern, because the land was bought in reference to them; and any legislation, whether State or Federal, that should have the effect to change these, would be inoperative, because disturbing vested rights.

4. In any case of disputed lines, unless the parties concerned settle the controversy by agreement, the determination of it is necessarily a

judicial act, and it must proceed upon evidence, and give full opportunity for a hearing. No arbitrary rules of survey or of evidence can be laid down whereby it can be adjudged.

The general duty of a surveyor in such a case is plain enough. He is not to assume that a monument is lost until after he has thoroughly sifted the evidence and found himself unable to trace it. Even then he should hesitate long before doing anything to the disturbance of settled possessions. Occupation, especially if long continued, often affords very satisfactory evidence of the original boundary when no other is attainable; and the surveyor should inquire when it originated, how, and why the lines were then located as they were, and whether a claim of title has always accompanied the possession, and give all the facts due force as evidence. Unfortunately, it is known that surveyors sometimes, in supposed obedience to the State statute, disregard all evidences of occupation and claim of title, and plunge whole neighborhoods into quarrels and litigation by assuming to "establish" corners at points with which the previous occupation cannot harmonize. It is often the case that where one or more corners are found to be extinct, all parties concerned have acquiesced in lines which were traced by the guidance of some other corner or landmark, which may or may not have been trustworthy; but to bring these lines into discredit when the people concerned do not question them not only breeds trouble in the neighborhood, but it must often subject the surveyor himself to annoyance and perhaps discredit, since in a legal controversy the law as well as common-sense must declare that a supposed boundary line long acquiesced in is better evidence of where the real line should be than any survey made after the original monuments have disappeared. *Stewart vs. Carleton*, 31 Mich. Reports, 270; *Diehl vs. Zanger*, 39 Mich. Reports, 601; *Dupont vs. Starring*, 42 Mich. Reports, 492. And county surveyors, no more than any others, can conclude parties by their surveys.

The mischiefs of overlooking the facts of possession must often appear in cities and villages. In towns the block and lot stakes soon disappear; there are no witness trees and no monuments to govern except such as have been put in their places, or where their places were supposed to be. The streets are likely to be soon marked off by fences, and the lots in a block will be measured off from these, without looking farther. Now it may perhaps be known in a particular case that a certain monument still remaining was the starting-point in the original survey of the town plat; or a surveyor settling in the town may take some central point as the point of departure in his surveys, and assuming the original plat to be accurate, he will then undertake to find all streets and all lots by course and distance according to the plat, measuring and estimating from his point of departure. This procedure might unsettle every line and every monument existing by acquiescence in the town; it would be very likely to change the lines of streets, and raise controversies everywhere. Yet this is what is sometimes done; the surveyor himself being the first person to raise the disturbing questions.

Suppose, for example, a particular village street has been located by acquiescence and use for many years, and the proprietors in a certain block have laid off their lots in reference to this practical location. Two lot-owners quarrel, and one of them calls in a surveyor that he may be sure that his neighbor shall not get an inch of land from him. This surveyor undertakes to make his survey accurate, whether the original was, or not, and the first result is, he notifies the lot-owners that there is error in the street line, and that all fences should be moved, say, one foot to the east. Perhaps he goes on to drive stakes through the block according to this conclusion. Of course, if he is right in doing this, all lines in the village will be unsettled; but we will limit our attention to the single block. It is not likely that the lot-owners generally will allow the new survey to unsettle their possessions, but there is always a probability of finding some one disposed to do so. We shall then have a lawsuit; and with what result?

It is a common error that lines do not become fixed by acquiescence in a less time than twenty years. In fact, by statute, road lines may become conclusively fixed in ten years; and there is no particular time that shall be required to conclude private owners, where it appears that they have accepted a particular line as their boundary, and all concerned have cultivated and claimed up to it. *McNamara vs. Seaton*, 82 Ill. Reports, 498; *Bunce vs. Bidwell*, 43 Mich. Reports, 542. Public policy requires that such lines be not lightly disturbed, or disturbed at all after the lapse of any considerable time. The litigant, therefore, who in such a case pins his faith on the surveyor, is likely to suffer for his reliance, and the surveyor himself to be mortified by a result that seems to impeach his judgment.

Of course nothing in what has been said can require a surveyor to conceal his own judgment, or to report the facts one way when he believes them to be another. He has no right to mislead, and he may rightfully express his opinion that an original monument was at one place, when at the same time he is satisfied that acquiescence has fixed the rights of parties as if it were at another. But he would do mischief if he were to attempt to "establish" monuments which he knew would tend to disturb settled rights; the farthest he has a right to go, as an officer of the law, is to express his opinion where the monument should be, at the same time that he imparts the information to those who employ him, and who might otherwise be misled, that the same authority that makes him an officer and entrusts him to make surveys, also allows parties to settle their own boundary lines, and considers acquiescence in a particular line or monument, for any considerable period, as strong, if not conclusive, evidence of such settlement. The peace of the community absolutely requires this rule. *Joyce vs. Williams*, 26 Mich. Reports, 332. It is not long since that, in one of the leading cities of the State, an attempt was made to move houses two or three rods into a street, on the ground that a survey under which the street had been located for many years had been found on more recent survey to be erroneous.

From the foregoing it will appear that the duty of the surveyor where boundaries are in dispute must be varied by the circumstances. 1. He is to search for original monuments, or for the places where they were originally located, and allow these to control if he finds them, unless he has reason to believe that agreements of the parties, express or implied, have rendered them unimportant. By monuments in the case of government surveys we mean of course the corner and quarter stakes: blazed lines or marked trees on the lines are not monuments; they are merely guides or finger-posts, if we may use the expression, to inform us with more or less accuracy where the monuments may be found. 2. If the original monuments are no longer discoverable, the question of location becomes one of evidence merely. It is merely idle for any State statute to direct a surveyor to locate or "establish" a corner, as the place of the original monument, according to some inflexible rule. The surveyor on the other hand must inquire into all the facts; giving due prominence to the acts of parties concerned, and always keeping in mind, *first*, that neither his opinion nor his survey can be conclusive upon parties concerned; *second*, that courts and juries may be required to follow after the surveyor over the same ground, and that it is exceedingly desirable that he govern his action by the same lights and rules that will govern theirs. On town plats if a surplus or deficiency appears in a block, when the actual boundaries are compared with the original figures, and there is no evidence to fix the exact location of the stakes which marked the division into lots, the rule of common-sense and of law is that the surplus or deficiency is to be apportioned between the lots, on an assumption that the error extended alike to all parts of the block. *O'Brien vs. McGrane*, 29 Wis. Reports, 446; *Quinnin vs. Reixers*, 46 Mich. Reports, 605.

It is always possible when corners are extinct that the surveyor may usefully act as a mediator between parties, and assist in preventing legal controversies by settling doubtful lines. Unless he is made for this purpose an arbitrator by legal submission, the parties, of course, even if they consent to follow his judgment, cannot, on the basis of mere consent, be compelled to do so; but if he brings about an agreement, and they carry it into effect by actually conforming their occupation to his lines, the action will conclude them. Of course it is desirable that all such agreements be reduced to writing; but this is not absolutely indispensable if they are carried into effect without.

Meander Lines.—The subject to which allusion will now be made is taken up with some reluctance, because it is believed the general rules are familiar. Nevertheless it is often found that surveyors misapprehend them, or err in their application; and as other interesting topics are somewhat connected with this, a little time devoted to it will probably not be altogether lost. The subject is that of meander lines. These are lines traced along the shores of lakes, ponds, and considerable rivers as the measures of quantity when sections are made fractional by such waters. These have determined the price to be paid when government lands were bought, and perhaps the impression still lingers in some minds that the meander lines are boundary lines, and all in front of them remains unsold. Of course this is erroneous. There was never any doubt that, except on the large navigable rivers, the boundary of the owners of the banks is the middle line of the river; and while some courts have held that this was the rule on all fresh-water streams, large and small, others have held to the doctrine that the title to the bed of the stream below low-water mark is in the State, while conceding to the owners of the banks all riparian rights. The practical difference is not very important. In this State the rule that the centre line is the boundary line is applied to all our great rivers, including the Detroit, varied somewhat by the circumstance of there being a distinct channel for navigation in some cases with the stream in the main shallow, and also sometimes by the existence of islands.

The troublesome questions for surveyors present themselves when the boundary line between two contiguous estates is to be continued from the meander line to the centre line of the river. Of course the original survey supposes that each purchaser of land on the stream has a water-front of the length shown by the field-notes; and it is presumable that he bought this particular land because of that fact. In many cases it now happens that the meander line is left some distance from the shore by the gradual change of course of the stream or diminution of the flow of water. Now the dividing line between two government subdivisions might strike the meander line at right angles, or obliquely; and in some cases, if it were continued in the same direction to the centre line of the river, might cut off from the water one of the subdivisions entirely, or at least cut it off from any privilege of navigation, or other valuable use of the water, while the other might have a water-front much greater than the length of a line crossing it at right angles to its side lines. The effect might be that, of two government subdivisions of equal size and cost, one would be of very great value as water-front property, and the other comparatively valueless. A rule which would produce this result would not be just, and it has not been recognized in the law.

Nevertheless it is not easy to determine what ought to be the correct rule for every case. If the river has a straight course, or one nearly so, every man's equities will be preserved by this rule: Extend the line of division between the two parcels from the meander line to the centre line of the river, as nearly as possible at right angles to the general course of the river at that point. This will preserve to each man the water front which the field-notes indicated, except as changes in the water may have affected it, and the only inconvenience will be that the division line between different subdivisions is likely to be more or less deflected where it strikes the meander line.

This is the legal rule, and it is not limited to government surveys, but applies as well to water lots which appear as such on town plats. *Bay City Gas Light Co. v. The Industrial Works*, 28 Mich. Reports, 182. It often happens, therefore, that the lines of city lots bounded on navigable streams are deflected as they strike the bank, or the line where the bank was when the town was first laid out.

When the stream is very crooked, and especially if there are short bends, so that the foregoing rule is incapable of strict application, it is sometimes very difficult to determine what shall be done; and in many cases the surveyor may be under the necessity of working out a rule for himself. Of course his action cannot be conclusive; but if he adopts one that follows, as nearly as the circumstances will admit, the general rule above indicated, so as to divide as near as may be the bed of the stream among the adjoining owners in proportion to their lines upon the shore, his division, being that of an expert, made upon the ground and with all available lights, is likely to be adopted as law for the case. Judicial decisions, into which the surveyor would find it prudent to look under such circumstances, will throw light upon his duties and may constitute a sufficient guide when peculiar cases arise. Each riparian lot-owner ought to have a line on the legal boundary, namely, the centre line of the stream, proportioned to the length of his line on the shore; and the problem in each case is, how this is to be given him. Alluvion, when a river imperceptibly changes its course, will be apportioned by the same rules.

The existence of islands in a stream, when the middle line constitutes a boundary, will not affect the apportionment unless the islands were surveyed out as government subdivisions in the original admeasurement. Wherever that was the case, the purchaser of the island divides the bed of the stream on each side with the owner of the bank, and his rights also extend above and below the solid ground, and are limited by the peculiarities of the bed and the channel. If an island was not surveyed as a government subdivision previous to the sale of the bank, it is of course impossible to do this for the purposes of government sale afterwards, for the reason that the rights of the bank owners are fixed by their purchase: when making that, they have a right to understand that all land between the meander lines, not separately surveyed and sold, will pass with the shore in the government sale; and having this right, anything which their purchase would include under it cannot afterward be taken from them. It is believed, however, that the federal courts would not recognize the applicability of this rule to large navigable rivers, such as those uniting the great lakes.

On all the little lakes of the State which are mere expansions near their mouths of the rivers passing through them—such as the Muskegon, Pere Marquette and Manistee—the same rule of bed ownership has been judicially applied that is applied to the rivers themselves; and the division lines are extended under the water in the same way. *Rice v. Ruddiman*, 10 Mich., 125. If such a lake were circular, the lines would converge to the centre; if oblong or irregular, there might be a line in the middle on which they would terminate, whose course would bear some relation to that of the shore. But it can seldom be important to follow the division line very far under the water, since all private rights are subject to the public rights of navigation and other use, and any private use of the lands inconsistent with these would be a nuisance, and punishable as such. It is sometimes important, however, to run the lines out for some considerable distance, in order to determine where one may lawfully moor vessels or rafts, for the winter, or cut ice. The ice crop that forms over a man's land of course belongs to him. *Lorman v. Benson*, 8 Mich., 18; *People's Ice Co. v. Steamer Excelsior*, recently decided.

What is said above will show how unfounded is the notion, which is sometimes advanced, that a riparian proprietor on a meandered river may lawfully raise the water in the stream without liability to the proprietors above, provided he does not raise it so that it overflows the meander line. The real fact is that the meander line has nothing to do with such a case, and an action will lie whenever he sets back the water upon the proprietor above, whether the overflow be below the meander lines or above them.

As regards the lakes and ponds of the State, one may easily raise questions that it would be impossible for him to settle. Let us suggest a few questions, some of which are easily answered, and some not:

1. To whom belongs the land under these bodies of water, where they are not mere expansions of a stream flowing through them?
2. What public rights exist in them?
3. If there are islands in them which were not surveyed out and sold by the United States, can this be done now?

Others will be suggested by the answers given to these.

It seems obvious that the rules of private ownership which are applied to rivers cannot be applied to the great lakes. Perhaps it should be held that the boundary is at low-water mark, but improvements beyond this would only become unlawful when they became nuisances. Islands in the great lakes would belong to the United States until sold, and might be surveyed and measured for sale at any time. The right to take fish in the lakes, or to cut ice, is public like the right of navigation, but is to be exercised in such manner as not to interfere with the rights of shore owners. But so far as these public rights can be the subject of ownership, they belong to the State, not to the United States; and, so it is believed, does the bed of a lake also. *Pollard v. Hagan*, 3 Howard's U. S. Reports. But such rights are not generally considered proper subjects of sale, but, like the right to make use of the public highways, they are held by the State in trust for all the people.

What is said of the large lakes may perhaps be said also of many of the interior lakes of the State; such, for example, as Houghton, Higgins, Cheboygan, Burt's, Mullet, Whitmore, and many others. But there are many little lakes or ponds which are gradually disappearing, and the shore proprietorship advances *pari passu* as the waters recede. If these are of any considerable size—say, even a mile across—there may be questions of conflicting rights which no adjudication hitherto made could settle. Let any surveyor, for example, take the case of a pond of irregular form, occupying a mile square or more of territory, and undertake to determine the rights of the shore proprietors to its bed when it shall totally disappear, and he will find he is in the midst of problems such as probably he has never grappled with, or reflected upon before. But the general rules for the extension of shore lines, which have already been laid down, should govern such cases, or at least should serve as guides in their settlmeent. *Note.*—Since this address was delivered some of these questions have received the attention of the Supreme Court of Michigan in the cases of *Richardson v. Prentiss*, 48 Mich. Reports, 88, and *Backus v. Detroit*, Albany Law Journal, vol. 26, p. 428.

Where a pond is so small as to be included within the lines of a private purchase from the government, it is not believed the public have any rights in it whatever. Where it is not so included, it is believed they have rights of fishery, rights to take ice and water, and rights of navigation for business or pleasure. This is the common belief, and probably the just one. Shore rights must not be so exercised as to disturb these, and the States may pass all proper laws for their protection. It would be easy with suitable legislation to preserve these little bodies of water as permanent places of resort for the pleasure and recreation of the people, and there ought to be such legislation.

If the State should be recognized as owner of the beds of these small lakes and ponds, it would not be owner for the purpose of selling. It would be owner only as a trustee for the public use; and a sale would be inconsistent with the right of the bank owners to make use of the water in its natural condition in connection with their estates. Some of them might be made salable lands by draining; but the State could not drain, even for this purpose, against the will of the shore owners, unless their rights were appropriated and paid for.

Upon many questions that might arise between the State as owner of the bed of a little lake and the shore owners, it would be presumptuous to express an opinion now, and fortunately the occasion does not require it.

I have thus indicated a few of the questions with which surveyors may now and then have occasion to deal, and to which they should bring good sense and sound judgment. Surveyors are not and cannot be judicial officers, but in a great many cases they act in a *quasi* judicial capacity with the acquiescence of parties concerned; and it is important for them to know by what rules they are to be guided in the discharge of their judicial functions. What I have said cannot contribute much to their enlightenment, but I trust will not be wholly without value.

APPENDIX B.

INSTRUCTIONS TO U. S. DEPUTY MINERAL SURVEYORS, FOR THE DISTRICT OF COLORADO. (1886.)

GENERAL RULES.

1. All official communications must be addressed to the Surveyor-General. You will always refer to the date and subject-matter of the letter to which you reply, and when a mineral claim is the subject of correspondence, you will give the name, ownership and survey number.

2. You should keep a complete record of each survey made by you, and the facts coming to your knowledge at the time, as well as copies of all your field-notes, reports and official correspondence, in order that such evidence may be readily produced when called for at any future time.

3. Field-notes and other reports must be written in a clear and legible hand, and upon the proper blanks furnished by this office. *No cut sheets, interlineations or erasures* will be allowed; and no abbreviations or symbols must be used, except such as are indicated in the specimen field-notes.

4. No return by you will be recognized as official unless made in pursuance of a special order from this office.

5. The claimant is required, in all cases, to make satisfactory arrangements with you for the payment for your services and those of your assistants in making the survey, as the United States will not be held responsible for the payment of the same. You will call the attention of applicants for mineral-survey orders to the requirements of the circular of this date in the appendix.

6. You will promptly notify this office of any change in your post-office address. Upon permanent removal from the State, you are expected to resign your appointment.

NOT TO ACT AS ATTORNEY.

7. You are precluded from acting, either directly or indirectly, as attorney in mineral claims. Your duty in any particular case ceases when you have executed the survey and returned the field-notes and preliminary plat, with your report to the Surveyor-General. You will not be allowed to prepare for the mining claimant the papers in support of his

application for patent, or otherwise perform the duties of an attorney before the land office in connection with a mining claim. You are not permitted to combine the duties of surveyor and notary-public in the same case by administering oaths to the parties in interest. In short, you must have absolutely nothing to do with the case except in your official capacity as surveyor. You will make no survey of a mineral claim in which you hold an interest.

THE FIELD-WORK.

8. The survey made and reported must, in every case, be an actual survey on the ground in full detail, made by you in person after the receipt of the order, and without reference to any knowledge you may have previously acquired by reason of having made the location-survey or otherwise, and must show the actual facts existing at the time. If the season of the year, or any other cause, renders such personal examination impossible, you will postpone the survey, and under no circumstances rely upon the statements or surveys of other parties, or upon a former examination by yourself.

The term *survey* in these instructions applies not only to the usual field-work, but also to the examinations required for the preparation of your affidavits of five hundred dollars expenditure, descriptive reports on placer-claims and all other reports.

SURVEY AND LOCATION.

9. The survey must be made in strict conformity with, or be embraced within, the lines of the record of location upon which the order is based. If the survey and location are identical, that fact must be clearly and distinctly stated in your field-notes. If not identical, a bearing and distance must be given from each established corner of the survey to the corresponding corner of the location. The lines of the location, as found upon the ground, must be laid down upon the preliminary plat in such manner as to contrast and show their relation to the lines of the survey.

10. If the record of location has been made prior to the passage of the mining act of May 10, 1872, and is not sufficiently definite and certain to enable you to make a correct survey therefrom, you are required, after reasonable notice in writing, to be served personally or through the United States mail on the applicant for survey and adjoining claimants, whose residence or post-office address you may know, or can ascertain by the exercise of reasonable diligence, to take testimony of neighboring claimants and other persons who are familiar with the boundaries thereof as originally located and asserted by the locators of the claim, and after having ascertained by such testimony the boundaries as originally established, you will make a survey in accordance therewith, and transmit full and correct returns of the survey, accompanied by the copy of the record of location, the testimony, and a copy of the notice served on the claimant and adjoining proprietors, certifying thereon when, in what manner, and on whom service was made.

11. If the location has been made subsequent to the passage of the

mining act of May 10, 1872, and the law has been complied with in the matter of marking the location on the ground and recording the same, and any question should arise in the execution of the survey as to the identity of monuments, marks, or boundaries which cannot be determined by a reference to the record, you are required to take testimony in the manner hereinbefore prescribed for surveys of claims located prior to May 10, 1872, and having thus ascertained the true and correct boundaries originally established, marked and recorded, you will make the survey accordingly.

12. In accordance with the principle that courses and distances must give way when in conflict with fixed objects and monuments, you will not, under any circumstances, change the corners of the location for the purpose of making them conform to the description in the record. If the difference from the location be slight, it may be explained in the field-notes, but if there should be a wide discrepancy, you will report the facts to this office and await further instructions.

INSTRUMENT.

13. All mineral surveys must be made with a SOLAR TRANSIT, or other instrument operating independently of the magnetic needle, and all courses must be referred to the true meridian. It is deemed best that a solar transit should be used under all circumstances. The variation should be noted at each corner of the survey.

CONNECTIONS.

14. Connect corner No. 1 of your survey by course and distance with some corner of the public survey or with a United States location-monument, if the claim lies within two miles of such corner or monument. If both are within the required distance, you will connect with the nearest corner of the public survey.

LOCATION-MONUMENTS.

15. In case your survey is situated in a district where there are no corners of the public survey and no monuments within the prescribed limits, you will proceed to establish a mineral monument, in the location of which you will exercise the greatest care to insure permanency as to site and construction.

16. The site, when practicable, should be some prominent point visible for a long distance from every direction, and should be so chosen that the permanency of the monument will not be endangered by snow, rock or land slides, or other natural causes.

17. The location-monument should consist of a post eight feet long and six inches square set three feet in the ground, and protected by a well-built conical mound of stone three feet high and six feet base. The letters U. S. L. M. followed by a *name* must be scribed on the post and also chiselled on a large stone in the mound, or on the rock in place that may form the base of the monument. There is no objection to the establishment of a location-monument of larger size, or of other material of equally durable character.

18. From the monument, connections by course and distance must

be taken to two or three bearing trees or rocks, and to any well-known natural and permanent objects in the vicinity, such as the confluence of streams, prominent rocks, buildings, shafts or mouths of adits. Bearings should also be taken to prominent mountain-peaks, and the approximate distance and direction ascertained from the nearest town or mining camp. A detailed description of the location-monument must be included in the field-notes of the survey for which it is established.

CORNERS.

19. Corners may consist of

First—A stone at least twenty-four inches long by six inches square set eighteen inches in the ground.

Second—A post at least four and a half feet long by four inches square set twelve inches in the ground and surrounded by a mound of stone or earth two and a half feet high and five feet base.

Third—A rock in place.

20. All corners must be established in a permanent and workmanlike manner, and the corner and survey number must be neatly chiselled or scribed on the sides facing the claim. When a rock in place is used its dimensions above ground must be stated, and a cross chiselled at the exact corner-point.

21. In case the point for the corner be inaccessible or unsuitable, you will establish a witness-corner, which must be marked with the letters W. C. in addition to the corner and survey number. The witness-corner should be located upon a line of the survey and as near as practicable to the true corner, with which it must be connected by course and distance. The reason for the establishment of a witness-corner must always be stated in the field-notes.

22. The identity of all corners should be perpetuated by taking courses and distances to bearing trees, rocks, and other objects, as prescribed in the establishment of location-monuments. If an official survey has been made within a reasonable distance in the vicinity, you will run a connecting line to some corner of the same, and connect in like manner with all conflicting surveys and claims.

TOPOGRAPHY.

23. Note carefully all topographical features of the claim, taking distances on your lines to intersections with all streams, gulches, ditches, ravines, mountain ridges, roads, trails, etc., with their widths, courses and other data that may be required to map them correctly. If the claim lies within a town-site, locate all municipal improvements, such as blocks, streets and buildings.

24. You are required also to locate all mining and other improvements upon the claim by courses and distances from corners of the survey, or by rectangular offsets from the centre line, specifying the dimensions and character of each in full detail.

CONFLICTS.

25. If in running the exterior boundaries of a claim, you find that two surveys conflict, you will determine the courses and distances from the

established corners at which the exterior boundaries of the respective surveys intersect each other, and run all lines necessary for the determination of the areas in conflict, both with surveyed and unsurveyed claims. You are not required, however, to show conflicts with unsurveyed claims unless the same are to be excluded.

26. When a placer-claim includes lodes, or when several lode-locations are included as one claim in one survey, you will preserve a consecutive series of numbers for the corners of the whole survey in each case. In the former case you will first describe the placer-claim in your field-notes.

PLACER-CLAIMS AND MILL-SITES.

27. The exterior lines of placer-claims cannot be extended over other claims, and the conflicting areas excluded as with lode-claims, it being the surface ground only, with side lines taken perpendicularly downward for which application is made. The survey must accurately define the boundaries of the *claim*. The same rule will apply to the survey of mill-sites.

28. If by reason of intervening surveys or claims a placer or mill-site survey should be divided into separate tracts, you will also preserve a consecutive series of numbers for the corners of the whole survey, and distinguish the detached portions as Lot No. 1, Lot No. 2, etc., connecting by course and distance a corner of each lot with some corner of the one previously described.

LODE AND MILL-SITE.

29. A lode and mill-site claim in one survey will be distinguished by the letters A and B following the number of the survey. The corners of the mill-site will be numbered independently of those of the lode. Corner No. 1 of the mill-site must be connected with a corner of the lode claim as well as with a corner of the public survey or U. S. location-monument.

FIELD-NOTES.

30. In order that the results of your survey may be reported to this office in a uniform manner, you will prepare your field-notes and preliminary plat in strict conformity with the specimen field-notes and plat, which are made part of these instructions. They are designed to furnish you with all needed information concerning the manner of describing the boundaries, corners, connections, intersections, conflicts and improvements, and stating the variation, area, location and other data connected with the survey of mineral claims, and contain forms of affidavits for the deputy-surveyor and his assistants.

In your first reference to any other mineral claim you will give the name, ownership, and if surveyed, the survey-number.

31. The total area of a lode-claim embraced by the exterior boundaries, and also the area in conflict with each intersecting survey or claim should be so stated, that the conflicts with any one or all of them may be included or excluded from your survey. This will enable the claimant to state in his application for patent the portions to be excluded in express terms, and to readily determine the *net area* of his claim.

32. You will state particularly whether the claim is upon surveyed or unsurveyed public lands, giving in the former case the quarter-section, township and range in which it is located, and in the latter the township, as near as can be determined.

33. The field-notes must contain the post-office address of the claimant or his authorized agent.

EXPENDITURE OF FIVE HUNDRED DOLLARS.

34. The claimant is required by law, either at the time of filing his application, or at any time thereafter, within the sixty days of publication, to file with the Register the certificate of the Surveyor-General that five hundred dollars' worth of labor has been expended or improvements made upon the claim by himself or grantors. The information upon which to base this certificate must be derived from the deputy who makes the actual survey and examination upon the premises, and such deputy is required to specify with particularity and full detail the character and extent of such improvements. See also Sec. 8.

35. When a survey embraces several locations or claims *held in common*, constituting one entire claim, whether lode or placer, an expenditure of five hundred dollars upon such entire claim embraced in the survey will be sufficient and need not be shown upon each of the locations included therein.

36. In case of a lode and mill-site claim in the same survey, an expenditure of five hundred dollars must be shown upon the lode-claim only.

37. Only *actual* expenditures and *mining* improvements, made by the claimant or his grantors, having a direct relation to the development of the claim, can be included in your estimate.

38. The expenditures required may be made from the surface, or in running a tunnel for the development of the claim. Improvements of any other character, such as buildings, machinery or roadways, must be excluded from your estimate unless you show clearly that they are associated with actual excavations, such as cuts, tunnels, shafts, etc., and are essential to the practical development of the survey-claim.

39. You will give in detail the value of each mining improvement included in your estimate of expenditure, and when a tunnel or other improvement has been made for the development of other claims in connection with the one for which survey is made, your report must give the name, ownership and survey-number, if any, of each claim to which a proportion or interest is credited, and the value of the proportion or interest credited to each. The value of improvements made upon other locations or by a former locator who has abandoned his claim cannot be included in your estimate.

40. In making out your certificate of the value of the improvements, you will follow the form prescribed in the specimen field-notes.

41. Following your certificate you will locate and describe all other improvements made by the claimant or other parties within the boundaries of the survey.

42. If the value of the labor and improvements upon a mineral claim

is less than five hundred dollars at the time of survey, you are authorized to file your affidavit of five hundred dollars expenditure at any time before the expiration of the sixty days of publication, but not afterwards unless by special instructions.

DESCRIPTIVE REPORTS ON PLACER-CLAIMS.

43. By General Land Office circular, approved September 23, 1882, you are required to make a full examination of all placer-claims at the time of survey, and file with your field-notes a descriptive report in which you will describe—

(*a*) The quality and composition of the soil, and the kind and amount of timber, and other vegetation.

(*b*) The *locus* and size of streams, and such other matters as may appear upon the surface of the claims.

(*c*) The character and extent of all surface and underground workings, whether placer or lode, for mining purposes.

(*d*) The proximity of centres of trade or residence.

(*e*) The proximity of well-known systems of lode deposits or of individual lodes.

(*f*) The use or adaptability of the claim for placer-mining, and whether water has been brought upon it in sufficient quantity to mine the same, or whether it can be procured for that purpose.

(*g*) What works or expenditures have been made by the claimant or his grantors for the development of the claim, and their situation and location with respect to the same as applied for.

(*h*) The true situation of all mines, salt-licks, salt-springs, and mill-seats, which come to your knowledge, or report that none exist on the claim, as the facts may warrant.

(*i*) Said report must be made under oath, and duly corroborated by one or more disinterested persons.

44. Descriptive reports upon placer-claims taken by legal subdivisions are authorized only by special order, and must contain a description of the claim in addition to the foregoing requirements.

PRELIMINARY PLAT.

45. You will file with your field-notes a preliminary plat on drawing-paper or tracing-muslin, protracted on a scale of two hundred feet to an inch, on which you will note accurately all the topographical features and details of the survey in conformity with the specimen plat herewith. Pencil sketches will not be accepted.

REPORT.

46. You will also submit with your return of survey a report upon the following matters incident to the survey, but not required to be embraced in the field-notes.

47. If the meridian from which your courses were deflected was established by other means than by the solar apparatus attached to your transit, you will state in detail your observations and calculations for the establishment of such meridian.

48. If any of the lines of the survey were determined by triangulation or traverse, you will give in full detail the calculations whereby you arrived at the results reported in your field-notes. You will also submit your calculations of areas of placer and mill-site claims or other irregular tracts.

49. You will mention in your report the discovery of any material errors in prior official surveys, giving the extent of the same.

ERRORS.

50. Whenever a survey has been reported in error, the deputy-surveyor who made it will be required to promptly make a thorough examination, upon the premises, and report the result under oath to this office. In case he finds his survey in error, he will report in detail all discrepancies with the original survey, and submit any explanation he may have to offer as to the cause. If, on the contrary, he should report his survey correct, a joint survey will be ordered to settle the differences with the surveyor who reported the error.

JOINT SURVEY.

51. A joint survey must be made within ten days after the date of order, unless satisfactory reasons are submitted, under oath, for a postponement.

52. The field-work must in every sense of the term be a *joint* and not a separate survey, and the observations and measurements taken with the same instrument and chain, previously tested and agreed upon.

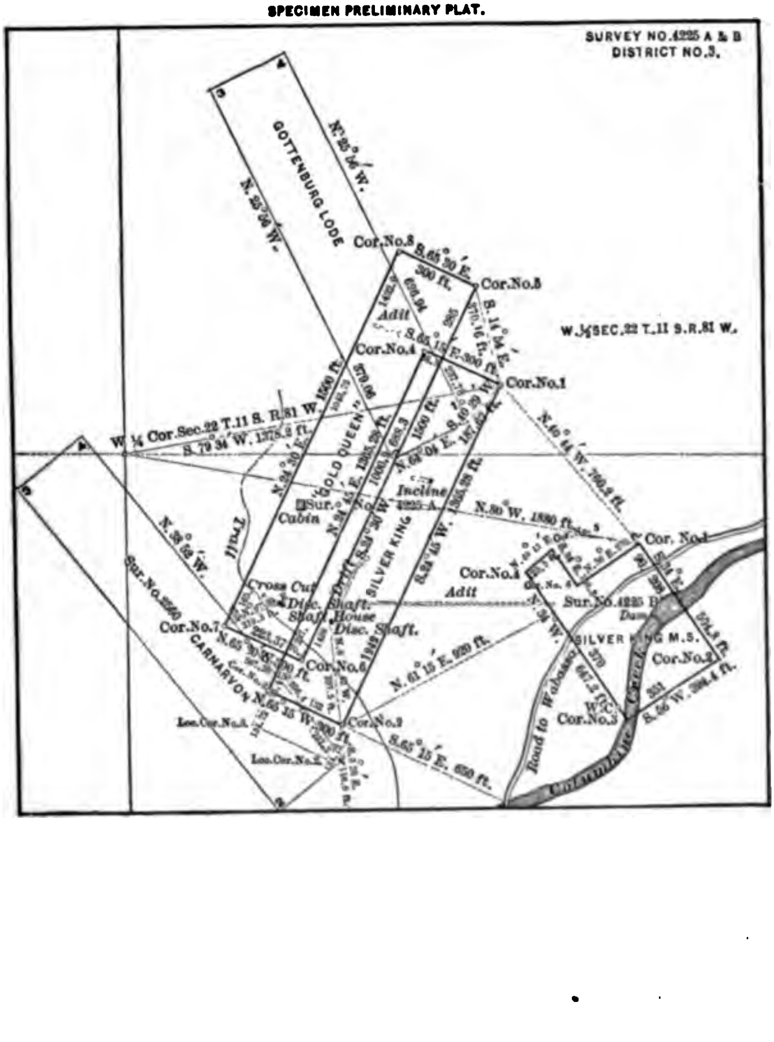
53. The deputy-surveyor found in error, or if both are in error, the one who reported the same will make out the field-notes of the joint survey, which, after being duly signed and sworn to by both parties, must be transmitted to this office.

54. The surveyor found in error will be required to pay all expenses of the joint survey and preliminary examinations incident thereto, including ten dollars per day to the surveyor whose work is proved to be substantially correct.

55. Your field-work must be accurately and properly performed, and your returns made in conformity with the foregoing instructions. Errors in the survey must be corrected at your own expense, and if the time required in the examination of your returns is increased by reason of your neglect or carelessness you will be required to make an additional deposit for office work. You will be held to a strict accountability for the faithful discharge of your duties, and will be required to observe fully the requirements and regulations in force as to making mineral surveys. If found incompetent as a surveyor, careless in the discharge of your duties, or guilty of a violation of said regulations, your appointment will be promptly revoked.

56. All former instructions inconsistent with the foregoing are hereby recalled.

SPECIMEN PRELIMINARY PLAT.



SPECIMEN FIELD-NOTES.

Survey No. 4225 A and B.
District No. 3.

FIELD-NOTES.

Of the survey of the claim of The Argentum Mining Company, upon the Silver King and Gold Queen lodes, and Silver King Mill site, in Alpine Mining District, Lake county, Colorado.

Surveyed by George Lightfoot, *U. S. Deputy Mineral Surveyor.* Survey begun April 22d, 1886, and completed April 24th, 1886.

Address of claimant, Wabasso, Colorado.

FEET.	SURVEY NO. 4225 A.
	SILVER KING LODE.
	Beginning at Cor. No. 1.
	Identical with Cor. No. 1 of the location.
	A spruce post, 5 ft. long, 4 ins. square, set 2 ft. in the ground, with mound of stone, marked $\frac{1}{4225}$ A, whence
	The W. $\frac{1}{4}$ cor. Sec. 22, T. 11 S., R. 81 W. of the 6th Principal Meridian, bears S. 79° 34′ W. 1378.2 ft.
	Cor. No. 1, Gottenburg lode (unsurveyed), Neals Mattson, claimant, bears S. 40° 29′ W. 187.67 ft.
	A pine 12 ins. dia., blazed and marked B. T. $\frac{1}{4225}$ A, bears S. 7° 25′ E. 22 ft.
	Mount Ouray bears N. 11° E.
	Hiawatha Peak bears N. 47° 45′ W.
	Thence S. 24° 45′ W.
	Va. 15° 12′ E.
1242.	To trail, course N. W. and S. E.
1365.28	To Cor. No. 2.
	A granite stone 25 x 9 x 6 ins., set 18 ins. in the ground, chiselled $\frac{2}{4225}$ A, whence
	Cor. No. 2 of the location bears S. 24° 45′ W. 134.72 ft.
	Cor. No. 1, Sur. No. 2560, Carnarvon lode, David Davies et al., claimants bears S. 3° 28′ E. 116.6 ft.
	North end of bridge over Columbine creek bears S. 65° 15′ E. 650 ft.
	Thence N. 65° 15′ W.
	Va. 15° 20′ E.
152	Intersect line 4–1, Sur. No. 2560, at N. 38° 52′ W., 231.2 ft. from Cor. No. 1.
300.	To Cor. No. 3.

FEET.

A cross at corner-point, and $\frac{3}{4225}$ A chiselled on a granite rock in place, 20 x 14 x 6 ft. above the general level, whence

Cor. No. 3 of the location bears S. 24° 45′ W. 134.72 ft.

A spruce 16 ins. dia., blazed and marked B. T. $\frac{3}{4225}$ A, bears S. 58° W. 18 ft.

Thence N. 24° 45′ E.
Va. 15° 20′ E.

73.4 Intersect line 4–1 Sur. No. 2560 at N. 38° 52′ W. 396.4 ft. from Cor. No. 1.

150. Intersect line 6–7 of this survey.

237. To trail, course N. W. and S. E.

1000.9 Intersect line 2–3, Gottenburg lode, at N. 25° 56′ W. 76.26 ft. from Cor. No. 2.

1365.28 To Cor. No. 4.

Identical with Cor. No. 4 of the location.

A pine post 4.5 ft. long 5 ins. square, set one foot in the ground, with mound of earth and stone, marked $\frac{4}{4225}$ A, whence

A cross chiselled on rock in place, marked B. R. $\frac{4}{4225}$ A, bears N. 28° 10′ E. 58.9 ft.

Thence S. 65° 15′ E.
Va. 15° 12′ E.

28.5 Intersect line 4–1, Gottenburg lode, at N. 25° 56′ W 285.15 ft. from Cor. No. 1.

65. Intersect line 5–6 of this survey.

300. To Cor. No. 1, the place of beginning.

GOLD QUEEN LODE.

Beginning at Cor. No. 5,

A pine post 5 ft. long, 5 ins. square, set 2 ft. in the ground, with mound of earth and stone, marked $\frac{5}{4225}$ A, whence

Cor. No. 1 of this survey bears S. 14° 54′ E. 370.16 ft.

A pine 18 in. dia. bears S. 33° 15′ W. 51 ft., and a silver spruce 13 ins. dia. bears N. 60° W. 23 ft., both blazed and marked B. T. $\frac{5}{4225}$ A.

Thence S. 24° 30′ W.
Va. 15° 14′ E.

285. Intersect line 4–1 of this survey.

315. Intersect line 4–1, Gottenburg lode, at N. 25° 56′ W. 237.78 ft. from Cor. No. 1.

688.3 Intersect line 1–2, Gottenburg lode, at N. 64° 04′ E. 12.23 ft. from Cor. No. 2.

1438. To trail, course N. W. and S. E.

1500. To cor. No. 6,

A granite stone 34 x 14 x 6 ins., set one foot in the ground to bedrock, with mound of stone, chiselled $\frac{6}{4225}$ A, whence

A cross chiselled on ledge of rock marked B. R. $\frac{6}{4225}$ A, bears due north 12 ft.

FEET.	
	Thence N. 65° 30′ W. Va. 15° 20′ E.
70.3	Intersect line 3–4 of this survey.
223.37	Intersect line 4–1, Sur. No. 2560 at N. 38° 52′ W, 567.28 ft. from Cor. No. 1.
300.	To Cor. No. 7. A cross at corner-point and $\frac{7}{4338}$ A chislled on a granite boulder 12 x 6 x 3 ft. above ground, whence A cross chiselled on vertical face of cliff, marked B. R. $\frac{7}{4338}$ A, bears N. 72° W. 56.2 ft. A pine 14 ins. dia., blazed and marked B. T. $\frac{7}{4338}$ A, bears N. 10° E. 39 ft.
	Thence N 24° 30′ E. Va. not determined on account of local attraction.
38.43	Intersect line 4–1, Sur. No. 2560, at N. 38° 52′ W. 653 ft. from Cor. No. 1.
165.	To trail, course N. W. and S. E.
1043.73	Intersect line 2–3, Gottenburg lode, at N. 25° 56′ W. 379.06 ft. from Cor. No. 2.
1432.90	Intersect line 4–1, Gottenburg lode, at N. 25° 56′ W. 626.94 ft. from Cor. No. 1.
1500.	To Cor. No. 8. A spruce post 6 ft. long, 5 ins. square, set 2.5 ft. in the ground with mound of stone, marked $\frac{8}{4338}$ A, whence A cross chislled on rock in place, marked B. R. $\frac{8}{4338}$ A, bears S. 9° 12′ E. 15.8 ft. A pine, 20 ins. dia., blazed and marked B. T. $\frac{8}{4338}$ A, bears N. 83° E. 28.5 ft.
	Thence S. 65° 30′ E. Va. 15° 16′ E.
300.	To Cor. No. 5, the place of beginning.

AREA.

Total area of Silver King lode.................. 9.403 acres
Less area in conflict with
Sur. No. 2560......... .124 acre
Gottenburg lode .....1.363 " 1.487 acres

Net area of Silver King lode............. 7.916 acres

Total area of Gold Queen lode ....10.331 acres
Area in conflict with
Sur. No. 2560.................. .034 "
Gottenburg lode .... 2.679 "
Silver King lode............... 1.887 "
Silver King lode (exclusive of conflict of said Silver King lode with the Gottenburg lode) 1.309 "

FEET. Total area of Gold Queen lode.................10.331 acres
Less area in conflict with
Sur. No. 2560.......... .034 acre
Gottenburg lode.......2.679 "
Silver King lode.......1.309 " 4.022 acres

Net area of Gold Queen lode............ 6.309 acres
" " Silver King lode............ 7.916 "

Net area of lode claim... ....14.225 acres

SURVEY NO. 4225 B.

SILVER KING MILL-SITE.

Beginning at Cor. No. 1,

A gneiss stone 32 x 8 x 6 ins., set 2 ft. in the ground, chiselled $\frac{1}{4225}$ B, whence W. $\frac{1}{4}$ cor. Sec. 22, T. 11 S, R. 81 W. of the 6th Principal Meridian, bears N. 80° W. 1880 ft.

Cor. No. 1, Sur. No. 4225 A, bears N. 40° 44′ W. 760.2 ft.

A cottonwood 18 ins. dia., blazed and marked $\frac{1}{4225}$ B, bears S. 5° 30′ E. 17 ft.

Thence S. 34° E.

90. Road to Wabasso, course N. E. and S. W.
208. Right bank of Columbine creek, 75 ft. wide, flows S. W.
504.8 To Cor. No. 2,

An iron bolt 18 ins. long, 1 in. dia., set one foot in rock in place, chiselled $\frac{2}{4225}$ B, whence

A cottonwood, blazed and marked B. T. $\frac{2}{4225}$ B, bears E. 182 ft.

Thence S. 56° W.

351. Left bank of Columbine creek.
394.4 To Cor. No. 3,

A point in bed of creek, unsuitable for the establishment of a permanent corner.

Thence N. 34° W.

15. Right bank of Columbine creek.
40. To witness-corner to Cor. No. 3.

A pine post 4.5 ft. long, 5 ins. square, set one foot in ground, with mound of stone, marked W. C. $\frac{3}{4225}$ B, whence

A cottonwood 15 ins. dia. bears N. 11° E. 16.5 ft. and a cottonwood 19 ins. dia. bears N. 83° W. 23 ft., both blazed and marked B. T. W. C. $\frac{3}{4225}$ B.

370. Road to Wabasso, course N. E. and S. W.
647.2 To Cor. No. 4,

A gneiss stone 24 x 10 x 4 ins., set 18 ins. in the ground, chiselled $\frac{4}{4225}$ B, whence

A cross chiselled on ledge of rock, marked B. R. $\frac{4}{4225}$ B, bears N. 85° 10′ E. 26.4 ft.

Thence N. 48° 43′ E.

FEET.

125.5 To Cor. No. 5,

A gneiss stone 30 x 8 x 5 ins., set 2 ft. in the ground, chiselled $\frac{5}{4225}$ B.

Thence S. 34° E.

158.3 To Cor. No. 6,

A pine post 5 ft. long, 5 ins. square, set 2 ft. in the ground with mound of earth and stone, marked $\frac{6}{4225}$ B, whence

A pine 12 ins. dia., blazed and marked B. T. $\frac{6}{4225}$ B, bears S. 33° E. 63.5 ft.

Thence N. 56° E.

270. To Cor. No. 1, the place of beginning,

Containing 5 acres.

Variation at all the corners, 15° 20′ E.

The surveys of the Gold Queen lode and Silver King mill site are identical with the respective locations.

LOCATION.

This claim is located in the W. ½ Sec. 22, T. 11 S., R. 81 W.

EXPENDITURE OF FIVE HUNDRED DOLLARS.

I certify that the value of the labor and improvements upon this claim, placed thereon by the claimant and its grantors, is not less than five hundred dollars, and that said improvements consist of

The discovery shaft of the Silver King lode, 6 x 3 ft., 10 ft. deep in earth and rock, which bears from Cor. No. 2 N. 6° 42′ W. 287.5 ft. Value $80.

An incline 7 x 5 ft., 45 ft. deep in coarse gravel and rock, timbered, course N. 58° 15′ W., dip 62°, the mouth of which bears from Cor. No. 2 N. 15° 37′ E. 908 ft. Value $550.

The discovery shaft of the Gold Queen lode, 5 x 5 ft., 18 ft. deep in rock, which bears from Cor. No. 7 N. 67° 39′ E. 219.3 ft., at the bottom of which is a cross-cut 6.5 x 4 ft. running N. 59° 26′ W. 75 ft.

Value of shaft and cross-cut, $1,000.

A log shaft-house 14 ft. square, over the last-mentioned shaft. Value $100.

Two-thirds interest in an adit 6.5 x 5 ft., running due west 835 ft., timbered, the mouth of which bears from Cor. No. 2 N. 61° 15′ E. 920 ft.

This adit is in course of construction for the development of the Silver King and Gold Queen lodes of this claim, and Sur. No. 2560, Carnarvon lode, David Davies et al., claimants, the remaining one-third interest therein having already been included in the estimate of five hundred dollars expenditure upon the latter claim, Total value of adit, $13,000.

A drift 6.5 x 4 ft. on the Silver King lode, beginning at a point in adit 800 ft. from the mouth, and running N. 20° 20′ E. 195 ft., thence N. 54° 15′ E. 40 ft. to breast. Value $2,800.

I further certify that no portion of the improvements claimed have been included in the estimate of five hundred dollars expenditure upon any other claim.

OTHER IMPROVEMENTS.

A log cabin 35 x 28 ft., the S. W. corner of which bears from Cor. No. 7 N. 30° 44′ E. 496 ft.

A dam 4 ft. high, 50 ft. long, across Columbine creek, the south end of which bears from Cor. No. 2 of the mill-site N. 58° 20′ W. 240 ft.

Said cabin and dam belong to The Argentum Mining Company.

An adit 6 x 4 ft., running N. 70° 50′ W. 100 ft., the mouth of which bears from Cor. No. 5 S. 58° 12′ W. 323 ft. belonging to Neals Mattson, claimant of the Gottenburg lode.

INSTRUMENT.

The survey was made with a Young & Sons mountain transit No. 5322, with Smith's solar attachment. The courses were deflected from the true meridian as determined by solar observations. The distances were measured with a 50 ft. steel tape.

EMPLOYÉ'S CERTIFICATE.

List of the names of individuals employed to assist in running, measuring and marking the lines and corners described in the foregoing field-notes of the survey of the claim of The Argentum Mining Company upon the Silver King and Gold Queen lodes and Silver King mill-site, in Alpine Mining District, Lake County, Colorado.

William Sharp,
Robert Talc.

We hereby certify that we assisted George Lightfoot U. S. Deputy Mineral Surveyor, in surveying the exterior boundaries and marking the corners of the claim of The Argentum Mining Company upon the Silver King and Gold Queen lodes and Silver King mill-site in Alpine Mining District, Lake County, Colorado, and that said survey has been in all respects, to the best of our knowledge and belief, well and faithfully surveyed and the boundary monuments planted according to the instructions furnished by the Surveyor-General.

William Sharp,
Robert Talc.

Subscribed and sworn to by the above-named persons before me, this 26th day of April, 1886.

[Seal] John Doolittle,
Notary Public.

SURVEYOR'S OATH.

I, George Lightfoot, U. S. Deputy Mineral Surveyor, do solemnly swear that in pursuance of an order from Jas. A. Dawson, Surveyor-General of the public lands in the State of Colorado, bearing date the 30th day of March 1886, and in strict conformity with the laws of the United States, and instructions furnished by said Surveyor-General, I have faithfully surveyed the claim of The Argentum Mining Company upon the Silver King and Gold Queen lodes and Silver King mill-site in Alpine Mining District, Lake County, Colorado, and do further solemnly swear that the foregoing are the true and original field-notes of such survey, and that the improvements are as therein stated.

George Lightfoot,
U. S. Deputy Mineral Surveyor.

Subscribed by said George Lightfoot, U. S. Deputy Mineral Surveyor, and sworn to before me this 26th day of April, 1886.

[Seal] John Doolittle,
Notary Public.

APPENDIX C.

FINITE DIFFERENCES.

THE CONSTRUCTION OF TABLES.

In the accompanying figure the ordinates are spaced at the uniform distance l apart. Let the successive values of these ordinates, and their several orders of differences, be represented by the following notation:

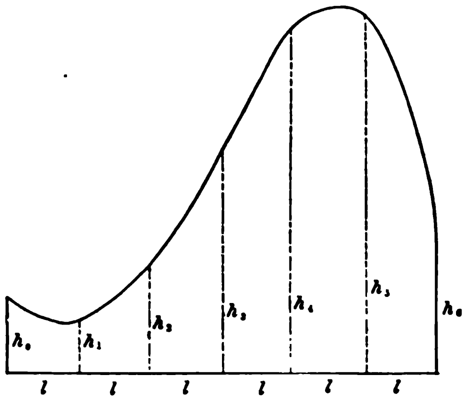


Fig. 152.

Values of the function,	h_0,	h_1,	h_2,	h_3,	h_4,	h_5,	h_6.
First order of differences,	$\Delta' h_0$,	$\Delta' h_1$,	$\Delta' h_2$,	$\Delta' h_3$,	$\Delta' h_4$,	$\Delta' h_5$.	
Second " "	$\Delta'' h_0$,	$\Delta'' h_1$,	$\Delta'' h_2$,	$\Delta'' h_3$,	$\Delta'' h_4$.		
Third " "	$\Delta''' h_0$,	$\Delta''' h_1$,	$\Delta''' h_2$,	$\Delta''' h_3$.			
Fourth " "	$\Delta^{iv} h_0$,	$\Delta^{iv} h_1$,	$\Delta^{iv} h_2$.				
etc.,	etc.						

We may now write

$$\left.\begin{array}{l} h_1 = h_0 + \Delta' h_0; \\ h_2 = h_1 + \Delta' h_1 = h_0 + \Delta' h_0 + \Delta' h_0 + \Delta'' h_0 = h_0 + 2\Delta' h_0 + \Delta'' h_0; \\ h_3 = h_2 + \Delta' h_2 = h_0 + 3\Delta' h_0 + 3\Delta'' h_0 + \Delta''' h_0; \\ h_4 = h_0 + 4\Delta' h_0 + 6\Delta'' h_0 + 4\Delta''' h_0 + \Delta^{\text{iv}} h_0; \\ h_n = h_0 + n\Delta' h_0 + \dfrac{n(n-1)}{1 \cdot 2}\Delta'' h_0 + \dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\Delta''' h_0 + \text{etc.} \end{array}\right\} \quad . \quad (1)$$

It is to be observed that the coefficients follow the law of the binomial development. It is also seen that the *first* of the successive orders of differences are alone sufficient to enable any term of the function to be computed. We will now proceed to find these first terms of the several orders of differences for any given equation.

Almost all functions of a single variable can be developed by the aid of Maclaurin's Formula, in the form

$$y_0 = C_0 + C_1 x_0 + C_2 x_0^2 + C_3 x_0^3 + C_4 x_0^4 + \text{etc.} \quad . \quad . \quad . \quad . \quad (2)$$

If x take an increment Δ_x, thus becoming x_1, the *change* in y_0 will be represented by Δ'_{y_0} and its value will be the new value of the function minus its initial value, or $\Delta'_{y_0} = y_1 - y_0$. By putting $x + \Delta_x$ for x in the above equation, developing, subtracting the original equation, and reducing, we would obtain

$$\begin{aligned} y_1 - y_0 = \Delta'_{y_0} &= (C_1 + 2C_2 x_0 + 3C_3 x_0^2 + 4C_4 x_0^3)\Delta_x \\ &+ (C_2 + 3C_3 x_0 + 6C_4 x_0^2)\Delta^2_x + (C_3 + 4C_4 x_0)\Delta^3_x + C_4 \Delta^4_x, \quad . \quad (3) \end{aligned}$$

assuming that the function stops with $C_4 x_0^4$.

If x_1 should now take another increment Δ_x, equal to the previous one, we would have $x_2 = x_1 + \Delta_x$ and $y_2 = y_1 + \Delta'_{y_1}$. Now Δ'_{y_1} is the value Δ'_y when x_0 has become x_1, and the *difference* between Δ'_{y_0} and Δ'_{y_1} is the *change* in the value of Δ'_{y_0} due to this change in x.

Hence $\qquad \Delta'_{y_1} - \Delta'_{y_0} = \Delta''_{y_0}$.

To find the value of Δ''_{y_0}, substitute $x + \Delta_x$ for x in equation (3), develop, subtract equation (3), reduce, and obtain

$$\Delta''_{y_0} = (2C_2 + 6C_3 x_0 + 12C_4 x_0^2)\Delta^2_x + (6C_3 + 24C_4 x_0)\Delta^3_x + 14C_4\Delta^4_x. \quad (4)$$

Similarly we find

$$\Delta'''_{y_0} = (6C_3 + 24C_4 x_0)\Delta^3_x + 36C_4\Delta^4_x. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\Delta^{\text{iv}}_{y_0} = 24C_4\Delta^4_x \text{ (a constant)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From the above development we see—

1. That *the number of orders of differences is equal to the highest exponent of the variable involved*, the last difference being a constant.

2. That if any initial value (x_0) of the variable be taken, the first of the several orders of differences can be obtained in terms of this initial value, its constant increment, and the constant coefficients. This furnishes a ready means of computing a table of values of the function, if it can be represented in the form of equation (1). Evidently if the initial value of the variable (x_0) be taken as zero, the evaluation for the several initial differences is much simplified, for then all the terms in x disappear. If the constant increment be also taken as unity, the labor is still further reduced.

Example.—Construct a table of values of the function

$$y = 50 - 40x + 20x^2 + 4x^3 - x^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let the initial value of the variable be zero and the increments unity.
Evaluating the initial differences by equations (3) to (6), we find, for $x_0 = 0$, and $\Delta_x = 1$,

$$\begin{aligned}
y_0 &= & &+ 50; \\
\Delta' y_0 &= C_1 + C_2 + C_3 + C_4 &&= - 17; \\
\Delta'' y_0 &= 2C_2 + 6C_3 + 14C_4 &&= + 50; \\
\Delta''' y_0 &= 6C_3 + 36C_4 &&= - 12; \\
\Delta^{iv} y_0 &= 24C_4 &&= - 24.
\end{aligned}$$

From these initial values we may readily construct the following table:

Values of x.	Values of y.*	1st Differences. Δ'_y.	2d Differences. Δ''_y.	3d Differences. Δ'''_y.	4th Differences. Δ^{iv}_y.
0	50				
		− 17			
1	33		+ 50		
		+ 33		− 12	
2	66		+ 38		− 24
		+ 71		− 36	
3	137		+ 2		− 24
		+ 73		− 60	
4	210		− 58		− 24
		+ 15		− 84	
5	225		− 142		− 24
		− 127		− 108	
6	98		− 250		− 24
		− 377		− 132	
7	− 279		− 382		etc.
		− 759		etc.	
8	− 1038		etc.		
		etc.			
etc.	etc.				

* Fig. 152 is the locus of this curve, the ordinates being taken from the y's column.

The initial values in all the columns being given, the table is made by continual additions, one column after another, working from right to left. Thus, the 4th difference being constant, the initial value, -24, is simply repeated indefinitely. The column of 3d differences is now computed by adding continuously -24 to the preceding value. The column of 2d differences is next made out, the quantity to be added each time being the intervening 3d difference, which is not constant. In a similar manner proceed with the column of 1st differences, and finally with the values of the function itself.

The above formulæ apply to all functions of a single variable not higher than the fourth degree. Evidently any of the C coefficients may be zero, and so cause one or more of the powers of x to entirely disappear. If the variable is involved to a higher degree than the fourth, a new development may be made, or the initial values of the successive orders of differences may be determined by simply evaluating the function for a series of successive values of the variable, one more in number than the degree of the equation, and then working out the successive columns of differences from these until the last, or constant, difference is found. The table may then be continued by combining these differences, as before. Thus in the above example the first five values of y might have been found by direct evaluation of the function for the corresponding values of x, and then the successive differences taken out until the constant fourth difference, -24, was found. This can always be done without resorting to any algebraic discussion as given above.

THE EVALUATION OF IRREGULAR AREAS.

The ordinates to any curve, as that in Fig. 152 for instance, may be represented by such an equation as the last of equations, (1), where the length of any ordinate is given in terms of its number from the initial ordinate, the value of this first ordinate, and the first of the successive orders of differences. This equation is

$$h_n = h_0 + n\Delta' h_0 + \frac{n(n-1)}{1 . 2}\Delta'' h_0 + \frac{n(n-1)(n-2)}{1 . 2 . 3}\Delta''' h_0 + \text{etc.},$$

where h_n is the nth, and therefore any ordinate to the curve. The constant distance between the ordinates apparently does not enter the equation, but it is really represented in the several Δ's.

By the calculus the area of any figure included between any curve, the axis of abscissas, and two extreme ordinates is $A = \int_0^x h dx$, where h is the general value of an ordinate, $= h_n$ in the above equation, where it is shown to be a function of n. Also $x = nl$ where l is the constant distance between ordinates, whence $dx = ldn$. Substituting these values of h and dx, we have

$$A = \int_0^x h\,dx = \int_0^n h_n\,l\,dn = l\int_0^n h_n\,dn = l\Big[h_0\int_0^n dn + \Delta' h_0\int_0^n n\,dn$$
$$+ \frac{\Delta'' h_0}{1.2}\int_0^n n(n-1)dn + \frac{\Delta''' h_0}{1.2.3}\int_0^n n(n-1)(n-2)dn$$
$$+ \frac{\Delta^{iv} h_0}{\lfloor 4}\int_0^n n(n-1)(n-2)(n-3)dn + \text{etc.}\Big] \quad . \quad . \quad . \quad . \quad (8)$$

Integrating this equation, we obtain

$$A = l\left\{\begin{array}{l} nh_0 + \frac{n^2}{2}\Delta' h_0 + \left(\frac{n^3}{6} - \frac{n^2}{4}\right)\Delta'' h_0 + \left(\frac{n^4}{24} - \frac{n^3}{6} + \frac{n^2}{6}\right)\Delta''' h_0 \\ + \left(\frac{n^5}{120} - \frac{n^4}{16} + \frac{11n^3}{72} - \frac{n^2}{8}\right)\Delta^{iv} h_0 + \left(\frac{n^6}{720} - \frac{n^5}{60} + \frac{7n^4}{96} - \frac{5n^3}{36} + \frac{n^2}{10}\right)\Delta^{v} h_0 \\ + \left(\frac{n^7}{5040} - \frac{n^6}{288} + \frac{17n^5}{720} - \frac{5n^4}{64} + \frac{137n^3}{1080} - \frac{n^2}{12}\right)\Delta^{vi} h_0 + \text{etc.} \end{array}\right\} \quad . \quad . \quad (9)$$

From the schedule of differences on p. 605 we may at once find the initial values of the several orders of differences in terms of the successive values of the function. Thus

$$\Delta' h_0 = h_1 - h_0;$$
$$\Delta'' h_0 = \Delta' h_1 - \Delta' h_0 = h_2 - 2h_1 + h_0;$$
$$\Delta''' h_0 = \Delta'' h_1 - \Delta'' h_0 = \Delta' h_2 - 2\Delta' h_1 + \Delta' h_0 = h_3 - 3h_2 + 3h_1 - h_0;$$
$$\Delta^{iv} h_0 = \Delta''' h_1 - \Delta''' h_0 = \Delta'' h_2 - 2\Delta'' h_1 + \Delta'' h_0 = \Delta' h_3 - 3\Delta' h_2 + 3\Delta' h_1 - \Delta' h_0$$
$$= h_4 - 4h_3 + 6h_2 - 4h_1 + h_0.$$

Again, the coefficients follow the law of the binomial development, and we may write

$$\Delta^n h_0 = h_n - nh_{n-1} + \frac{n(n-1)}{1.2}h_{n-2} - \frac{n(n-1)(n-2)}{1.2.3}h_{n-3} + \text{etc.} \quad . \quad (10)$$

By the aid of this equation we may now substitute for the several initial differences in equation (9) their values in terms of the successive values of the function. Also for any area divided into n sections by ordinates, uniformly spaced a distance l apart, equation (9) will give the area in terms of l, n, and the several ordinates, when these latter are substituted for the Δ's by means of eq. (10).

Thus, for $n = 1$, equation (9) becomes

$$A = l(h_0 + \tfrac{1}{2}\Delta' h_0) = \frac{l}{2}(h_0 + h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which is the *Trapezoidal Rule.*

For $n = 2$,

$$A = l(2h_0 + 2\Delta'h_0 + (\tfrac{4}{3} - 1)\Delta''h_0) = l(\tfrac{1}{3}h_0 + \tfrac{4}{3}h_1 + \tfrac{1}{3}h_2) = \frac{l}{3}(h_0 + 4h_1 + h_2), \quad (12)$$

which is called *Simpson's* $\tfrac{1}{3}$ *Rule.*

If $l' = 2l =$ total length of figure, this formula becomes

$$A = \frac{l'}{6}(h_0 + 4h_1 + h_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

which is the well-known *form* of the *Prismoidal Formula,* and it would be that formula if areas were substituted for ordinates.

If $n = 3$,

$$A = \frac{3l}{8}(h_0 + 3h_1 + 3h_2 + h_3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which is called *Simpson's* $\tfrac{3}{8}$ *Rule.*

If $n = 4$,

$$A = \frac{2l}{45}[7(h_0 + h_4) + 32(h_1 + h_3) + 12h_2]. \quad . \quad . \quad . \quad . \quad (14)$$

If $n = 6$,

$$A = l[6h_0 + 18\Delta'h_0 + 27\Delta''h_0 + 24\Delta'''h_0 + \tfrac{123}{10}\Delta^{\mathrm{iv}}h_0 + \tfrac{33}{10}\Delta^{\mathrm{v}}h_0 + \tfrac{41}{140}\Delta^{\mathrm{iv}}h_0].$$

If now the coefficient of $\Delta^{\mathrm{vi}}h_0$ be changed from $\frac{41}{140}$ to $\frac{42}{140}$, which would not affect curves of a degree less than the sixth, the resulting equation, when the h's are substituted for the Δ's, takes the following very simple form :

$$A = \frac{3l}{10}[h_0 + h_2 + h_4 + h_6 + 5(h_1 + h_3 + h_5) + h_3], \quad . \quad . \quad (15)$$

which is called *Weddel's Rule.*

For a greater number of ordinates than seven, it is best to use either equation (12), (13), or (15) several times, as the formulæ become very complicated for $n > 6$.

APPENDIX D.

DERIVATION OF FORMULÆ FOR COMPUTING GEOGRAPHICAL COÖRDINATES AND FOR THE PROJECTION OF MAPS.*

Let Fig. 153 represent a distorted meridian section of the earth.
Let a = the major and b the minor semi-axes.

Then $$e = \frac{a - b}{a} = \text{the ellipticity.}$$

The eccentricity is given by

$$e^2 = \frac{a^2 - b^2}{a^2}, \qquad \text{whence} \qquad 1 - e^2 = \frac{b^2}{a^2}.$$

The line $nm = N$ is the *normal* to the curve at n;

the angle $ncd = \lambda$ is the *geocentric latitude*;

while $nld = L$ is the *geodetic latitude*.

The geodetic latitude is always understood, as it is the latitude obtained from astronomical observations.

It is desirable to find the length of the line nl, of the normal nm, and of the radius of curvature $p'r'$, all in terms of e, L, and a. Also to find the geocentric latitude in terms of a, b, and L.

To find nl, we have

$$nl = \sqrt{\overline{nd}^2 + \overline{dl}^2} = \sqrt{y^2 + \frac{y^2 dy^2}{dx^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the ellipse, $$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y};$$

whence $$nl = \sqrt{y^2 + \frac{b^4}{a^4}x^2} = \sqrt{y^2 + (1 - e^2)^2 x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

* See Chapters XIV. and XV. for the use of the formulæ.

But the equation of the ellipse in terms of its eccentricity is

$$x^2 = a^2 - \frac{y^2}{1 - e^2};$$

whence $$nl = \sqrt{y^2e^2 + a^2(1 - e^2)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

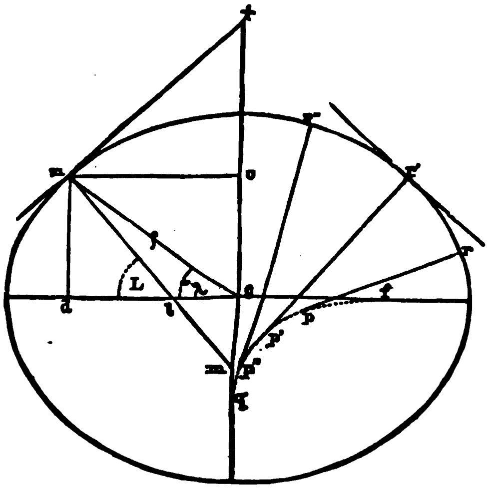


FIG. 153.

Squaring, remembering that $y = nl \sin L$, we have, after reducing,

$$nl = \frac{a(1-e^2)}{(1 - e^2 \sin^2 L)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

To find the length of the normal $nm = N$, we have

$$nm : nl :: x : dl.$$

But $$dl = \overline{nd} \tan dnl = y\frac{dy}{dx} = \frac{b^2}{a^2}x = (1 - e^2)x; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whence $$nm = N = \frac{nl}{1 - e^2} = \frac{a}{(1 - e^2 \sin^2 L)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

To find λ, the geocentric latitude in terms of a, b, and L, we have

$$\lambda = ncd; \qquad L = nld.$$

Since both have the common ordinate nd, we may write

$$\tan \lambda : \tan L :: dl : dc.$$

But $$dl = \frac{b^2}{a^2} x \text{ from (4)}, \qquad \text{and } dc = x,$$

whence $$\tan \lambda = \frac{b^2}{a^2} \tan L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

To find the radius of curvature, R, we have, in general,

$$R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For the ellipse, $\dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}$, and $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$;

whence $$R = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To get this in terms of a, e, and L, we have, from Fig. 153,

$$y^2 = \overline{nl}^2 \sin^2 L = \frac{a^2(1 - e^2)^2 \sin^2 L}{1 - e^2 \sin^2 L}.$$

Also from the equation of the ellipse in terms of its eccentricity we have

$$x^2 = a^2 - \frac{y^2}{1 - e^2} = \frac{a^2(1 - \sin^2 L)}{1 - e^2 \sin^2 L}.$$

We may now find

$$a^4y^2 + b^4x^2 = \frac{a^2b^4}{1 - e^2 \sin^2 L},$$

or
$$(a^4y^2 + b^4x^2)^{\frac{3}{2}} = \frac{a^3b^6}{(1 - e^2 \sin^2 L)^{\frac{3}{2}}}. \quad \ldots \ldots \quad (7)$$

Substituting this in (6), we obtain

$$R = \frac{b^2}{a} \cdot \frac{1}{(1 - e^2 \sin^2 L)^{\frac{3}{2}}} = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 L)^{\frac{3}{2}}}. \quad \ldots \ldots \quad (D)$$

The radius of curvature of the meridian, R, and the radius of curvature of the great circle perpendicular to a given meridian at the point where they intersect, which is the normal, N, are the most important functions in geodetic formulæ. We will now derive the equations used

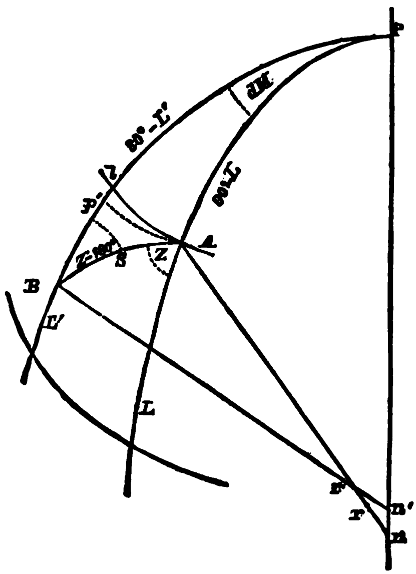


FIG. 154.

on the U. S. Coast and Geodetic Survey for computing geodetic positions from the results of a primary triangulation.

In Fig. 154, let A and B be two points on the surface of the earth, which were used as adjacent triangulation-stations. The distance between them, the azimuth of the line AB at one of the stations, and the latitude

and longitude of one station are supposed to be known; the latitude and longitude of the other station, and the back azimuth of the line joining them, are to be found.

Let L' = known latitude of B;
L = unknown latitude of A;
K = known length of line AB reduced to sea-level;
s = length of arc $AB = \frac{K}{N}$;
Z' = known azimuth of BA at B;
Z = unknown azimuth of AB at A;
M' = known longitude of B;
M = unknown longitude of A.

The angle APB formed by the two meridional planes through A and B is the difference of longitude $M - M' = \Delta M$.

The difference of latitude is, $L - L' = \Delta L = Bl$ in the figure. Al is the trace of a parallel of latitude through A and l is its intersection with the meridian through B. AP' is the trace of a great circle through A perpendicular to the meridian through B, and P' is the point of its intersection with that meridian.

The normals are $Bn' = N'$ and $An = N$. The radii of curvatvre are $Br' = R'$ and $Ar = R$.

The latitude and longitude of A, and the azimuth of the line AB from A towards B, can now be found by solving the spherical triangle APB. Thus $L = 90° - AP$; $M = M' - Al$; and $Z = 180° - PAB$.

Although the line AB lies on the surface of a spheroid, if a sphere be conceived such that its surface is tangent internally to the surface of the spheroid on the parallel of latitude passing through the middle point of the line AB, then this line will lie so nearly in the surface of the sphere, that no appreciable error is made by assuming it to be in its surface. The triangle ABP then becomes a triangle on the surface of the tangent sphere, and hence is a true spherical triangle. The sphere is defined by taking its radius equal to the normal to the meridian at the mean latitude of the points A and B. Since this mean latitude is unknown, the formulæ are first derived for the latitude of B, L', and then a correction applied to reduce it to the mean latitude.

THE DIFFERENCE OF LATITUDE.

Let it first be required to find L from L', or find $\Delta L = L - L'$.

If we write l, l', for the co-latitudes of L, L', and z' for $Z' - 180°$, we have, from the spherical triangle ABP,

$$\cos l = \cos l' \cos s + \sin l' \sin s \cos z'. \quad \ldots \ldots \quad (8)$$

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By means of Taylor's Formula we may find the value of l in ascending powers of s, and since s is always a very small arc in terms of the radius, usually from 20 to 60 minutes, the series will be rapidly converging

By means of Taylor's Formula, we may at once write

$$l = l' + \frac{dl'}{ds}s + \frac{1}{2}\frac{d^2l'}{ds^2}s^2 + \frac{1}{6}\frac{d^3l'}{ds^3}s^3 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We will use but the first three terms of this development, the fourth term being used only in the largest primary triangles.

The derivation of the successive differential coefficients of l with respect to s is the most difficult portion of this general development. If s be supposed to vary, then l and z both must vary, and they are all implicit functions of each other. These coefficients are therefore best found geometrically, as follows: in Fig. 155,

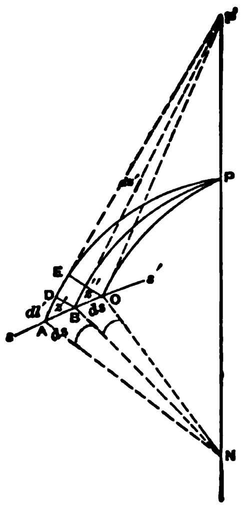


FIG. 155.

Let $AB = BC = ds =$ differential portions of the line $AB = s$ in Fig. 154;
$AD = - dl' =$ change in AP ($= l'$) due to the change $+ ds$ in s.

Let the angle $PAB = z'$ and $PBC = z''$, z'' being greater than z' by the convergence of the meridians shown by the angle $AP'B$.

The lines BD and CE are parallels of latitude through the points B and C. They cut all meridians at right angles.

Since the triangle ABD is a differential one on the surface of the sphere, it may be treated as a plane triangle, and we may at once write

$$\frac{dl'}{ds} = \frac{AD}{AB} = -\cos z', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

the minus sign indicating that l and s are inverse functions of each other.

Differentiating this equation and dividing both sides by ds we obtain

$$\frac{d^2l'}{ds^2} = \sin z' \frac{dz'}{ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Now the angle dz' is the angle $AP'B$, subtended by the arc BD with radius BP'. But this arc, with radius BN gives the angle $ds \sin z'$.

Therefore

$$dz' = \sin z' ds \frac{BN}{BP'} = \sin z' ds \tan L' = \sin z' \cot l' ds,$$

or

$$\frac{dz'}{ds} = \sin z' \cot l'.$$

Substituting this in (11) we obtain

$$\frac{d^2l'}{ds^2} = \sin^2 z' \cot l'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Substituting these values in (9), we have

$$l - l' = -s\cos z' + \tfrac{1}{2}s^2 \sin^2 z' \cot l' + \text{etc.}$$

Now, replacing l, l', and z, by L, L', and Z', we have

$$L' - L = s\cos Z' + \tfrac{1}{2}s^2 \sin^2 Z' \tan L'. \quad . \quad . \quad . \quad . \qquad (13)$$

Here s is expressed in arc to a radius of unity.

Referring it now to the radius N, we have $s = \frac{K}{N}$, where K is the length of the arc s in any unit, N being the length of the normal nm in Fig. 153, given in the same unit.

Substituting these in (13), we have

$$L' - L = \frac{K \cos Z'}{N'} + \frac{1}{2}\frac{K^2 \sin^2 Z' \tan L'}{N'^2}. \quad \ldots \ldots \quad (14)$$

This gives the difference of latitude in units of arc in terms of radius N. But differences of latitude are measured on a sphere whose radius is the radius of curvature of the meridian at the middle latitude. Since we do not yet know the middle latitude, we can use the known latitude L' and afterwards correct to $\frac{L + L'}{2}$.

Changing to a sphere, whose radius is R', and dividing by the arc of 1″ in order to get the result in seconds, we have

$$L' - L = - \delta L = \frac{K}{R' \text{ arc } 1''} \cos Z' + \frac{1}{2}\frac{K^2}{R'N' \text{ arc } 1''} \sin^2 Z' \tan L'. \quad (15)$$

If we let $\qquad B = \frac{1}{R' \text{ arc } 1''}$, and $C = \frac{\tan L'}{2R'N' \text{ arc } 1''}$,

we may write $\qquad -\delta L = K \cos Z' \cdot B + K^2 \sin^2 Z' \cdot C. \quad \ldots \ldots \quad (16)$

To reduce this to what it would be if the mean latitude had been used we have to correct it for the difference in the radii of curvature, R_L' and R_m, at the latitude L' and the middle latitude respectively. If ΔL be the true difference of latitude when R_m is used, and δL be the difference when R_L' is used, we would have

$$\Delta L : \delta L :: R_L' : R_m,$$

or

$$\Delta L = \delta L \frac{R_L'}{R_m} = \delta L \left(1 + \frac{R_L' - R_m}{R_m}\right) = \delta L \left(1 + \frac{dR_m}{R_m}\right).$$

To reduce δL to ΔL, therefore, we must add the quantity $\delta L \cdot \frac{dR_m}{R_m}$.

Now

$$R' = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 L')^{\frac{3}{2}}},$$

whence

$$dR = \frac{a(1 - e^2)(3e^2 \sin L' \cos L')}{(1 - e^2 \sin^2 L')^{\frac{5}{2}}} dL.$$

Here dL is the difference in latitude between one extremity of the line s and its middle point, or $dL = \frac{1}{2}\delta L$, as given in eq. (16), hence

$$\delta L\left(\frac{dR}{R'}\right) = \frac{\frac{3}{4}e^2 \sin L' \cos L'}{1 - e^2 \sin^2 L'}\,\delta^2 L.^* \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If we put
$$D = \frac{\frac{3}{4}e^2 \sin L' \cos L' \sin 1''}{1 - e^2 \sin^2 L'},$$

the corrective term becomes

$$\delta L \frac{dR_m}{R_m} = (\delta L)^2 D;$$

whence we finally obtain

$$-\Delta L = K \cos Z \cdot B + K^2 \sin^2 Z \cdot C + (\delta L)^2 \cdot D, \quad . \quad . \quad . \quad . \quad (D)$$

where δL is given by (16), or it is the value of the first two terms in the right member of this final equation. For distances less than 12 miles the first term only may be used as giving the value of δL.

The values of the constants B, C, and D are given for every minute of latitude from 23° to 65° in Appendix No. 7 of the U. S. Coast and Geodetic Report for 1884. This Appendix can be obtained by applying to the Superintendent.

For distances of 12 miles or less, using the first term only for δL, equation (18) becomes

$$\Delta L = K \cos Z\ (B + K \cos Z \cdot D) + K^2 \sin^2\ Z \cdot C. \quad . \quad . \quad . \quad (E')$$

THE DIFFERENCE OF LONGITUDE.

In the triangle APB, Fig. 154, the three sides and the angle at the known station B are known. To find ΔM = angle APB, we have, therefore,

$$\sin PA : \sin AB :: \sin PBA : \sin APB,$$

or
$$\sin l : \sin s :: \sin z : \sin \Delta M.$$

* In the U. S. Coast and Geodetic Survey Report for 1884, Appendix 7, p. 326, this term is given with its denominator raised to the $\frac{3}{2}$ power, and the tabular values of D are computed accordingly. The development there given is laborious and approximate, but the error is not more than 0.001 of the value of this term, which is itself very small.

But $s = \frac{K}{N'}$ where N' is the normal Bn', Fig. 154; and if we assume that the arc s is proportional to its sine, we have

$$\Delta M = \frac{K \sin Z}{N \cos L' \text{ arc } 1''}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where ΔM is expressed in seconds of arc.

If we put $$A = \frac{1}{N' \text{ arc } 1''},$$

this equation becomes $$\Delta M = \frac{K \sin Z . A}{\cos L'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (F)$$

In order to correct for the assumption that the arc is proportional to its sine, a table of the differences of the logarithms of arcs and sines is given in the U. S. C. and G. Report for 1884, p. 373, with instructions for its use on p. 327.

THE DIFFERENCE OF AZIMUTH.

In the spherical triangle APB, Fig. 154, we have, from spherical trigonometry,

$$\cot \tfrac{1}{2}(PBA + PAB)^* = \tan \tfrac{1}{2}BPA \frac{\cos \frac{1}{2}(BP + AP)}{\cos \frac{1}{2}(BP - AP)},$$

or $$\cot \tfrac{1}{2}(z' + z) = \tan \tfrac{1}{2}(- \Delta M^\dagger) \frac{\cos \frac{1}{2}(l' + l)}{\cos \frac{1}{2}(l' - l)}$$

$$= - \tan \tfrac{1}{2}\Delta M \frac{\sin \frac{1}{2}(L' + L)}{\cos \frac{1}{2}(L' - L)}.$$

But $$z = 180° - Z,$$

therefore $$\cot \tfrac{1}{2}(180° - Z + z') = \tan \tfrac{1}{2}(Z - z') = \tan \tfrac{1}{2}(\Delta Z),$$

whence $$- \tan \tfrac{1}{2}\Delta Z = \tan \tfrac{1}{2}\Delta M \frac{\sin \frac{1}{2}(L' + L)}{\cos \frac{1}{2}(L' - L)}. \quad . \quad . \quad . \quad . \quad (19)$$

* Chauvenet's Spherical Trigonometry, eq. (127).
† Increments of M are measured positively towards the west.

It will be seen that since the azimuth Z of a line is measured from the south point in the direction S.W.N.E., the azimuth of the line BA from B towards A (forward azimuth) is the angle $PBA + 180° = Z'$, while the azimuth of the same line from A is $180° - PAB = Z$. Also, that $\Delta Z = Z + 180° - Z'$.

Assuming that the tangents $\frac{1}{2}\Delta Z$ and $\frac{1}{2}\Delta M$ are proportional to their arcs, and putting L_m for the middle latitude, we have

$$- \Delta Z = \Delta m \frac{\sin L_m}{\cos \frac{1}{2}\Delta L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (G)$$

The U. S. Coast Survey Tables are based on the following semi-diameters:

$$a = 6\ 378\ 206 \text{ metres},$$
$$b = 6\ 356\ 584 \quad ''$$

or

$$a : b :: 294.98 : 293.98.$$

See Appendix No. 7, U. S. Coast and Geodetic Survey, for tabular values of constants and forms for reduction.

APPENDIX E.

GEOGRAPHICAL POSITIONS OF BASE-LINES AND PRINCIPAL MERIDIANS GOVERNING THE PUBLIC SURVEYS.

Since the adoption of the rectangular system of public surveys, May 20, 1785, twenty-four initial points, or the intersection of the principal bases with surveying meridians, have been brought into requisition to secure the certainty and brevity of description in the transfer of public lands to individual ownership. From the principal bases townships of six miles square are run out and established, with regular series of numbers counting north and south thereof, and from the surveying meridians a like series of ranges are numbered both east and west of the principal meridians.

During the period of one hundred years since the organization of the system the following numerical and independent principal meridians and bases have been initiated, to wit:

The first principal meridian divides the States of Ohio and Indiana, having for its base the Ohio River, the meridian being coincident with 84° 51′ of longitude west from Greenwich. The meridian governs the surveys of public lands in the State of Ohio.

The second principal meridian coincides with 86° 28′ of longitude west from Greenwich, starts from the confluence of the Little Blue River with the Ohio, runs north to the northern boundary of Indiana, and governs the surveys in Indiana and a portion of those in Illinois.

The third principal meridian starts from the mouth of the Ohio River and extends to the northern boundary of the State of Illinois, and governs the surveys in said State *east* of the meridian, with the exception of those projected from the second meridian, and the surveys on the *west* to the Illinois River. This meridian coincides with 89° 10′ 30″ of longitude west from Greenwich.

The fourth principal meridian begins in the middle of the channel of the mouth of the Illinois River, in latitude 38° 58′ 12″ north and longitude 90° 29′ 56″ west from Greenwich, and governs the surveys in Illinois west of the Illinois River and west of the third principal meridian lying north of the river. It also extends due north through Wisconsin and northeastern Minnesota, governing all the surveys in the former and those in the latter State lying east of the Mississippi and the third guide meridian (west of the fifth principal meridian) north of the river.

The fifth principal meridian starts from the mouth of the Arkansas River, and, with a common base-line running due west from the mouth of the Saint Francis River, in Arkansas, governs the surveys in Arkansas, Missouri, Iowa, Minnesota west of the Mississippi, and the third guide meridian north of the river, and in Dakota Territory east of the Missouri River. This meridian is coincident with 90° 58' longitude west from Greenwich.

The sixth principal meridian coincides with longitude 97° 22' west from Greenwich, and, with the principal base-line intersecting it on the 40th degree of north latitude, extends north to the intersection of the Missouri River and south to the 37th degree of north latitude, controlling the surveys in Kansas, Nebraska, that part of Dakota lying south and west of the Missouri River, Wyoming, and Colorado, excepting the valley of the Rio Grande del Norte, in southwestern Colorado, where the surveys are projected from the New Mexico meridian.

In addition to the foregoing six principal meridians and bases governing public surveys, there have been established the following meridians and bases, viz.:

The Michigan meridian, in longitude 84° 19' 09" west from Greenwich, with a base-line on a parallel seven miles north of Detroit, governing the surveys in Michigan.

The Tallahassee meridian, in longitude 84° 18' west from Greenwich, runs due north and south from the point of intersection with the base-line at Tallahassee, and governs the surveys in Florida.

The Saint Stephen's meridian, longitude 88° 02' west from Greenwich, starts from Mobile, passes through Saint Stephen's, intersects the base-line on the 31st degree of north latitude, and controls the surveys of the southern district in Alabama and of the Pearl River district lying east of the river and south of township 10 north in the State of Mississippi.

The Huntsville meridian, longitude 86° 31' west from Greenwich, extends from the northern boundary of Alabama as a base, passes through the town of Huntsville, and governs the surveys of the northern district in Alabama.

The Choctaw meridian, longitude 89° 10' 30" west from Greenwich, passes two miles west of the town of Jackson, in the State of Mississippi, starting from the base-line twenty-nine miles south of Jackson, and terminating on the south boundary of the Chickasaw cession, controlling the surveys east and west of the meridian and north of the base.

The Washington meridian, longitude 91° 05' west from Greenwich, seven miles east of the town of Washington, in the State of Mississippi, with the base-line corresponding with the 31st degree of north latitude, governs the surveys in the southwestern angle of the State.

The Saint Helena meridian, 91° 11' longitude west from Greenwich, extends from the 31st degree of north latitude, as a base, due south, and passing one mile east of Baton Rouge, controls the surveys in the Greensborough and the southeastern districts of Louisiana, both lying east of the Mississippi.

The Louisiana meridian, longitude 92° 20' west from Greenwich, intersects the 31st degree north latitude at a distance of forty-eight miles west of the eastern bank of the Mississippi River, and, with the base-line coincident with the said parallel of north latitude, governs the surveys in Louisiana west of the Mississippi.

The New Mexico meridian, longitude 106° 52' 09" west from Greenwich, intersects the principal base-line on the Rio Grande del Norte, about ten miles below the mouth of the Puerco River, on the parallel of 34° 19' north latitude, and controls the surveys in New Mexico, and in the valley of the Rio Grande del Norte, in Colorado.

The Great Salt Lake meridian, longitude 111° 53' 47" west from Greenwich, intersects the base-line at the corner of Temple Block, in Salt Lake City, Utah, on the parallel of 40° 46' 04" north latitude, and governs the surveys in the Territory of Utah.

The Boisé meridian, longitude 116° 20' west from Greenwich, intersects the principal base between the Snake and Boisé Rivers, in latitude 43° 26' north. The initial monument, at the intersection of the base and meridian, is nineteen miles distant from Boisé City, on a course of south 29° 30' west. This meridian governs the surveys in the Territory of Idaho.

The Mount Diablo meridian, California, coincides with longitude 121° 54' west from Greenwich, intersects the base-line on the summit of the mountain from which it takes its name, in latitude 37° 53' north, and governs the surveys of all central and northeastern California and the entire State of Nevada.

The San Bernardino meridian, California, longitude 116° 56' west from Greenwich, intersects the base-line at Mount San Bernardino, latitude 34° 06' north, and governs the surveys in southern California lying east of the meridian, and that part of the surveys situated west of it which are south of the eighth standard parallel south of the Mount Diablo base-line.

The Humboldt meridian, longitude 124° 11' west from Greenwich, intersects the principal base-line on the summit of Mount Pierce, in latitude 40° 25' 30" north, and controls the surveys in the northwestern corner of California lying west of the coast range of mountains and north of township 5 south of the Humboldt base.

The Willamette meridian is coincident with longitude 122° 44' west from Greenwich, its intersection with the base-line is on the parallel of 45° 30' north latitude, and it controls the public surveys in Oregon and Washington Territory.

The Montana meridian extends north and south from the initial monument established on the summit of a limestone hill, eight hundred feet high, longitude 111° 40' 54" west from Greenwich. The base-line runs east and west from the monument on the parallel of 45° 46' 27" north latitude. The surveys for the entire Territory of Montana are governed by this meridian.

The Gila and Salt River meridian intersects the base-line on the south side of the Gila River, opposite the mouth of Salt River, in longitude 112° 15' 46" west from Greenwich, and latitude 33° 22' 57" north, and governs the public surveys in the Territory of Arizona.

The Indian meridian intersects the base-line at Fort Arbuckle, Indian Territory, in longitude 97° 15' 56" west from Greenwich, latitude 34° 31' north, and governs the surveys in that Territory.

APPENDIX F.

NOTE.—The following instructions, issued by the Mississippi River Commission in 1891, embody the continuous experience of some twenty years' work on the United States surveys of the Great Lakes, and of twelve years' work on the survey of the Mississippi River, and are believed to represent the best practice in secondary triangulation, precise leveling, topographic and hydrographic work. J. B. J.

May, 1892.

INSTRUCTIONS FOR SECONDARY TRIANGULATION, PRECISE LEVEL, AND TOPOGRAPHICAL AND HYDROGRAPHICAL FIELD WORK UNDER THE MISSISSIPPI RIVER COMMISSION, 1891.

INSTRUCTIONS FOR SECONDARY TRIANGULATION.

Locating stations.—In locating stations it is desirable to fix them at such points as give good conditioned triangles. The smallest angle in any triangle should never be less than 30 degrees, and but few of these should be permitted to enter into the system. The triangles should lie in such a way that pointings can be made from any station to the stations immediately above and below on the same side of the river. That is, blind lines should always be avoided. Other things being equal, stations should be set where they can be readily found and where they will not be disturbed.

Reading angles.—The angles will be read with T. & S. theodolites, Nos. 1 and 2. The instruments will be mounted firmly and protected from sun and wind when in use. The value of the angle will be determined by eight combined results read as follows:

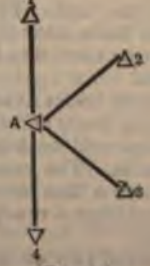


FIG. 156.

The instrument being at A, carefully leveled and in good adjustment. With the vertical circle to the right, or telescope direct, and lower motion fixed, point successively to △ 1, 2, 3, and 4, recording the reading of both micrometers for each pointing. This gives a positive result for each angle. Then point to △ 4, 3, 2, 1, and record readings as before. This gives a negative result for each angle. A mean of the two gives one combined result. The readings in a positive and negative direction will eliminate twist of station or instrument, provided the readings occupy but a short period of time during which the twist, if any, is uniform.

For the next combined result. The telescope will now be reversed, that is, revolved through, leaving the pivots in the same wyes, and the whole will be revolved 180 degrees in azimuth. The vertical circle will then be on the left; the limb will be shifted 22½ degrees, and the stations will be read forward and back as before. The notes for this series will be headed circle left. Reversing the telescope will eliminate errors of collimation, small level errors, and inequality of pivots. Shifting the limb so as to read the angles at equal intervals around the circle will eliminate periodic errors and errors of graduation.

The same programme is followed until all the results are obtained, the limb being shifted and the telescope reversed after each combined result.

The micrometers should be adjusted so the run will be nearly zero. This should, however, be tested at the beginning of each day's work, and entered in the note-book.

Closing triangles.—The error in closing a triangle should rarely reach and never exceed 6 seconds, and the average closure should be much below this. This will require great care in the centering of instruments and targets. A discrepancy of one-third of an inch will give an error of a second in a distance of 1 mile. A transparent cloth, phaseless target will be used, the size varying with the length of triangle sides.

Base lines.—Base lines will be measured at intervals of about 75 miles. This will be done with the 300-foot steel tape. The line should be carefully staked out, and its grade determined instrumentally. Supporting stakes will be driven at intervals of 30 feet. The stakes marking the extremities of each tape will be firmly set and free from any disturbing influence due to tension of tape or otherwise. On these tapes strips of zinc will be fastened and remain until the whole measurement is completed. The temperature of the tape will be determined by three thermometers placed near the ends and in the middle of the tape. They will be attached to suitable supports and placed with their bulbs near the tape when measurements are being taken. Observers must be careful to keep sufficiently far away so as not to affect the thermometers.

The tape will be suspended in hooks at intervals of 30 feet, and attached in such a way that it may swing freely and eliminate friction as far as practicable. The tension of the tape will be kept uniform while measuring by attaching a weight of 16 pounds. The extremity of each tape length will be marked on the zinc strip with a fine line and suitably numbered. The preservation of these strips furnishes a ready means of comparison of each tape length at any future time.

The line should be measured two or more times, with a discrepancy when reduced of not more than one in 250,000. This can readily be done if measurements are made on cloudy days or at night.

Observations for azimuth.—The azimuth of each base line will be determined by observing, with a triangulation instrument, two closely circumpolar stars at elongation on two different nights.

The instrument and light should preferably be at the extremities of the base or a triangle side. The following order of observing will be used: *

First.	Second.	Third.	Fourth.
Circle right. Point to light. Point to star and note time. Read level direct and reverse. Point to star and note time. Point to light.	Shift limb 45 degrees. Circle left. Point to light. Point to star and note time. Read level direct and reverse. Point to star and note time. Point to light.	Shift limb 45 degrees. Circle left. Point to light. Point to star and note time. Read level direct and reverse. Point to star and note time. Point to light.	Shift limb 45 degrees. Circle right. Point to light. Point to star and note time. Read level direct and reverse. Point to star and note time. Point to light.

On the second night repeat this programme, starting with a reading of limb 45 degrees greater than the last reading of previous evening.

It will probably be found most convenient in these observations to use Polaris, δ Ursæ Minoris, λ Ursæ Minoris and 51 Cephei.

The time will be determined by observing the meridian passage of high and low stars.

Stone line bench-marks.—At intervals of about 3 miles along the river, lines of pipe and tile marks will be set for future surveys.

These lines will be numbered and located about as shown on maps on file in this office.

The marks nearest the river will be far enough back to be safe from erosion for many years; the others will be half a mile farther back.

In cases where the bluffs are near the river the rear marks may be omitted. The marks will preferably be placed at property corners, along public roads, or on property lines, in places where they can be readily found, and where they will not be liable to disturbance.

It is desirable to determine the azimuth and distance between the successive marks on the same line when practicable. The marks should also be as nearly in a line as the conditions of location above named will admit.

The marks will be connected directly with the secondary triangulation, where practicable, by 3 pointings from 2 or more secondary stations, and an equal number from the point to be located to 2 stations that will give a fairly good triangle.

Where the points cannot be located directly from the secondary work, a tertiary system may be used, starting and closing on a secondary line. In this work the angles may be read with a good 10-second transit, and the

* If a mercury surface be used and alternate readings be taken on the star and on the image, the bubble readings may be dispensed with, as all errors from this source are eliminated. J. B. J.

triangles should close within 15 seconds. A steel tape or chain may also be used, where desirable, in locating the point which is farthest from the river.

Cutting timber.—Cutting timber to clear the lines of sight or for material with which to build stations should be avoided as far as practicable. Where cutting is necessary, a strict account must be kept of the number of trees cut, their size, and kind of timber.

Descriptions of stations.—A minute description of each station will be made and entered in notebook kept for that purpose. This description will be complete for each station, and will show what the Geodetic point is and how marked. Its location with reference to surrounding objects will be shown by an accurate sketch giving azimuth and distance to bearing trees, houses, or other prominent objects.

A similar record will also be kept of the stone line marks.

INSTRUCTIONS FOR PRECISE LEVELS.

1. Before commencing operations the constants of the instruments will be determined. The most important of these is the value of one division of the level tube. This can best be determined by means of a level trier. It can also be determined in the field as follows:

Set up the instrument firmly, if possible mounting it on a wooden post, or, better still, on a stone pier. Set up a rod in its tripod at such a distance that it can be distinctly read through the telescope. The distance should be at least 50 metres, or if the air is very still 100 metres, and should be carefully measured. Adjust the instrument carefully, taking such length of bubble in the level tube that its ends will be about the middle or tenth graduated line on each side. Direct the telescope to the rod, and by means of the elevation screw cause the bubble to run to near one end of the level. Carefully note the position of the three wires on the rod and the reading of the level. Now, by means of the elevation screw cause the bubble to run to near the other end of the tube, and note the reading of the wire and bubble as before. One result for value of 1 division of level can then be obtained. This operation should be repeated 10 times. The elevation of the rod should be changed occasionally between sets, in order to avoid estimating the same part of the same centimetre on the rod. It will be sufficient to run the bubble 5 divisions each side of its central position.

If

k = distance from instrument to rod,

d, d^1 = distance through which eye and object ends of bubble move when run from near eye end to near object end,

$\frac{d+d^1}{2}$ = amount of displacement of bubble between 2 readings.

r, r^1 = corresponding means of 3 thread readings on rod, and

v = value of 1 division of level in seconds of arc.

Then

$$v = \frac{2\,(r^1 - r)}{k \sin 1'' \,(d + d^1)}$$

2. With the value of 1 division of the level, tables will be constructed showing the correction to be applied to a rod-reading for an observed inclination of the level, and for a distance determined by interval between extreme threads.

If the level-bubble is well ground, equal displacements of the bubble, say of 2 divisions, will correspond to equal displacements on the rod.

3. Before using the level, or determining its value, the fastening of the tube in its case should be examined. One end should be clamped down just tight enough to prevent the tube from moving easily, but not tight enough to strain the glass. The other end should be lightly clamped so that the tube may be free to expand and contract with temperature changes. The cotton packing at the ends should not exert a lateral strain on the tube. All level tubes will be numbered and have their numbers marked upon them.

4. In order to determine the inequality in the telescope rings, the instrument should be mounted on a stone pier or other firm support and carefully leveled. The level should be carefully adjusted and the instrument clamped to prevent its moving in azimuth. Now, with the eyepiece of the telescope over the elevating screw, note the reading of the bubble when level is set on telescope, both in direct and reversed position. Now reverse the telescope in the wyes, and read the level as before. Several sets of observations should be made.

Let b, b^1 = inclination of telescope as denoted by means of level readings with telescope direct and reversed, then the inequality of rings $p = \frac{b - b^1}{4}$.

Sixteen determinations of the value of p of two instruments in use on the lake survey gave probably errors of $\pm 0''.046$ and $\pm 0''.041$.

The inequality may be expressed in seconds of arc if desired, but for purposes of computation is best expressed in terms of level divisions, as it can then be combined directly with the error of adjustment of level.

5. The centering of the object glass will be examined. This may be done as follows:

Draw out the eyepiece until the threads are no longer visible. Direct the telescope upon some well-defined object, and while looking at it rotate the telescope in its wyes. If the object remains steady, the object glass is sufficiently well centered. Should the object appear unsteady, the fault can only be remedied by a maker. The objective should be firmly screwed into the telescope.

6. The values of the wire intervals will be determined as follows: Set up a rod at carefully measured distances of 10, 20, 30, to 100 metres from the instrument. Read the rod ten times at each distance. The rod may be altered in elevation, the level may be caused to change, and the telescope may be rotated 180 degrees (inverted) in order to change the position of the threads on the rod.

Taking the mean of the ten observed differences of readings of the extreme threads at each station occupied by the rod, a table will be constructed giving in metres the distance of the rod from the instrument for any observed difference of reading between extreme wires.

7. Unless the rods used have been previously compared with some

known standard, they will be compared with each other and their relative lengths determined. This may be done by establishing two fixed points, or two foot plates, at equal distances from the instrument and differing in elevation about 2.7 metres. The distance should be about 10 metres. Determine the difference of elevation of the points by reading each rod on each point. A comparison of the resulting differences of elevation will give relative lengths of metres on rods. Ten measurements with each rod will be determined. The elevation of the instrument will be slightly changed between each set in order to eliminate errors in estimating the millimetres. Each rod will be numbered and have its number marked on it. The rods should also be kept dry and provided with canvas covers to protect them while being carried to and from work.

The distance of the zero graduation above the steel spur on which the rod stands will be well determined. This may be done with a right angle triangle and rule. It may also be determined by means of another leveling rod, the graduations of which commence at the foot of the rod, by determining the height of the instrument above some fixed point and subtracting it from the reading of the rod to be determined. The relative lengths of the rods must be known.

Whenever a bench-mark is connected with in such a way that the rod is not placed directly on the bench-mark, this quantity (a) enters into the computation of difference of elevation.

8. Before commencing work at any time all adjustments will be carefully made.

(a) The telescope will be collimated by having a rod set up at a distance of 50 metres and noting the position of the wires on the rod when the telescope is normal and when inverted or rotated 180 degrees about its axis. The collimation error of the mean of the horizontal thread must not exceed 1.25 millimetres at a distance of 50 metres.

(b) The horizontality of the horizontal wires will be examined by moving the telescope in azimuth so that the rod shall appear to move through the field of the telescope. If the threads are horizontal the reading on the rod will be the same, the position of the level, which should be closely watched, remaining the same. If the threads are found to be not horizontal they will be made so by turning the telescope a small amount in the wyes. When the thread wires have once been made horizontal, small screws which abut against projection of wye above elevating screw should be so adjusted that when they press against this projection the wires are horizontal. If the vertical thread is then inclined, as shown by the plumb line attached to the rod, it must remain so.

(c) To make the axis of the level parallel to the upper surface of the rings, it is necessary to make the vertical planes passing through them parallel (lateral adjustment), and to make them equally inclined to the horizon (vertical adjustment).

To make the lateral adjustment, raise the clips fastening the level to the telescope, and revolve the level about the telescope a short distance each side of the vertical. If the bubble runs in opposite directions when on opposite sides of the vertical, the level is to be adjusted by means of the opposing horizontal screws at one end of the level until such is not the case.

To make the vertical adjustment, raise one of the clips and read the level in its direct position and also when it is reversed on the telescope. The difference between the differences of end readings in each position is four times the error of adjustment, and is to be corrected by the opposing vertical screws at one end of the level case. The error of adjustment must not be allowed to exceed two divisions of the level. Care must be taken that the telescope rings are free from dust while adjusting the level. After having made the vertical adjustment it will be necessary to examine the lateral adjustment again, since making one of these adjustments affects the other.

(*d*) To make the level and vertical axis of revolution perpendicular to each other, loosen the small clamp screw at one end of the horizontal bar fastened to the vertical axis and by means of the elevating screw raise or lower that end of the upper horizontal bar until the telescope can be rotated 180 degrees from any position and have the level reading the same in both positions.

(*e*) To adjust the level attached to the rod, set up the rod in its tripod in such a position that when a plumb line is attached to the small hook near the top of the rod, the point of the plumb bob shall coincide with the point of a small cone attached to the rod near its foot. Now bring the level bubble to the center by means of the leveling screws. In making this adjustment the rod should not be exposed to the wind, as the plumb line is influenced thereby. This adjustment will be made at least once each day.

Each time that the instrument is placed on a station, its axis will first be made vertical by means of the leveling screws in such manner that the telescope may be turned around the horizon without the bubble of the level running a great number of divisions. The telescope is finally made horizontal by means of the elevating screw. The inclination at the moment of observing must not ordinarily exceed three divisions of the level, and never five divisions.

The instrument when in use ought always to be sheltered from the sun and wind. It is carried from station to station without being dismounted, but the level should be taken off and carried in the hand. The small clamp screw at the end of horizontal bar, and the large screw which fastens the instrument immovably to the tripod, should both be turned tight before moving the instrument.

The rods must be placed on the plates which accompany them and held in a vertical position as indicated by the spherical level attached. It is advisable to always use the same rod with the same foot plate. In placing the foot plates great care should be taken that they be horizontal, on firm ground, and not liable to change. The surface of the ground, if not firm or level, should be removed.

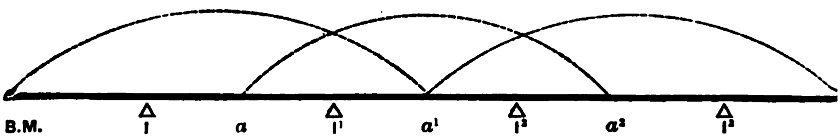
The errors of adjustment will be determined at beginning and end of each series of observations; that is to say, after having mounted the instrument and before dismounting it, and in all cases at least once each day. If the instrument has been deranged by a jar the corrections must be determined anew.

The error of collimation will be determined by two readings of the rod at a distance of 50 metres when the telescope is in its normal position and two when it is rotated 180 degrees in the wyes. The difference between the means of the two readings, after being corrected for the inclination of the level, must not exceed 2.5 millimetres at that distance, and commonly should not exceed 1 millimetre. The error of the adjustment of the level (inclination) will be determined by reading the level four times when direct and four times when reversed on the telescope, reversing it between each reading.

The error of adjustment must not exceed two level divisions, and commonly should not exceed one. All the details of the determination of the errors of adjustment must be entered in the note book in their proper place. It is always advisable to have the errors of adjustment as small as possible, and necessary that they be well determined. The time of making these determinations will be recorded in the note book.

In all work along the main line of levels each observer will duplicate his own work by running over the line in opposite directions, preferably under similar conditions as to illuminations, etc.

While connecting two bench-marks the order of using the rods will be as follows :



In the above figure let $I\,I^1$, I^2——etc., represent the successive stations occupied by the instrument. *B. M.* a^1, a^3——etc., the positions occupied by Rod 1, and a, a^2——etc., the positions occupied by Rod 2. The instrument having been set up at *I*, Rod 1 is placed on *B. M.* and Rod 2 at *a*, making the distance *I—a* equal to *I—B. M.* Rod 1 is then read, and immediately afterward Rod 2. The time elapsing between these readings commonly will not exceed one minute and should not exceed 5 minutes. The instrument is then carried to I^1, and Rod 1 to a^1, the distances a—1^1, and 1^1—a^1, being equal. Rod 2 will then be read, and immediately afterward Rod 1.

The instrument will then be taken to I^2 and the rods read in the order 1, 2. Work will be continued in this manner until the other bench-mark is reached. Rod 1 must be placed upon this bench-mark, which will be the regular order if there have been an even number of instrument stations. If there have been an odd number of instrument stations, at the last station use Rod 1 for both backsight and foresight. While leveling the rate of progress in favorable weather will be about one kilometre per hour.

After having properly leveled the instrument at any station and having made the vertical thread coincide with the center line of the rod, the observation will be made and recorded in the following order : * First the level will be read, the tenths of the division being estimated ; then the position

* It is preferable to keep the bubble in the center while threads are being read.

of the threads on the rod will be read, the millimetres being estimated; and finally the level will be read again. The observer will then read the rod a second time to make sure that no error has been made. The recorder will then take the differences between the readings of the middle and extreme wires to guard against errors, and if these differences denote any error the observations must be repeated. If an error exists it will be shown by too great a difference between the differences. This is a most important check and must not be neglected. These differences will also serve as a check upon the distances between the instrument and rods.

The recorder should also check the level readings to make sure that errors of whole divisions have not been made. This may be done by summing up the readings and noticing the length of the bubble. In reading the level by means of the mirror care should be taken that the position of the eye is such that there will be no parallax. Such positions can be determined once for all when the mirror is at its greatest angle of elevation, by a second person reading the level directly while the observer finds the position from which the reading of the level in the mirror is the same. The notes will be kept in the form given in note books. When once a number has been written down it must not be erased or made illegible. If wrong a line will be drawn through it and the correct number written above.

The lengths of sights taken will depend upon the condition of the atmosphere, but the rods should always be near enough to be seen distinctly. It will be seldom that lengths of sights greater than 150 metres can be taken. The backsight and foresight corresponding to any instrument station must not differ in length by more than ten metres, and the sum of the lengths of the backsights and foresights between any two bench-marks should be equal.

Whenever it is necessary that the line of levels should cross a river or other wide obstruction, a narrow place should be chosen. Firm points should be set upon the two banks; levels in good adjustment are set upon posts about 10 metres from each bench-mark, and both levels go through the same operation.

The error of adjustment is first accurately determined.—Call one of the levels *A*. *A* first reads on the bench-mark near it, once with the telescope normal and once with the telescope inverted, and then on the rod across the river five times with the telescope normal and five times with the telescope inverted. The error of adjustment of the level is again accurately determined. The rod across the river will need an extra vane. *B* performs the same operation simultaneously. *A* and *B* change places and repeat the observation at these new stations. The simultaneous levels eliminate refraction, the change of station eliminates curvature and small instrumental errors. Unless good results are obtained the levels should be repeated. If but one level can be used the operation will be performed in the same order, but the time occupied in crossing must be as small as possible. With a single Kern level this process has given for a river 815 metres wide five results, the mean of which has a probable error of $\pm 0^{mm}.5$. (Ohio River, Cairo, Ill.)

Permanent bench-marks will be established at intervals of 3 miles along the river and 5 miles on lines connecting the river line proper with the other levels or bench-marks.

These bench-marks will consist of a thoroughly verified tile 4 inches by 18 inches by 18 inches placed 3 feet below the surface of the ground and surmounted by a 4-inch wrought-iron pipe as a surface mark. The tile should have time to settle before leveling to it. Both tile and pipe will be suitably marked to designate the character of the point. In the center of the upper surface of the tile a copper bolt will be leaded, the upper surface of which will be the point of reference. These bench-marks will be placed where they can be easily found and where they will not be disturbed. Property corners should be utilized where practicable.

In addition to the above, benches should be established on permanent brick or stone structures by leading into them a horizontal copper bolt, with the letters *U. S. P. B. M.*, and the number of the bench-mark cut near it. A small hole in the center of the bolt will be the point of reference.

In connecting with a bench-mark if the bolt is vertical the foot of the rod is placed directly upon it. If the bolt is horizontal in the wall of a building or other structure, it may be best connected in the following manner: Set up the instrument in such a position and at such an elevation that the small hole in the bolt may be bisected by the middle thread without displacing the level by more than five divisions, using the elevating screw for making this bisection. Since the instrument can be raised or lowered about two centimetres by means of the leveling screws, the instrument can be placed in such a position by two or three trials.

Now bisect the bench-mark with the telescope normal and also inverted, noting the reading of the level. Read the rod on the plate with the telescope in both positions. It is necessary to eliminate collimation by inverting the telescope, since the collimation of the middle wire is not the same as that of the three wires. The quantity A (distance of zero above foot of rod) must be taken into account when a bench-mark is connected with in this manner. The distance of bench-mark from instrument must be determined and recorded.

Whenever work is stopped at least two temporary bench-marks should be established. These will consist of large nails or spikes driven their entire length vertically into the base of trees, or in the tops of sound stumps.

When not in the vicinity of trees or stumps, wooden posts may be firmly set in the ground with their tops flush with the surface and nails driven into them. When near the river, temporary bench-marks should be set every two kilometres. Every bench-mark will be fully described in a note book kept for that purpose. Its position with reference to the most prominent objects near it should be given by distance and direction. Public buildings, such as depots, court-houses, churches, etc., are the best positions for permanent bench-marks. In a village or town several permanent bench-marks should be established to secure some one against loss.

If a railroad is crossed the elevation of the foot of the rail will be determined, and if leveling along a railroad, the elevation of the foot of the rail at depots will be determined.

The elevation of the zeros of all water gauges and also the gauge bench-marks will be determined.

The datum planes of cities along the line of levels will be connected with and their elevations deduced.

Frequent connections will also be made with the United States Engineer bench-marks between St. Paul and Grafton.

In reducing the observations the nearest tenth of a millimetre will be retained. The distance will be taken out from the table to the nearest metre.

The limit of discrepancy in closing a polygon will be—

$$3^{mm} \sqrt{\text{Distance in kilometres.}}$$

The distance referred to is the entire length of the polygon from bench-mark 1 to bench-mark 2 and back to bench-mark 1, and the limit of discrepancy refers to the polygons between successive bench-marks. If the discrepancy exceeds the prescribed limit, then the entire polygon must be re-run one or more times, or until the difference of the means of the direct and reverse results is within the limit.

The notes will be kept in the following form :

BACK-SIGHT.

[Left-hand page.]

Thread readings.	Mean.	Difference of threads.	Level. Eye.	Level. Object.	Rod	Remarks
7.95	....	207	....	......	......	
10.02	1001.7	206	11.1	11.1	12	
12.08	....	413	....	....	....	

FORE SIGHT.

[Right hand page.]

Thread readings.	Mean.	Difference of threads.	Level. Eye.	Level. Object.	Rod.	Remarks.
18.69	....	187	......	......	....	
20.56	2055.7	186	11.4	11.4	1[illegible]	
22.42	....	373	...	......	..	

INSTRUCTIONS FOR TOPOGRAPHICAL AND HYDROGRAPHICAL FIELD WORK.

The objects of the survey of the Mississippi River are to obtain sufficient data for an accurate topographical and hydrographical map which may be used in studying the physical characteristics of the river, planning improvements, and also serve as a basis for future surveys, by means of which the changes in bed and banks may be ascertained and their causes and effects studied. The importance of having the work accurately done and the information embodied therein reliable is therefore apparent.

The experience derived in the surveys from Cairo to Donaldsonville, covering a period of several years, suggests the following instructions relating to the scope of the work and the methods to be employed. Other points will suggest themselves as the work progresses and new difficulties are met with.

General instructions.—A record will be kept showing the daily progress of the party. It will contain at the beginning the organization of the party, and the names and rates of pay of all persons connected with it. It will also give a detailed account of all occurrences of any importance which may in any way be of use in reducing the work or in settling accounts.

At the beginning of each day's work each note book in use will give locality of work, date, name of observer and recorder, number of instrument used, and corrections, if any, to readings of distance and azimuth.

In recording notes hard pencils will be used, and when an entry has once been made it should never be erased. Where an error has been made the record will be corrected by drawing a line through the first value and writing the new value above it. Corrections that are made after the work is done should be marked with the date of the change and the name of the person making the change.

All notes should be so full and plain that they could be readily reduced by one who has not seen the ground. This will require careful attention to details which may seem of trifling importance in the field.

All available information concerning the river and its adjacent banks which will aid in the proper representation of the characteristic features on the map or be valuable in the study of their changes will be fully noted.

Local names of bars, bends, streams, or other features will be carefully noted and the proper spelling of all names to appear on the map will be ascertained.

Permanent marks, as reference points for future surveys, will be established at intervals of about 3 miles along the river. There will be two on each side of the river nearly in a line, normal to the stream. The two nearest the river will be placed where they will be safe from the erosion of the banks for 20 years or more, and the others will be a half mile further back. Where the bluffs are near the river the outer marks may be omitted. These marks should, when practicable, be placed near roads, property lines, or other places where they can be easily found at any time. The marks will consist of flat tiles bearing wrought iron pipes (see instructions for secondary triangulation), the tops of which should project not less than a foot above the ground.

Note books will be fully indexed at the end of each day's work. Each note book will be marked on the outside with a title giving locality of work, date, names of chief of party and observer. All note books will be entered on the office files and properly numbered as soon as parties return from the field.

The chief of party being responsible for the accuracy of the work done, should see that the work of each member of the party is properly checked and fully covers the ground required.

Tertiary triangulation.—Where the secondary stations are more than 3 miles apart a tertiary system will be carried giving points on either bank at intervals of a mile or less. This system will begin on a triangle side of the secondary system or a carefully measured base, and all of the available secondary stations will be used in the tertiary chain. The tertiary work will also close on a line of known length as a check on its accuracy. The discrepancy should not exceed 1 in 3,000. The system should be laid out and the angles read in advance of the topographers, so that the azimuths and lengths of sides can be used in checking stadia work.

The station point may be marked by a pole 2 inches in diameter stuck into the ground, and bearing a red and white flag to distinguish it from the ordinary sounding flags. A strip of white cloth wrapped near the bottom of the pole will admit of the pointings being made so low down that errors arising from disturbance of the pole by the wind will be inappreciable.

For observing, the instrument may be placed on an ordinary tripod centered over the hole after the pole has been removed.

The angles should be read with a 10-second instrument in good adjustment, and should be repeated at least three times on different parts of the line to check errors of reading.

It is desirable to have the first series read on azimuth. Having pointed to the first station, read to all of the others in succession. Pointings should also be made to all of the sounding flags in the vicinity, as well as prominent objects on land, such as chimneys, houses, etc., the location of which will serve to check the topographical work.

For the second series slip the lower limb 60 degrees and read to the stations in the opposite order from the first series. Slip the limb the same amount again and read the third series.

The river ends of the stone lines will be made points in the tertiary system, and whenever practicable the stones should also be located trigonometrically.

Tertiary points which are likely to remain undisturbed for some time should be plainly marked with a strong stake 2 feet high, the number of the point, the initials of the observer and date being marked on it with red chalk.

Topography.—The detailed topography will cover a belt on each side of the river, which, in wooded country or on the bluffs, will be about one-half to three-fourths mile wide and in open country may reach about $1\frac{1}{4}$ miles. In this area there will be located, with transit and stadia, all points needed to plat accurately the important features on a scale of 1:10,000. In all work the scale of the plat should be borne in mind, so that only such points be instrumentally located as can be readily platted.

Beyond the above limits outline surveys will be made defining streams, lakes, and the foot and main crests of bluffs with approximate elevations of same within a limit of 10 miles of the river. This work will be run with the transit or compass and stadia, and will frequently be connected with the detailed topography.

Within the limits of the detailed area there will be located the top and bottom of the river bank proper, the shore line of islands and bars, the banks and water lines of all waterways and lakes, with elevations of their water surfaces and depths, the points where the slope of the ground changes either in direction or inclination, the limits of rock ledges, the approximate limits and kinds of cultivation, and forests, roads, levees, fences, houses, etc., and in fact everything that may be necessary to a truthful representation of the section surveyed.

A sufficient number of elevations will be determined on the bottom lands to admit of putting in contours 5 feet apart. In a wooded area this will require cross sections at intervals of 500 metres or less. These should preferably be the continuation of lines sounded across the river. The space between the lines should also be examined and if any important features are found they should be located.

When the trees are too close together to admit of long sights it will be more expeditious and sufficiently accurate to use the compass needle for obtaining the direction, as it will then only be necessary to set up at alternate stakes.

The bluffs within the detailed area will be shown by contour lines 20 feet apart, the bluff curves being all some multiple of 10. The bluffs in the outline area may be shown by hachures.

Boundary lines, such as State, county, township, etc., coming within the limits of the survey, will be carefully located.

Section or township corners, where they are well identified, will also be connected with.

Great care must be taken in running out the stone lines. The azimuth of the lines must be accurately determined, and all distances will be carefully read, both forward and back. Each stone will be occupied instrumentally, and when practicable, the azimuth from one stone to the next will be read, and readings will be made to surrounding objects both as checks on the located positions of the stones and to aid in finding them in future time. A careful sketch and minute description will be given of each bench-mark thus located.

All sounding flags, water gauges, and bench-marks will be located.

In running the main transit line along the shore sufficient check shots will be made to known points on the opposite shore, to prove the accuracy of the positions given for the transit stakes. Such check shots should in fact be made use of in all parts of the work, so that errors of azimuth and distance may be detected and located. They also furnish means of correcting errors of position if any occur.

The error of level carried with the transit should never exceed one foot for the longest distances. In good work the discrepancies will rarely reach 0.5 feet. The work will be frequently checked by starting and closing on points whose elevations are known.

Distances and vertical angles between transit stakes will be read from each end of the line. On transit lines the distances between stakes, read with stadia, should never exceed 500 metres at a single reading unless they can be checked by intersections or other means. Single shots to distant objects may be read as far as the figures of the rod are distinguishable.

The transit stakes of each observer will be numbered consecutively in the same reach and each stake will be marked with its proper number and the initial of the observer so that it can be readily identified when connected with by others.

Careful sketches will be made in the field of the entire area surveyed and the located points will be indicated on the sketches and numbered to correspond with the pointings in the notes so there will be no difficulty in connecting the points properly on the field plats. The character of the immediate river bank will be frequently noted so as to show whether it is rock in place, loose rock, sand or silt, steep or sloping, caving or stable.

Checks on azimuth and elevation should be frequent, and when obtained should be marked in the notebook in such a way that the amount of error will be plainly shown and where the correction should be applied. Notes that are not full in this particular will always be open to suspicion which will throw doubt on the observer's honesty and the reliability of his work.

Discrepancies in closing on triangulation points should never exceed $0^{\circ}\ 05'$ in azimuth or 1 in 500 in distance. As a rule the discrepancies should be far within these limits.

The notes will be kept in the following form on the left-hand page of the notebook, the other page being reserved for reductions, sketches and remarks.

[Left-hand page.]

[Vicinity of Phillips Landing, Nov. 3, 1882. Inst. Wurd, No. 154—Dis. Short 1 in 100, Az. Cor, F. B. Maltby, observer. F. P. Gibbs, recorder.]

Objects.	No. of pointing.	Ver. A.	Ver. B.	Distance.		Vertical angle.
				Read.	Corrected.	
		At Δ 1. ° ′	° ′			′
Δ Everett..........	1	95 10	275 10	100	101	+0.10
Top of bank...	2	127 16	....	250	252	+0.8

[Right-hand page.]

Diff. of elev.	Elev.
....	290.3
+0.96	291.2
+1.9	292.2

Ordinary levels.—There will be a line of levels run along each bank of the river, the ordinary Y level being used for that purpose.

All turning points will be numbered so that the topographers can readily identify the points connected with.

The ordinary level lines will connect with all precise bench-marks in their vicinity.

The errors of closure should not in any case exceed 0.2 feet for the longest intervals. The two lines will check on each other at intervals of not more than 3 miles or in the vicinity of each stone line.

The elevations of stone-line bench-marks will be determined by duplicate lines of levels, the discrepancies between which should not exceed 0.05 feet. The adopted elevation will be the mean of the two determinations.

The elevations of all permanent stations near the river, except those on the bluffs, will be determined. Bench-marks will also be established on each bank at intervals of about a mile. These may be placed on buildings, trees, or other permanent objects near the river. A careful description, sketch of location, and corrected elevations of all bench-marks will be made and entered in a book kept for that purpose. These notes should be so full as to enable one not familiar with the ground to find the marks even after the lapse of several years' time.

All water gauges will be connected with by duplicate lines of levels from the nearest bench-marks and the elevations of their zero points entered in the gauge book. The elevation of the zero should be tested whenever there is a probability that the gauge has been disturbed.

The elevations of the water surface will be determined at the extremities of sounding lines at intervals of not more than 400 metres, and the time of the observation will also be entered in the notebook. This, when corrected for change of stage, as shown by the local gauge readings, will give the slope, and also serve to check large errors in leveling.

Elevations of transit stakes, high-water marks, and surface of ground at sounding flags will be determined whenever it is practicable.

Level notes will be kept in the following form on the left-hand page of the notebook, the other page being reserved for sketches and remarks.

[R. B. near Grand Tower, December 12, 1884. Inst. B. & B. 140. M. Greenwood, observer.]

Stations.	B. S.	Ht. Inst.	F. S.	Elevation.
T P. 113	4.206	340.527		336.321
⊙ 51			1.400	339.127
T. P. 114	4.031	341.399	3.159	337.368
Δ 28			2.380	339.019

River crossings for connecting the two lines of levels will be made by the two observers taking ten simultaneous readings across the river in opposite directions. Then the observers and instruments should change places and repeat the observations.

The instruments should be in good adjustment, and when once focussed for the long distance should not be changed until the observations are completed. The mean of the values thus determined will be taken as the true value.

*Hydrography.**—A continuous record of the stage of the river will be

* For smaller rivers the entire survey, including location of soundings, may well be done by the stadia. See paper by J. L. Van Ornum, C. E., in *Journal of the Association of Engineering Societies*, vol. xiv., p. 219.

derived from a suitable gauge read three times a day, its zero being referred to a known bench-mark as soon after it is set as practicable.

Sounding lines will be run normal to the stream at intervals of 250 metres, and the soundings on these lines will be as close together as practicable. These lines will be numbered consecutively.

A continuous longitudinal line passing through the deepest water on each section will be sounded.

On all crossings there will be sufficient soundings to determine the least channel depth between the pools.

As many soundings as practicable will be located by means of angles read simultaneously between located points on shore, with two sextants in the sounding boat. Intermediate soundings can be interpolated by taking them at equal intervals of time.

The character of the bottom will be frequently determined by means of a tallowed lead.

In water less than 10 feet deep it is convenient to use a pole divided to feet and tenths. A 10-pound lead is suitable in water from 10 to 40 feet deep. In greater depths it is desirable to use a lead weighing from 15 to 20 pounds.

A firmly twisted or braided hemp lead line three-eighths of an inch in diameter should be used. It should be marked with leather or cloth tags at intervals of one foot, the 10-foot marks being made conspicuous. The length of the lead line from the end of the lead to each 10-foot mark must be tested at the beginning and end of each day's work and the result entered in the notebook. The lead line should be accurately marked, so as to avoid corrections as far as practicable.

The beginning of each line sounded will be headed in the notebook with the number of the section. A description of the character of the banks at intervals of about half a mile will also be entered in the notebook.

The elevation of the water surface at the time of sounding may be determined for each line by means of the levels as already described under the head of ordinary levels.

The notes will be kept in the following forms:

Sounding in the vicinity of St. Louis, November 15, 1889.

[E. L. Harmann, G. W. Wisner, observers; D. E. Perkins, recorder; J. Stott, leadsman.]

[Left-hand page.]

Time.	Observed depths.	Angles and ranges.	Character of bottom.

[Right-hand page.]

Correction. Lead line and stage.	Corrected depth.	Remarks.

The angles read to locate soundings will be numbered consecutively for each day's work, and the soundings located will be marked with corresponding numbers.

Computing and platting.—The coördinates of all tertiary stations and stone-line bench-marks will be computed, and, together with the secondary stations, will form the basis for platting the topographical detail. The results of these computations will be kept in suitable form and preserved for future reference.

The work will be platted on a scale of 1 : 10,000. The field plats will be 26 by 30 inches in size, on which, near the center, will be printed a 12-inch circle divided to 15-minute spaces to facilitate the platting of polar coördinates.

Parallels and meridians, 1 minute apart, will be projected on the field plats and shown by fine red lines properly numbered. From these the △ stations will be platted. As this is the ground work for subsequent detail it should be carefully done and checked over to insure its accuracy. All △ stations, stadia stakes, and sounding flags and their elevations should be marked on the plats in red ink before the detailed work is put in.

All of the detail must be carefully platted and positions verified by check shots when such are available.

The contour lines and other outlines should preferably be put in by the observer who located them in the field.

Field plats will be laid out in such a way as to show both banks of the river with the adjacent topography on the same sheet whenever it is practicable. If the sheet is not large enough, plat the remaining work on a new sheet rather than enlarge by pasting pieces to the first sheet.

Banks that are too steep to admit of drawing in the contours, and abrupt banks of less than 5 feet in height, will be shown by hachures.

The elevations of water surfaces for each day's work will be plainly written on the plats.

All field plats must be completed in the field at least far enough to detect any instrumental errors in the field work before the platted area is out of reach. Hard pencils will be used in platting.

Each field plat will bear a legend giving locality and date, the names of the chief of the party, the observers and draftsman, the numbers and pages of the notebooks from which the notes were derived, and any other information that may be useful in the final reduction of the work. This data should be noted as the work is platted.

Care must be taken at the edges of the sheets to have the detail on successive plats join properly, and make sure that the ground is fully covered by the survey.

Nomenclature.—The word lake will be confined to the larger bodies of water, which are seldom, if ever, dry. They usually have a local name which should always be noted. Smaller and temporary bodies of water having no local names will be called ponds.

The word swamp will be applied only to ground which is covered with a growth of grass, cypress, elbow brush, willows, or such other vegetation as indicates that the area is generally wet, soft, or spongy.

The terms bayou, or creek, will be applied only to main watercourses which connect lakes and swamps or other drainage areas with the river and carry water to or from the latter, as the stage varies.

Minor swampy conduits will be called sloughs. This applies only to such as are not designated by local names.

The character of the material composing the bars and banks of the river will be frequently noted and carefully described.

The names of property owners or residents, of landings, wood-yards, fields, patches of timber, islands, chutes, bends, bars, points, and other local names necessary to a full description of the section surveyed will be fully noted and entered on the field plats. The following signs and abbreviations will be used: secondary stations, ◬; tertiary stations, △; transit stakes, [·]; sounding flags, ⊙. Turning points in leveling notes will be written T. P.; bench-marks, B. M.; temporary bench-marks, T. B. M.; and precise bench-marks, P. B. M.

On the field plats the precise bench-marks, with their numbers and elevations, will be written thus: P. B. M. 27 ⊙ 218'.032; stone-line bench marks thus: B. M. $\frac{10}{3}$ ⊡ 219'.23, in which the numerator is the number of the stone-line, and the denominator the number of the stone on the line reckoned from the outer stone on the left bank.

The stone lines will be numbered consecutively up stream, beginning with number 1, near Cairo, Ill.

All elevations in the topographical work will be referred to the Memphis datum plane.

To reduce elevations from the Cairo datum to the Memphis datum subtract 13.13 feet from Cairo datum elevations.

The approximate mean Gulf level is 8.13 feet above the Memphis datum plane.

APPENDIX G.

THE OWNERSHIP OF SURVEYS, AND WHAT CONSTITUTES A SURVEY AND MAP.*

A GREAT difference of opinion seems to exist among surveyors as to how much of the information obtained and work done in making a survey should be furnished to the individual for whom the survey is made. It is believed that some surveyors have mistaken notions as to what constitutes a survey and map and as to the ownership of the same. Many surveyors keep what are called private notes. All men doing business as surveyors must keep notes of all surveys in a convenient form for ready reference. The extent to which these notes are "private," however, seems to the writer not to have been fully comprehended by all surveyors, and hence has arisen the difference of opinion mentioned.

The present article is an attempt to present a side of this question that has not, so far as the observation of the writer has extended, been heretofore fully considered. An endeavor has also been made to point out to the young surveyor a line of action which it is believed he will find to his advantage to follow, as well as to that of the community in which he works.

In this discussion, the question of what constitutes a survey arises at once, and the answer obviously depends on the object of the survey. The discussion will be confined to land surveys; that is, surveys made for the purpose of subdividing a large tract of land into smaller parcels to be sold, or surveys made for the purpose of determining the boundary of a tract the description of which is known, or surveys made to determine the description when the boundaries are known.

The principle to be enunciated applies to any other survey as well, be it railroad, canal, bridge, or topographical survey. Indeed, it is well understood in all such surveys, but seems to be ignored by many engineers having to do with land surveys.

A survey is "the operation of finding the contour, dimensions, position, or *other particulars* of any part of the earth's surface, . . . tract of land, etc., and *representing the same on paper*."

In making a survey it is necessary to set certain points, called monuments or corners, and to determine a description of these points. These items therefore become a part of the survey.

Then a map must be drawn. This map, to be a faithful representation

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* By William G. Raymond, C.E., Professor of Geodesy, Road Engineering, and Topographical Drawing, Rensselaer Polytechnic Institute. The original paper, of which this is a slight modification, appeared in *The Polytechnic*, the student Journal of the Rensselaer Polytechnic Institute, January, 1894.

of the ground and the work done, should, together with the notes, show all of the items mentioned.

The object of establishing monuments or corners and describing them is a double one, viz.: the marking on the ground of the boundaries of the tract, and the securing of definite information as to the location of the tract with reference to other points or tracts, so that from this information the land may at a future time be found.

The survey is not complete, therefore, till the corners are fixed, information that will preserve their location obtained, and the same delineated on a map and its accompanying notes.

The doing of all this, then, constitutes the survey. To whom belongs the survey?

It would appear to be evident that it belongs to the individual who pays to have it made. It is not readily seen in what way the survey, or any part of it, becomes the sole property of the surveyor.

The surveyor may keep a copy of his notes to facilitate his future work, but he has not the shadow of a claim to a single note the time for taking which has been paid for by his employer.

If his charge for his work is on a time basis, there can be no question as to the correctness of the above proposition. If he takes the work for a definite sum for the entire job, he may take as much time as he likes and as many private notes, but he is bound in honor to return to his employer the complete survey, and if he does so, it is not obvious that the private notes would thereafter be of great assistance to him in securing further work, particularly when it is remembered that professional men of repute do not bid against each other for professional work.

His reputation for accuracy and honesty will be a far more potent factor in securing employment than any set of private notes fairly obtained.

It is true that a great many surveyors hold a different opinion, and purposely return their maps and notes in such condition that, while they may answer the purpose for which they are primarily made, they do not tell the whole story, nor enough to make it easily possible for another surveyor to re-locate the tract surveyed. When this is done, the person ordering the survey does not receive what he pays for. Something is withheld. It seems to need no argument to show that this is radically wrong.

But there is another reason for condemning this practice.

The correct and permanent location of all public land lines, as streets, alleys, etc., as well as the permanent location of party lines between private owners, is a matter of the gravest importance, and no information that will at all serve to definitely fix such lines in their correct positions for all time should be withheld from the owner who pays for the survey, be it private citizen, municipality, county, or State.

The records of monuments and street lines made by a city engineer are no more his private property than are the records in the offices of the clerk, auditor, or treasurer the property of the individuals who hold office at the time the records are made.

The correctness of the position assumed has been indicated by court decisions. A great deal of laxity is shown in the conduct of offices of city engineers and county surveyors.

The methods of regulating the pay of these officers has doubtless had much to do with this. It is not uncommonly the case that the surveyor receives no salary, but is allowed to collect certain specified fees for work performed, and this gives color to his claim that his work is private work and belongs to him.

That this is not true concerning the public work he does, is believed to be evident from what has preceded. That the records of work done for private citizens are not the property of the public, needs no demonstration, but it is true that such work belongs to those citizens for whom it was done.

The writer believes that a different policy should be pursued with regard to these offices.

He believes that in every case such office should be a salaried one, with such salaried assistants as may be necessary, and that certain fees should be prescribed for performing the various kinds of work that the surveyor may be called upon to do within the limits of the territory of the political division whose servant he is. These fees should cover all work connected with public construction or public or private land lines, and should be returned to the public treasury.

Their amount may be regulated from time to time so that they shall aggregate a sum sufficient to pay the expenses of the office. They should, of course, not cover work of a private character, not having to do with land lines. But the entire public is interested in the permanency of land lines, and all records concerning them made by a public official should become public property. The writer has had in the past some experience in this class of work, and never declined to furnish a competitor with any information in his possession that would help the competitor to arrive at the truth in surveys he might have under way.

The writer believes that the permanency of land lines is too important a matter to be subject to avaricious and jealous rivalry, and he believes that all the surveyors in a given district should coöperate to preserve in their correct places all lines within the district.

To this end the returns of every surveyor made to the owner should be thoroughly complete. Maps made for filing as public records should be so finished as to enable any surveyor to re-locate the land without the least uncertainty as to the correctness of his work. That this is done in very few instances is well known to every surveyor who has had occasion to examine public records for data for surveys he has been called upon to make.

Because of the fact that in most cases neither owners nor attorneys have been fully posted, nor could they be expected to be, as to what constitutes a complete description, sufficient for re-location, and because surveyors have been willing to let matters stand as they were, great carelessness has arisen in the practice of making and filing maps for record.

* While in some States good laws exist prescribing what shall appear on a map before it will be received as a public record, in more States

* What follows is a modification of some notes on this subject prepared by the writer for the Technical Society of the Pacific Coast.

there is nothing whatever to guide either owner, surveyor, attorney, or recorder in the matter. In the county records in such States anything that is made up of lines and figures, either drawn by hand, photo-lithographed, or simply printed with "rule" and type and labelled "This is a map," is considered a sufficient basis for the correct description and location of the property it purports to represent. The records are full of auctioneers' circulars, manufactured in a printing-office from information, coming from nobody knows where, filed at the request of the auctioneer's clerk, with no name of owner or other interested party attached, except as the name of the auctioneer appears in the accompanying advertisement. Further than this, these maps are frequently purposely distorted to create a favorable impression of the property to be sold. Wide streets are shown where only narrow ones exist, streets opened for the full width where they have been opened for but half their width, rectangular subdivisions that really may not be even parallelograms, etc., etc. Such maps as these frequently form the only basis for the description and location of the property they are supposed to represent. This circular business is bad, *very* bad for those who buy; but is the information given by these circulars much worse than that furnished by many of the maps made by surveyors and filed at the request of the owners?

On these plats, if of "additions," we find lines indicating the boundaries of blocks and lots, all of which blocks and lots are numbered; the names of streets appear in neat letters; a few dimensions, possibly all linear dimensions, will be given; the streets or blocks may be tinted with soft and delicate tints, and the whole set off with an elegant border and title.

As an exhibition of the draughtsman's skill these maps are perhaps valuable. As a source of information as to the location of the lines they purport to show, they are worth about as much as the auctioneer's circular.

Perhaps they have a few more figures, and the presumption may be a little stronger that the figures are correct.

Examine one of these maps closely. There will be found no evidence that a monument has been set in the field; not an angle recorded, though the lines may cross at all sorts of angles; and dimensions given that do not agree among themselves, so that the angles cannot be calculated.

There will be found no name signed, except, possibly, that of the surveyor, who thus advertises either his stupidity or something worse. Let us be kindly, and call it stupidity.

Frequently no monuments are set except small stakes at the corners of the blocks; but the fact that even such stakes have been set is not recorded on the plot.

One who is acquainted with the practice of surveyors in a given district knows at what points to look for such stakes, and if they have been set and not pulled out to make room for a fence post or building, he may succeed in finding them. Some surveyors have a practice of setting stakes a certain distance away from the point the stake is supposed to mark, but no mention of this fact appears on the map. In fact the map is so drawn that no one but a surveyor who made it can write a description of any one of the parcels of land shown, nor correctly locate it on the ground. Furthermore, the surveyor himself finds it impossible, after the lapse of a few years and the destruction of his "private marks," to re-run any one of the lines exactly as originally laid out.

It is easy to see to what this leads: impossible descriptions of property giving opportunity for differences in judgment as to interpretation of what was intended; disputes as to position of party lines; costly litigation and expensive movement of structures begun or completed, and the actual shifting of lines back and forth by different surveyors, or even the same surveyor, honestly trying to locate the lines properly.

The writer has seen enough of trouble of this sort to indicate to him that a radical change is needed in the field work and mapping of cities, towns, and additions, not to mention farms and other tracts of land that it may be necessary to lay out and describe.

So long as fallible man is responsible for the accuracy of surveys, maps, and descriptions of properties, so long will there be errors; but that it is possible to greatly reduce their number by proper regulation, the writer is fully persuaded. What we have been describing are not maps at all, or at most they are very imperfect maps, and "What constitutes a map?" thus seems to be a very pertinent question.

A map of a city, town, or addition, or other tract of land, serving as a basis for the description of property, should furnish all the information necessary to the proper description and location of the various parcels shown, and also of the whole piece. It should further show the exact location of the whole tract to the lands immediately adjoining; particularly should this be done when an offset or angle in a street line occurs. To accomplish these things there should appear on the map the following items:

1. The lengths of all lines shown.
2. The exact angle made by all intersecting lines.
3. The exact position and character of all monuments set, with notes of reference points.
4. The number of each block and lot.
5. The names of all streets, streams or bodies of water, and recognized landmarks.
6. The scale.
7. The direction of the meridian, and a note as to whether the true or magnetic meridian is shown—it should be the true meridian.
8. The angles of intersection made by the lines of adjoining property with the boundaries of the tract mapped.
9. The exact amount of offset in lines that may extend from the outside through the tract mapped.
10. A simple, complete, and explicit title, including the date and the name of the surveyor.

Thus much to make the map valuable for *description and location* of the property it represents.

Of course monuments will not be shown if none have been set, and very frequently none are set, either from carelessness on the part of the surveyor, or an unwillingness on the part of the owner to pay their cost. Monuments of a permanent character should be set at each corner of a tract surveyed, and at least two, visible the one from the other, on the line of each street. If these monuments are not placed on the centre lines of the streets, they should be placed at *uniform* distances from the centre or property lines. If placed with reference to the centre line, they should all be placed on the same side of the centre. In streets extending east and west the monuments should all be on the north of the centre, or they should all be on the south, and at uniform distance. In streets extending north and south the monuments should all be on the east of the centre, or all on the west.

Uniformity in such practice saves a vast amount of time.

Monuments may be set at uniform distances from the block lines, in the sidewalk area, and this is an excellent practice.

The stakes or monuments set at the corners of the blocks in additions, or town sites, should never be the *only* stakes or monuments set in the tract.

That the map may be reliable there should appear on it the following:

1. The certificate of the surveyor that he has carefully surveyed the land, that the map is a correct representation of the tract, and that he has set monuments (to be described) at the points indicated on the map.

2. The acknowledged signature of all persons possessing title to any of the land shown in the tract, and, if possible, those of adjoining owners.

3. If of an addition, the acknowledged dedication to public use forever of all areas shown as streets or roads.

4. If a street of full width, whose centre line is a boundary of the tract, is shown, the acknowledged signature of the owner of the adjoining property, unless his half of the street has been previously dedicated.

It has been already stated that, in some States, a map may be filed at the request of any person, and without signature.

This practice frequently leads to trouble. The writer knows of cases in which owners of large tracts of land have had those tracts subdivided and have taken land of adjoining non-resident owners for street purposes without the consent or knowledge of those owners. When at a later day the owners of the land so taken have objected and attempted to close half of the street, trouble of a serious character has arisen. The same trouble has occurred where streets have been run through narrow gores of land and have subsequently been completely closed, leaving houses built on the mapped property without outlet. Time and again have cases of this sort come to the knowledge of the writer.

Having pointed out certain evils, it remains to suggest a remedy.

It lies in the enactment of a law in each State governing these matters.

There should appear on the statutes of every State a law explicitly defining what shall appear on every map filed for reference, and making it a misdemeanor to file a map that does not strictly conform to the definition.

In the absence of such laws it is believed that the young surveyor can assist greatly in a much-needed reform, by following the principles suggested in this paper as the correct ones, and avoiding the errors here indicated.

It is hoped that those graduates of our engineering schools who drift into this class of work will be guided by a higher principle than that which actuates the surveyor who covers up his tracks, at the expense of his employer, in order to secure a monopoly of the business of his locality.

The young surveyor can spend his energies to greater advantage in devising new and better methods of work than in inventing ways for hiding information that it has been endeavored to show belongs to his employer.

Certainly a thorough education should so broaden the young man's views as to make it impossible for him to be controlled by those meaner instincts which, indulged, lead ever to narrow his vision and prevent him from perceiving the greater problems that continually present themselves for solution.

TABLES.

NOTE.

The following Tables, with explanatory notes, and the whole of Chapter XIII. on the *Measurement of Volumes*, are issued by the Publishers in a separate volume, bound in cloth. Price, $1.25.

TABLE I.

Trigonometric Formulæ.

Trigonometric Functions.

Let A (Fig. 107) = angle BAC = arc BF, and let the radius $AF = AB = AH = 1$.

We then have

$\sin A$	$= BC$
$\cos A$	$= AC$
$\tan A$	$= DF$
$\cot A$	$= HG$
$\sec A$	$= AD$
$\operatorname{cosec} A$	$= AG$
$\operatorname{versin} A$	$= CF = BE$
$\operatorname{covers} A$	$= BK = HL$
$\operatorname{exsec} A$	$= BD$
$\operatorname{coexsec} A$	$= BG$
$\operatorname{chord} A$	$= BF$
$\operatorname{chord} 2A$	$= BI = 2BC$

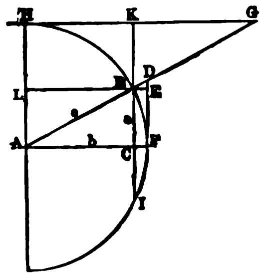

Fig. 107.

In the right-angled triangle ABC (Fig. 107)

Let $AB = c$, $AC = b$, and $BC = a$.

We then have:

1. $\sin A = \dfrac{a}{c} = \cos B$
2. $\cos A = \dfrac{b}{c} = \sin B$
3. $\tan A = \dfrac{a}{b} = \cot B$
4. $\cot A = \dfrac{b}{a} = \tan B$
5. $\sec A = \dfrac{c}{b} = \operatorname{cosec} B$
6. $\operatorname{cosec} A = \dfrac{c}{a} = \sec B$
7. $\operatorname{vers} A = \dfrac{c-b}{c} = \operatorname{covers} B$
8. $\operatorname{exsec} A = \dfrac{c-b}{b} = \operatorname{coexsec} B$
9. $\operatorname{covers} A = \dfrac{c-a}{c} = \operatorname{versin} B$
10. $\operatorname{coexsec} A = \dfrac{c-a}{a} = \operatorname{exsec} B$
11. $a = c \sin A = b \tan A$
12. $b = c \cos A = a \cot A$
13. $c = \dfrac{a}{\sin A} = \dfrac{b}{\cos A}$
14. $a = c \cos B = b \cot B$
15. $b = c \sin B = a \tan B$
16. $c = \dfrac{a}{\cos B} = \dfrac{b}{\sin B}$
17. $a = \sqrt{(c+b)(c-b)}$
18. $b = \sqrt{(c+a)(c-a)}$
19. $c = \sqrt{a^2 + b^2}$
20. $C = 90° = A + B$

21. area $= \dfrac{ab}{2}$

TABLE I.—*Continued.*

TRIGONOMETRIC FORMULÆ.

SOLUTION OF OBLIQUE TRIANGLES.

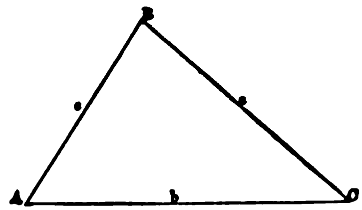

FIG. 108.

	GIVEN.	SOUGHT.	FORMULÆ.
22	A, B, a	C, b, c	$C = 180° - (A + B)$, $b = \frac{a}{\sin A} . \sin B$, $c = \frac{a}{\sin A} \sin (A + B)$
23	A, a, b	B, C, c	$\sin B = \frac{\sin A}{a} . b$, $C = 180° - (A + B)$, $c = \frac{a}{\sin A} . \sin C$.
24	C, a, b	$\frac{1}{2}(A + B)$	$\frac{1}{2}(A + B) = 90° - \frac{1}{2} C$
25		$\frac{1}{2}(A - B)$	$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B)$
26		A, B	$A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B)$, $B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B)$
27		c	$c = (a + b) \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} = (a - b) \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)}$
28		area	$K = \frac{1}{2} a b \sin C$.
29	a, b, c	A	Let $s = \frac{1}{2}(a + b + c)$; $\sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}$
30			$\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}$; $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
31			$\sin A = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$; $\text{vers } A = \frac{2(s-b)(s-c)}{bc}$
32		area	$K = \sqrt{s(s-a)(s-b)(s-c)}$
33	A, B, C, a	area	$K = \frac{a^2 \sin B . \sin C}{2 \sin A}$

TABLE I.—*Continued.*

Trigonometric Formulæ.

	General Formulæ.
34	$\sin A = \dfrac{1}{\operatorname{cosec} A} = \sqrt{1-\cos^2 A} = \tan A \cos A$
35	$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A = \operatorname{vers} A \cot \tfrac{1}{2} A$
36	$\sin A = \sqrt{\tfrac{1}{2} \operatorname{vers} 2A} = \sqrt{\tfrac{1}{2}(1-\cos 2A)}$
37	$\cos A = \dfrac{1}{\sec A} = \sqrt{1-\sin^2 A} = \cot A \sin A$
38	$\cos A = 1 - \operatorname{vers} A = 2\cos^2 \tfrac{1}{2} A - 1 = 1 - 2 \sin^2 \tfrac{1}{2} A$
39	$\cos A = \cos^2 \tfrac{1}{2} A - \sin^2 \tfrac{1}{2} A = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} \cos 2A}$
40	$\tan A = \dfrac{1}{\cot A} = \dfrac{\sin A}{\cos A} = \sqrt{\sec^2 A - 1}$
41	$\tan A = \sqrt{\dfrac{1}{\cos^2 A} - 1} = \dfrac{\sqrt{1-\cos^2 A}}{\cos A} = \dfrac{\sin 2A}{1+\cos 2A}$
42	$\tan A = \dfrac{1-\cos 2A}{\sin 2A} = \dfrac{\operatorname{vers} 2A}{\sin 2A} = \operatorname{exsec} A \cot \tfrac{1}{2} A$
43	$\cot A = \dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A} = \sqrt{\operatorname{cosec}^2 A - 1}$
44	$\cot A = \dfrac{\sin 2A}{1-\cos 2A} = \dfrac{\sin 2A}{\operatorname{vers} 2A} = \dfrac{1+\cos 2A}{\sin 2A}$
45	$\cot A = \dfrac{\tan \tfrac{1}{2} A}{\operatorname{exsec} A}$
46	$\operatorname{vers} A = 1 - \cos A = \sin A \tan \tfrac{1}{2} A = 2 \sin^2 \tfrac{1}{2} A$
47	$\operatorname{vers} A = \operatorname{exsec} A \cos A$
48	$\operatorname{exsec} A = \sec A - 1 = \tan A \tan \tfrac{1}{2} A = \dfrac{\operatorname{vers} A}{\cos A}$
49	$\sin \tfrac{1}{2} A = \sqrt{\dfrac{1-\cos A}{2}} = \sqrt{\dfrac{\operatorname{vers} A}{2}}$
50	$\sin 2A = 2 \sin A \cos A$
51	$\cos \tfrac{1}{2} A = \sqrt{\dfrac{1+\cos A}{2}}$
52	$\cos 2A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A$

TABLE I.—*Continued.*

TRIGONOMETRIC FORMULÆ.

GENERAL FORMULÆ.

53. $\tan \frac{1}{2} A = \dfrac{\tan A}{1 + \sec A} = \operatorname{cosec} A - \cot A = \dfrac{1 - \cos A}{\sin A} = \sqrt{\dfrac{1 - \cos A}{1 + \cos A}}$

54. $\tan 2 A = \dfrac{2 \tan A}{1 - \tan^2 A}$

55. $\cot. \frac{1}{2} A = \dfrac{\sin A}{\operatorname{vers} A} = \dfrac{1 + \cos A}{\sin A} = \dfrac{1}{\operatorname{cosec} A - \cot A}$

56. $\cot 2 A = \dfrac{\cot^2 A - 1}{2 \cot A}$

57. $\operatorname{vers} \frac{1}{2} A = \dfrac{\frac{1}{2} \operatorname{vers} A}{1 + \sqrt{1 - \frac{1}{2} \operatorname{vers} A}} = \dfrac{1 - \cos A}{2 + \sqrt{2(1 + \cos A)}}$

58. $\operatorname{vers} 2 A = 2 \sin^2 A$

59. $\operatorname{exsec} \frac{1}{2} A = \dfrac{1 - \cos A}{(1 + \cos A) + \sqrt{2(1 + \cos A)}}$

60. $\operatorname{exsec} 2 A = \dfrac{\tan^2 A}{1 - \tan^2 A}$

61. $\sin (A \pm B) = \sin A . \cos B \pm \sin B . \cos A$

62. $\cos (A \pm B) = \cos A . \cos B \mp \sin A . \sin B$

63. $\sin A + \sin B = 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B)$

64. $\sin A - \sin B = 2 \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$

65. $\cos A + \cos B = 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B)$

66. $\cos B - \cos A = 2 \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$

67. $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin (A + B) \sin (A - B)$

68. $\cos^2 A - \sin^2 B = \cos (A + B) \cos (A - B)$

69. $\tan A + \tan B = \dfrac{\sin (A + B)}{\cos A . \cos B}$

70. $\tan A - \tan B = \dfrac{\sin (A - B)}{\cos A . \cos B}$

TABLE II.

For Converting Metres, Feet, and Chains.

Metres to Feet.		Feet to Metres and Chains.			Chains to Feet.	
Metres.	Feet.	Feet.	Metres.	Chains.	Chains.	Feet.
1	3.28087	1	0.304797	0.0151	0.01	0.66
2	6.56174	2	0.609595	.0303	.02	1.32
3	9.84261	3	0.914392	.0455	.03	1.98
4	13.12348	4	1.219189	.0606	.04	2.64
5	16.40435	5	1.523986	.0758	.05	3.30
6	19.68522	6	1.828784	.0909	.06	3.96
7	22.96609	7	2.133581	.1061	.07	4.62
8	26.24695	8	2.438378	.1212	.08	5.28
9	29.52782	9	2.743175	.1364	.09	5.94
10	32.80869	10	3.047973	.1515	.10	6.60
20	65.61739	20	6.095946	.3030	20	13.20
30	98.42609	30	9.143918	.4545	.30	19.80
40	131.2348	40	12.19189	.6061	.40	26.40
50	164.0435	50	15.23986	.7576	.50	33.00
60	196.8522	60	18.28784	.9091	.60	39.60
70	229.6609	70	21.33581	1.0606	.70	46.20
80	262.4695	80	24.38378	1.2121	.80	52.80
90	295.2782	90	27.43175	1.3636	.90	59.40
100	328.0869	100	30.47973	1.5151	1	66.00
200	656.1739	100	60.95946	3.0303	2	132
300	984.2609	300	91.43918	4.5455	3	198
400	1312.348	400	121.9189	6.0606	4	264
500	1640.435	500	152.3986	7.5756	5	330
600	1968.522	600	182.8784	9.0909	6	396
700	2296.609	700	213.3581	10.606	7	462
800	2624.695	800	243.8378	12.121	8	528
900	2952.782	900	274.3175	13.636	9	594
1000	3280.869	1000	304.7973	15.151	10	660
2000	6561.739	2000	609.5946	30.303	20	1320
3000	9842.609	3000	914.3918	45.455	30	1980
4000	13123.48	4000	1219.189	60.606	40	2640
5000	16404.35	5000	1523.986	75.756	50	3300
6000	19685.22	6000	1828.784	90.909	60	3960
7000	22966.09	7000	2133.581	106.06	70	4620
8000	26246.95	8000	2438.378	121.21	80	5280
9000	29527.82	9000	2743.175	136.36	90	5940

TABLE III.

Logarithms of Numbers. § 173.

Nat. Nos.	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
10	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374	4	8	12	17	21	25	29	33	37
11	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755	4	8	11	15	19	23	26	30	34
12	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106	3	7	10	14	17	21	24	28	31
13	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430	3	6	10	13	16	19	23	26	29
14	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732	3	6	9	12	15	18	21	24	27
15	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014	3	6	8	11	14	17	20	22	25
16	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279	3	5	8	11	13	16	18	21	24
17	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529	2	5	7	10	12	15	17	20	22
18	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765	2	5	7	9	12	14	16	19	21
19	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989	2	4	7	9	11	13	16	18	20
20	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201	2	4	6	8	11	13	15	17	19
21	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404	2	4	6	8	10	12	14	16	18
22	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598	2	4	6	8	10	12	14	15	17
23	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784	2	4	6	7	9	11	13	15	17
24	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962	2	4	5	7	9	11	12	14	16
25	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133	2	3	5	7	9	10	12	14	15
26	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298	2	3	5	7	8	10	11	13	15
27	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456	2	3	5	6	8	9	11	13	14
28	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609	2	3	5	6	8	9	11	12	14
29	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757	1	3	4	6	7	9	10	12	13
30	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900	1	3	4	6	7	9	10	11	13
31	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038	1	3	4	6	7	8	10	11	12
32	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172	1	3	4	5	7	8	9	11	12
33	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302	1	3	4	5	6	8	9	10	12
34	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428	1	3	4	5	6	8	9	10	11
35	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551	1	2	4	5	6	7	9	10	11
36	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670	1	2	4	5	6	7	8	10	11
37	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786	1	2	3	5	6	7	8	9	10
38	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899	1	2	3	5	6	7	8	9	10
39	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010	1	2	3	4	5	7	8	9	10
40	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1	2	3	4	5	6	8	9	10
41	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222	1	2	3	4	5	6	7	8	9
42	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325	1	2	3	4	5	6	7	8	9
43	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425	1	2	3	4	5	6	7	8	9
44	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522	1	2	3	4	5	6	7	8	9
45	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618	1	2	3	4	5	6	7	8	9
46	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712	1	2	3	4	5	6	7	7	8
47	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803	1	2	3	4	5	5	6	7	8
48	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893	1	2	3	4	4	5	6	7	8
49	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981	1	2	3	4	4	5	6	7	8
50	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067	1	2	3	3	4	5	6	7	8
51	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152	1	2	3	3	4	5	6	7	8
52	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235	1	2	2	3	4	5	6	7	7
53	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316	1	2	2	3	4	5	6	6	7
54	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396	1	2	2	3	4	5	6	6	7

TABLE III.—*Continued.*

LOGARITHMS OF NUMBERS.

Nat. Nos.	0	1	2	3	4	5	6	7	8	9	Proportional Parts 1	2	3	4	5	6	7	8	9
55	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474	1	2	2	3	4	5	5	6	7
56	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551	1	2	2	3	4	5	5	6	7
57	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627	1	2	2	3	4	5	5	6	7
58	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701	1	1	2	3	4	4	5	6	7
59	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774	1	1	2	3	4	4	5	6	7
60	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846	1	1	2	3	4	4	5	6	6
61	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917	1	1	2	3	4	4	5	6	6
62	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987	1	1	2	3	3	4	5	6	6
63	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055	1	1	2	3	3	4	5	5	6
64	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122	1	1	2	3	3	4	5	5	6
65	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189	1	1	2	3	3	4	5	5	6
66	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254	1	1	2	3	3	4	5	5	6
67	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319	1	1	2	3	3	4	5	5	6
68	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382	1	1	2	3	3	4	4	5	6
69	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445	1	1	2	2	3	4	4	5	6
70	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506	1	1	2	2	3	4	4	5	6
71	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567	1	1	2	2	3	4	4	5	5
72	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627	1	1	2	2	3	4	4	5	5
73	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686	1	1	2	2	3	4	4	5	5
74	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745	1	1	2	2	3	4	4	5	5
75	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802	1	1	2	2	3	3	4	5	5
76	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859	1	1	2	2	3	3	4	5	5
77	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915	1	1	2	2	3	3	4	4	5
78	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971	1	1	2	2	3	3	4	4	5
79	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025	1	1	2	2	3	3	4	4	5
80	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079	1	1	2	2	3	3	4	4	5
81	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133	1	1	2	2	3	3	4	4	5
82	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186	1	1	2	2	3	3	4	4	5
83	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238	1	1	2	2	3	3	4	4	5
84	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289	1	1	2	2	3	3	4	4	5
85	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340	1	1	2	2	3	3	4	4	5
86	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390	1	1	2	2	3	3	4	4	5
87	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440	0	1	1	2	2	3	3	4	4
88	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489	0	1	1	2	2	3	3	4	4
89	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538	0	1	1	2	2	3	3	4	4
90	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586	0	1	1	2	2	3	3	4	4
91	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633	0	1	1	2	2	3	3	4	4
92	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680	0	1	1	2	2	3	3	4	4
93	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727	0	1	1	2	2	3	3	4	4
94	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773	0	1	1	2	2	3	3	4	4
95	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818	0	1	1	2	2	3	3	4	4
96	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863	0	1	1	2	2	3	3	4	4
97	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908	0	1	1	2	2	3	3	4	4
98	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952	0	1	1	2	2	3	3	4	4
99	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996	0	1	1	2	2	3	3	3	4

TABLE III.

LOGARITHMS OF NUMBERS. § 173.

Nat. Nos.	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
10	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374	4	8	12	17	21	25	29	33	37
11	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755	4	8	11	15	19	23	26	30	34
12	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106	3	7	10	14	17	21	24	28	31
13	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430	3	6	10	13	16	19	23	26	29
14	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732	3	6	9	12	15	18	21	24	27
15	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014	3	6	8	11	14	17	20	22	25
16	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279	3	5	8	11	13	16	18	21	24
17	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529	2	5	7	10	12	15	17	20	22
18	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765	2	5	7	9	12	14	16	19	21
19	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989	2	4	7	9	11	13	16	18	20
20	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201	2	4	6	8	11	13	15	17	19
21	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404	2	4	6	8	10	12	14	16	18
22	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598	2	4	6	8	10	12	14	15	17
23	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784	2	4	6	7	9	11	13	15	17
24	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962	2	4	5	7	9	11	12	14	16
25	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133	2	3	5	7	9	10	12	14	15
26	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298	2	3	5	7	8	10	11	13	15
27	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456	2	3	5	6	8	9	11	13	14
28	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609	2	3	5	6	8	9	11	12	14
29	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757	1	3	4	6	7	9	10	12	13
30	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900	1	3	4	6	7	9	10	11	13
31	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038	1	3	4	6	7	8	10	11	12
32	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172	1	3	4	5	7	8	9	11	12
33	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302	1	3	4	5	6	8	9	10	12
34	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428	1	3	4	5	6	8	9	10	11
35	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551	1	2	4	5	6	7	9	10	11
36	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670	1	2	4	5	6	7	8	10	11
37	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786	1	2	3	5	6	7	8	9	10
38	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899	1	2	3	5	6	7	8	9	10
39	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010	1	2	3	4	5	7	8	9	10
40	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1	2	3	4	5	6	8	9	10
41	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222	1	2	3	4	5	6	7	8	9
42	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325	1	2	3	4	5	6	7	8	9
43	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425	1	2	3	4	5	6	7	8	9
44	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522	1	2	3	4	5	6	7	8	9
45	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618	1	2	3	4	5	6	7	8	9
46	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712	1	2	3	4	5	6	7	7	8
47	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803	1	2	3	4	5	5	6	7	8
48	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893	1	2	3	4	4	5	6	7	8
49	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981	1	2	3	4	4	5	6	7	8
50	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067	1	2	3	3	4	5	6	7	8
51	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152	1	2	3	3	4	5	6	7	8
52	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235	1	2	2	3	4	5	6	7	7
53	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316	1	2	2	3	4	5	6	6	7
54	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396	1	2	2	3	4	5	6	6	7

TABLE III.—*Continued.*

Logarithms of Numbers.

Nat. Nos.	0	1	2	3	4	5	6	7	8	9	P.P. 1	2	3	4	5	6	7	8	9
55	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474	1	2	2	3	4	5	5	6	7
56	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551	1	2	2	3	4	5	5	6	7
57	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627	1	2	2	3	4	5	5	6	7
58	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701	1	1	2	3	4	4	5	6	7
59	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774	1	1	2	3	4	4	5	6	7
60	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846	1	1	2	3	4	4	5	6	6
61	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917	1	1	2	3	4	4	5	6	6
62	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987	1	1	2	3	3	4	5	6	6
63	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055	1	1	2	3	3	4	5	5	6
64	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122	1	1	2	3	3	4	5	5	6
65	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189	1	1	2	3	3	4	5	5	6
66	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254	1	1	2	3	3	4	5	5	6
67	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319	1	1	2	3	3	4	5	5	6
68	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382	1	1	2	3	3	4	4	5	6
69	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445	1	1	2	2	3	4	4	5	6
70	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506	1	1	2	2	3	4	4	5	6
71	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567	1	1	2	2	3	4	4	5	5
72	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627	1	1	2	2	3	4	4	5	5
73	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686	1	1	2	2	3	4	4	5	5
74	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745	1	1	2	2	3	4	4	5	5
75	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802	1	1	2	2	3	3	4	5	5
76	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859	1	1	2	2	3	3	4	5	5
77	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915	1	1	2	2	3	3	4	4	5
78	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971	1	1	2	2	3	3	4	4	5
79	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025	1	1	2	2	3	3	4	4	5
80	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079	1	1	2	2	3	3	4	4	5
81	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133	1	1	2	2	3	3	4	4	5
82	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186	1	1	2	2	3	3	4	4	5
83	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238	1	1	2	2	3	3	4	4	5
84	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289	1	1	2	2	3	3	4	4	5
85	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340	1	1	2	2	3	3	4	4	5
86	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390	1	1	2	2	3	3	4	4	5
87	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440	0	1	1	2	2	3	3	4	4
88	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489	0	1	1	2	2	3	3	4	4
89	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538	0	1	1	2	2	3	3	4	4
90	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586	0	1	1	2	2	3	3	4	4
91	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633	0	1	1	2	2	3	3	4	4
92	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680	0	1	1	2	2	3	3	4	4
93	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727	0	1	1	2	2	3	3	4	4
94	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773	0	1	1	2	2	3	3	4	4
95	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818	0	1	1	2	2	3	3	4	4
96	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863	0	1	1	2	2	3	3	4	4
97	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908	0	1	1	2	2	3	3	4	4
98	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952	0	1	1	2	2	3	3	4	4
99	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996	0	1	1	2	2	3	3	3	4

TABLE IIIA.
LOGARITHMS OF SINES AND TANGENTS.

	0°				1°				
	Sin.	Cos.	Tan.	Cot.	Sin.	Cos.	Tan.	Cot.	
0′		0.0000			8.2419	9.9999	8.2419	1.7581	60′
1	6.4637	.0000	6.4637	3.5363	.2490	.9999	.2491	.7509	59
2	.7648	.0000	.7648	.2352	.2561	.9999	.2562	.7438	58
3	6.9408	.0000	6.9408	3.0592	.2630	.9999	.2631	.7369	57
4	7.0658	.0000	7.0658	2.9342	.2699	.9999	.2700	.7300	56
5	.1627	.0000	.1627	.8373	.2766	.9999	.2767	.7233	55
6	.2419	.0000	.2419	.7581	.2832	.9999	.2833	.7167	54
7	.3088	.0000	.3088	.6912	.2898	.9999	.2899	.7101	53
8	.3668	.0000	.3668	.6332	.2962	.9999	.2963	.7037	52
9	.4180	.0000	.4180	.5820	.3025	.9999	.3026	.6974	51
10	.4637	.0000	.4637	.5363	.3088	.9999	.3089	.6911	50
11	.5051	.0000	.5051	.4949	.3150	.9999	.3150	.6850	49
12	.5429	.0000	.5429	.4571	.3210	.9999	.3211	.6789	48
13	.5777	.0000	.5777	.4223	.3270	.9999	.3271	.6729	47
14	.6099	.0000	.6099	.3901	.3329	.9999	.3330	.6670	46
15	.6398	.0000	.6398	.3602	.3388	.9999	.3389	.6611	45
16	.6678	.0000	.6678	.3322	.3445	.9999	.3446	.6554	44
17	.6942	.0000	.6942	.3058	.3502	.9999	.3503	.6497	43
18	.7190	.0000	.7190	.2810	.3558	.9999	.3559	.6441	42
19	.7425	.0000	.7425	.2575	.3613	.9999	.3614	.6386	41
20	.7648	.0000	.7648	.2352	.3668	.9999	.3669	.6331	40
21	.7859	.0000	.7860	.2140	.3722	.9999	.3723	.6277	39
22	.8061	.0000	.8062	.1938	.3775	.9999	.3776	.6224	38
23	.8255	.0000	.8255	.1745	.3828	.9999	.3829	.6171	37
24	.8439	.0000	.8439	.1561	.3880	.9999	.3881	.6119	36
25	.8617	.0000	.8617	.1383	.3931	.9999	.3932	.6068	35
26	.8787	.0000	.8787	.1213	.3982	.9999	.3983	.6017	34
27	.8951	.0000	.8951	.1049	.4032	.9999	.4033	.5967	33
28	.9109	.0000	.9109	.0891	.4082	.9999	.4083	.5917	32
29	.9261	.0000	.9261	.0739	.4131	.9999	.4132	.5868	31
30	.9408	.0000	.9409	.0591	.4179	.9999	.4181	.5819	30
31	.9551	.0000	.9551	.0449	.4227	.9998	.4229	.5771	29
32	.9689	.0000	.9689	.0311	.4275	.9998	.4276	.5724	28
33	.9822	.0000	.9823	.0177	.4322	.9998	.4323	.5677	27
34	7.9952	.0000	7.9952	2.0048	.4368	.9998	.4370	.5630	26
35	8.0078	.0000	8.0078	1.9922	.4414	.9998	.4416	.5584	25
36	.0200	.0000	.0200	.9800	.4459	.9998	.4461	.5539	24
37	.0319	.0000	.0319	.9681	.4504	.9998	.4506	.5494	23
38	.0435	.0000	.0435	.9565	.4549	.9998	.4551	.5449	22
39	.0548	.0000	.0548	.9452	.4593	.9998	.4595	.5405	21
40	.0658	.0000	.0658	.9342	.4637	.9998	.4638	.5362	20
41	.0765	.0000	.0765	.9235	.4680	.9998	.4682	.5318	19
42	.0870	.0000	.0870	.9130	.4723	.9998	.4725	.5275	18
43	.0972	.0000	.0972	.9028	.4765	.9998	.4767	.5233	17
44	.1072	.0000	.1072	.8928	.4807	.9998	.4809	.5191	16
45	.1169	.0000	.1170	.8830	.4848	.9998	.4851	.5149	15
46	.1265	.0000	.1265	.8735	.4890	.9998	.4892	.5108	14
47	.1358	.0000	.1359	.8641	.4930	.9998	.4933	.5067	13
48	.1450	.0000	.1450	.8550	.4971	.9998	.4973	.5027	12
49	.1539	.0000	.1540	.8460	.5011	.9998	.5013	.4987	11
50	.1627	.0000	.1627	.8373	.5050	.9998	.5053	.4947	10
51	.1713	.0000	.1713	.8287	.5090	.9998	.5092	.4908	9
52	.1797	0.0000	.1798	.8202	.5129	.9998	.5131	.4869	8
53	.1880	9.9999	.1880	.8120	.5167	.9998	.5170	.4830	7
54	.1961	.9999	.1962	.8038	.5206	.9998	.5208	.4792	6
55	.2041	.9999	.2041	.7959	.5243	.9998	.5246	.4754	5
56	.2119	.9999	.2120	.7880	.5281	.9998	.5283	.4717	4
57	.2196	.9999	.2196	.7804	.5318	.9997	.5321	.4679	3
58	.2271	.9999	.2272	7728	.5355	.9997	.5358	.4642	2
59	.2346	.9999	.2346	.7654	.5392	.9997	.5394	.4606	1
60	8.2419	9 9999	8.2419	1.7581	8.5428	9.9997	8.5431	1.4569	0
	Cos.	Sin.	Cot.	Tan.	Cos.	Sin.	Cot.	Tan.	
	89°				88°				

TABLE IIIA.—*Continued.*

Logarithms of Sines and Tangents.

	2°				3°				4°				
	Sin.	Cos.	Tan.	Cot.	Sin.	Cos.	Tan.	Cot.	Sin.	Cos.	Tan.	Cot.	
0'	8.5428	9.9997	8.5431	1.4569	8.7188	9.9994	8.7194	1.2806	8.8436	9.9989	8.8446	1.1554	60'
1	.5464	.9997	.5467	.4533	.7212	.9994	.7218	.2782	.8454	.9989	.8465	.1535	59
2	.5500	.9997	.5503	.4497	.7236	.9994	.7242	.2758	.8472	.9989	.8483	.1517	58
3	.5535	.9997	.5538	.4462	.7260	.9994	.7266	.2734	.8490	.9989	.8501	.1499	57
4	.5571	.9997	.5573	.4427	.7283	.9994	.7290	.2710	.8508	.9989	.8518	.1482	56
5	.5605	.9997	.5608	.4392	.7307	.9994	.7313	.2687	.8525	.9989	.8536	.1464	55
6	.5640	.9997	.5643	.4357	.7330	.9994	.7337	.2663	.8543	.9989	.8554	.1446	54
7	.5674	.9997	.5677	.4323	.7354	.9994	.7360	.2640	.8560	.9989	.8572	.1428	53
8	.5708	.9997	.5711	.4289	.7377	.9994	.7383	.2617	.8578	.9989	.8589	.1411	52
9	.5742	.9997	.5745	.4255	.7400	.9993	.7406	.2594	.8595	.9989	.8607	.1393	51
10	.5776	.9997	.5779	.4221	.7423	.9993	.7429	.2571	.8613	.9989	.8624	.1376	50
11	.5809	.9997	.5812	.4188	.7445	.9993	.7452	.2548	.8630	.9988	.8642	.1358	49
12	.5842	.9997	.5845	.4155	.7468	.9993	.7475	.2525	.8647	.9988	.8659	.1341	48
13	.5875	.9997	.5878	.4122	.7491	.9993	.7497	.2503	.8665	.9988	.8676	.1324	47
14	.5907	.9997	.5911	.4089	.7513	.9993	.7520	.2480	.8682	.9988	.8694	.1306	46
15	.5939	.9997	.5943	.4057	.7535	.9993	.7542	.2458	.8699	.9988	.8711	.1289	45
16	.5972	.9997	.5975	.4025	.7557	.9993	.7565	.2435	.8716	.9988	.8728	.1272	44
17	.6003	.9997	.6007	.3993	.7580	.9993	.7587	.2413	.8733	.9988	.8745	.1255	43
18	.6035	.9996	.6038	.3962	.7602	.9993	.7609	.2391	.8749	.9988	.8762	.1238	42
19	.6066	.9996	.6070	.3930	.7623	.9993	.7631	.2369	.8766	.9988	.8778	.1222	41
20	.6097	.9996	.6101	.3899	.7645	.9993	.7652	.2348	.8783	.9988	.8795	.1205	40
21	.6128	.9996	.6132	.3868	.7667	.9993	.7674	.2326	.8799	.9987	.8812	.1188	39
22	.6159	.9996	.6163	.3837	.7688	.9992	.7696	.2304	.8816	.9987	.8829	.1171	38
23	.6189	.9996	.6193	.3807	.7710	.9992	.7717	.2283	.8833	.9987	.8845	.1155	37
24	.6220	.9996	.6223	.3777	.7731	.9992	.7739	.2261	.8849	.9987	.8862	.1138	36
25	.6250	.9996	.6254	.3746	.7752	.9992	.7760	.2240	.8865	.9987	.8878	.1122	35
26	.6279	.9996	.6283	.3717	.7773	.9992	.7781	.2219	.8882	.9987	.8895	.1105	34
27	.6309	.9996	.6313	.3687	.7794	.9992	.7802	.2198	.8898	.9987	.8911	.1089	33
28	.6339	.9996	.6343	.3657	.7815	.9992	.7823	.2177	.8914	.9987	.8927	.1073	32
29	.6368	.9996	.6372	.3628	.7836	.9992	.7844	.2156	.8930	.9987	.8944	.1056	31
30	.6397	.9996	.6401	.3599	.7857	.9992	.7865	.2135	.8946	.9987	.8960	.1040	30
31	.6426	.9996	.6430	.3570	.7877	.9992	.7886	.2114	.8962	.9986	.8976	.1024	29
32	.6454	.9996	.6459	.3541	.7898	.9992	.7906	.2094	.8978	.9986	.8992	.1008	28
33	.6483	.9996	.6487	.3513	.7918	.9992	.7927	.2073	.8994	.9986	.9008	.0992	27
34	.6511	.9996	.6515	.3485	.7939	.9992	.7947	.2053	.9010	.9986	.9024	.0976	26
35	.6539	.9996	.6544	.3456	.7959	.9992	.7967	.2033	.9026	.9986	.9040	.0960	25
36	.6567	.9996	.6571	.3429	.7979	.9991	.7988	.2012	.9342	.9986	.9056	.0944	24
37	.6595	.9995	.6599	.3401	.7999	.9991	.8008	.1992	.9057	.9986	.9071	.0929	23
38	.6622	.9995	.6627	.3373	.8019	.9991	.8028	.1972	.9073	.9986	.9087	.0913	22
39	.6650	.9995	.6654	.3346	.8039	.9991	.8048	.1952	.9089	.9986	.9103	.0897	21
40	.6677	.9995	.6682	.3318	.8059	.9991	.8067	.1933	.9104	.9986	.9118	.0882	20
41	.6704	.9995	.6709	.3291	.8078	.9991	.8087	.1913	.9119	.9985	.9134	.0866	19
42	.6731	.9995	.6736	.3264	.8098	.9991	.8107	.1893	.9135	.9985	.9150	.0850	18
43	.6758	.9995	.6762	.3238	.8117	.9991	.8126	.1874	.9150	.9985	.9165	.0835	17
44	.6784	.9995	.6789	.3211	.8137	.9991	.8146	.1854	.9166	.9985	.9180	.0820	16
45	.6810	.9995	.6815	.3185	.8156	.9991	.8165	.1835	.9181	.9985	.9196	.0804	15
46	.6837	.9995	.6842	.3158	.8175	.9991	.8185	.1815	.9196	.9985	.9211	.0789	14
47	.6863	.9995	.6868	.3132	.8194	.9991	.8204	.1796	.9211	.9985	.9226	.0774	13
48	.6889	.9995	.6894	.3106	.8213	.9990	.8223	.1777	.9226	.9985	.9241	.0759	12
49	.6914	.9995	.6920	.3080	.8232	.9990	.8242	.1758	.9241	.9985	.9256	.0744	11
50	.6940	.9995	.6945	.3055	.8251	.9990	.8261	.1739	.9256	.9985	.9272	.0728	10
51	.6965	.9995	.6971	.3029	.8270	.9990	.8280	.1720	.9271	.9984	.9287	.0713	9
52	.6991	.9995	.6996	.3004	.8289	.9990	.8299	.1701	.9286	.9984	.9302	.0698	8
53	.7016	.9994	.7021	.2979	.8307	.9990	.8317	.1683	.9301	.9984	.9316	.0684	7
54	.7041	.9994	.7046	.2954	.8326	.9990	.8336	.1664	.9315	.9984	.9331	.0669	6
55	.7066	.9994	.7071	.2929	.8345	.9990	.8355	.1645	.9330	.9984	.9346	.0654	5
56	.7090	.9994	.7096	.2904	.8363	.9990	.8373	.1627	.9345	.9984	.9361	.0639	4
57	.7115	.9994	.7121	.2879	.8381	.9990	.8392	.1608	.9359	.9984	.9376	.0624	3
58	.7140	.9994	.7145	.2855	.8400	.9990	.8410	.1590	.9374	.9984	.9390	.0610	2
59	.7164	.9994	.7170	.2830	.8418	.9989	.8428	.1572	.9388	.9984	.9405	.0595	1
60	8.7188	9.9994	8.7194	1.2806	8.8436	9.9989	8.8446	1.1554	8.9403	9.9983	8.9420	1.0580	0
	Cos.	Sin.	Cot.	Tan.	Cos.	Sin.	Cot.	Tan.	Cos.	Sin.	Cot.	Tan.	
	87°				86°				85°				

TABLE IIIA—*Continued.*

LOGARITHMS OF SINES AND TANGENTS.

Arc.	Sin.	Df.	Cos.	Df.	Tan.	Df.	Cot.	Arc.	Arc.	Sin.	Df.	Cos.	Df.	Tan.	Df.	Cot.	Arc.
° '								° '	° '								° '
5 0	8.9403	142	9.9983	1	8.9420	143	1.0580	85 0	15 0	9.4130	47	9.9849	3	9.4281	50	0.5719	75 0
10	.9545	137	.9982	1	.9563	138	.0437	50	10	.4177	46	.9846	3	.4331	50	.5669	50
20	.9682	134	.9981	1	.9701	135	.0299	40	20	.4223	46	.9843	4	.4381	49	.5619	40
30	.9816	129	.9980	1	.9836	130	.0164	30	30	.4269	45	.9839	3	.4430	49	.5570	30
40	8.9945	125	.9979	2	8.9966	127	1.0034	20	40	.4314	45	.9836	4	.4479	48	.5521	20
50	9.0070	122	.9977	1	9.0093	123	0.9907	10	50	.4359	44	.9832	4	.4527	48	.5473	10
6 0	.0192	119	.9976	1	.0216	120	.9784	84 0	16 0	.4403	44	.9828	3	.4575	47	.5425	74 0
10	.0311	115	.9975	2	.0336	117	.9664	50	10	.4447	44	.9825	4	.4622	47	.5378	50
20	.0426	113	.9973	1	.0453	114	.9547	40	20	.4491	42	.9821	4	.4669	47	.5331	40
30	.0539	109	.9972	1	.0567	111	.9433	30	30	.4533	43	.9817	3	.4716	46	.5284	30
40	.0648	107	.9971	2	.0678	108	.9322	20	40	.4576	42	.9814	4	.4762	46	.5238	20
50	.0755	104	.9969	1	.0786	105	.9214	10	50	.4618	41	.9810	4	.4808	45	.5192	10
7 0	.0859	102	.9968	2	.0891	104	.9109	83 0	17 0	.4659	41	.9806	4	.4853	45	.5147	73 0
10	.0961	99	.9966	2	.0995	101	.9005	50	10	.4700	41	.9802	4	.4898	45	.5102	50
20	.1060	97	.9964	1	.1096	98	.8904	40	20	.4741	40	.9798	4	.4943	44	.5057	40
30	.1157	95	.9963	2	.1194	97	.8806	30	30	.4781	40	.9794	4	.4987	44	.5013	30
40	.1252	93	.9961	2	.1291	94	.8709	20	40	.4821	40	.9790	4	.5031	44	.4969	20
50	.1345	91	.9959	1	.1385	93	.8615	10	50	.4861	39	.9786	4	.5075	43	.4925	10
8 0	.1436	89	.9958	2	.1478	91	.8522	82 0	18 0	.4900	39	.9782	4	.5118	43	.4882	72 0
10	.1525	87	.9956	2	.1569	89	.8431	50	10	.4939	38	.9778	4	.5161	42	.4839	50
20	.1612	85	.9954	2	.1658	87	.8342	40	20	.4977	38	.9774	4	.5203	42	.4797	40
30	.1697	84	.9952	2	.1745	86	.8255	30	30	.5015	37	.9770	5	.5245	42	.4755	30
40	.1781	82	.9950	2	.1831	84	.8169	20	40	.5052	38	.9765	4	.5287	42	.4713	20
50	.1863	80	.9948	[illegible]	.1915	82	.8085	10	50	.5090	36	.9761	4	.5329	41	.4671	10
9 0	[illegible]	79	.9946	2	.1997	81	.8003	81 0	19 0	.5126	37	.9757	5	.5370	41	.4630	71 0
10	.2022	78	.9944	2	[illegible]	80	.7922	50	10	.5163	36	.9752	4	.5411	40	.4589	50
20	.2100	76	.9942	2	.2158	78	.7842	40	20	.5199	36	.9748	5	.5451	40	.4549	40
30	.2176	75	.9940	2	.2236	77	.7764	30	30	.5235	35	.9743	4	.5491	40	.4509	30
40	.2251	73	.9938	2	.2313	76	.7687	20	40	.5270	36	.9739	5	.5531	40	.4469	20
50	.2324	73	.9936	2	.2389	74	.7611	10	50	.5306	35	.9734	4	.5571	40	.4429	10
10 0	.2397	71	.9934	3	.2463	73	.7537	80 0	20 0	.5341	34	.9730	5	.5611	39	.4389	70 0
10	.2468	70	.9931	2	.2536	73	.7464	50	10	.5375	34	.9725	4	.5650	39	.4350	50
20	.2538	68	.9929	2	.2609	71	.7391	40	20	.5409	34	.9721	5	.5689	38	.4311	40
30	.2606	68	.9927	3	.2680	70	.7320	30	30	.5443	34	.9716	5	.5727	39	.4273	30
40	.2674	66	.9924	2	.2750	69	.7250	20	40	.5477	33	.9711	5	.5766	38	.4234	20
50	.2740	66	.9922	3	.2819	68	.7181	10	50	.5510	33	.9706	4	.5804	38	.4196	10
11 0	.2806	64	.9919	2	.2887	66	.7113	79 0	21 0	.5543	33	.9702	5	.5842	37	.4158	69 0
10	.2870	64	.9917	3	.2953	67	.7047	50	10	.5576	33	.9697	5	.5879	38	.4121	50
20	.2934	63	.9914	2	.3020	65	.6980	40	20	.5609	32	.9692	5	.5917	37	.4083	40
30	.2997	61	.9912	3	.3085	64	.6915	30	30	.5641	32	.9687	5	.5954	37	.4046	30
40	.3058	61	.9909	2	.3149	63	.6851	20	40	.5673	31	.9682	5	.5991	37	.4009	20
50	.3119	60	.9907	3	.3212	63	.6788	10	50	.5704	32	.9677	5	.6028	36	.3972	10
12 0	.3179	59	.9904	3	.3275	61	.6725	78 0	22 0	.5736	31	.9672	5	.6064	36	.3936	68 0
10	.3238	58	.9901	2	.3336	61	.6664	50	10	.5767	31	.9667	6	.6100	36	.3900	50
20	.3296	57	.9899	3	.3397	61	.6603	40	20	.5798	30	.9661	5	.6136	36	.3864	40
30	.3353	57	.9896	3	.3458	59	.6542	30	30	.5828	31	.9656	5	.6172	36	.3828	30
40	.3410	56	.9893	3	.3517	59	.6483	20	40	.5859	30	.9651	5	.6208	35	.3792	20
50	.3466	55	.9890	3	.3576	58	.6424	10	50	.5889	30	.9646	6	.6243	36	.3757	10
13 0	.3521	54	.9887	3	.3634	57	.6366	77 0	23 0	.5919	29	.9640	5	.6279	35	.3721	67 0
10	.3575	54	.9884	3	.3691	57	.6309	50	10	.5948	30	.9635	6	.6314	34	.3686	50
20	.3629	53	.9881	3	.3748	56	.6252	40	20	.5978	29	.9629	5	.6348	35	.3652	40
30	.3682	52	.9878	3	.3804	55	.6196	30	30	.6007	29	.9624	6	.6383	34	.3617	30
40	.3734	52	.9875	3	.3859	55	.6141	20	40	.6036	29	.9618	5	.6417	35	.3583	20
50	.3786	51	.9872	3	.3914	54	.6086	10	50	.6065	28	.9613	6	.6452	34	.3548	10
14 0	.3837	50	.9869	3	.3968	53	.6032	76 0	24 0	.6093	28	.9607	5	.6486	34	.3514	66 0
10	.3887	50	.9866	3	.4021	53	.5979	50	10	.6121	28	.9602	6	.6520	33	.3480	50
20	.3937	49	.9863	4	.4074	53	.5926	40	20	.6149	28	.9596	6	.6553	34	.3447	40
30	.3986	49	.9859	3	.4127	52	.5873	30	30	.6177	28	.9590	6	.6587	33	.3413	30
40	.4035	48	.9856	3	.4178	51	.5822	20	40	.6205	27	.9584	5	.6620	34	.3380	20
50	.4083	47	.9853	4	.4230	51	.5770	10	50	.6232	27	.9579	6	.6654	33	.3346	10
15 0	9.4130	47	9.9849	3	9.4281	50	0.5719	75 0	25 0	0.6259	27	9.9573	7	9.6687	33	0.3313	65 0
			Sin.	Df.	Cot.	Df.	Tan.	Arc.	Arc.	Cos.	Df.	Sin.	Df.	Cot.	Df.	Tan.	Arc.

TABLE IIIA—*Continued.*

LOGARITHMS OF SINES AND TANGENTS.

Arc.	Sin.	Df.	Cos.	Df.	Tan.	Df.	Cot.	Arc.	Arc.	Sin.	Df.	Cos.	Df.	Tan.	Df.	Cot.	Arc.
° ′								° ′	° ′								° ′
25 0	9.6259	27	9.9573	6	9.6687	33	0.3313	65 0	35 0	9.7586	18	9.9134	9	9.8452	27	0.1548	55 0
10	.6286	27	.9567	6	.6720	32	.3280	50	10	.7604	18	.9125	9	.8479	27	.1521	50
20	.6313	27	.9561	6	.6752	33	.3248	40	20	.7622	18	.9116	9	.8506	27	.1494	40
30	.6340	26	.9555	6	.6785	32	.3215	30	30	.7640	17	.9107	9	.8533	26	.1467	30
40	.6366	26	.9549	6	.6817	33	.3183	20	40	.7657	18	.9098	9	.8559	27	.1441	20
50	.6392	26	.9543	6	.6850	32	.3150	10	50	.7675	17	.9089	9	.8586	27	.1414	10
26 0	.6418	26	.9537	7	.6882	32	.3118	64 0	36 0	.7692	18	.9080	10	.8613	26	.1387	54 0
10	.6444	26	.9530	6	.6914	32	.3086	50	10	.7710	17	.9070	9	.8639	27	.1361	50
20	.6470	25	.9524	6	.6946	31	.3054	40	20	.7727	17	.9061	9	.8666	26	.1334	40
30	.6495	26	.9518	6	.6977	32	.3023	30	30	.7744	17	.9052	10	.8692	26	.1308	30
40	.6521	25	.9512	7	.7009	31	.2991	20	40	.7761	17	.9042	9	.8718	27	.1282	20
50	.6546	24	.9505	6	.7040	32	.2960	10	50	.7778	17	.9033	10	.8745	26	.1255	10
27 0	.6570	25	.9499	7	.7072	31	.2928	63 0	37 0	.7795	16	.9023	9	.8771	26	.1229	53 0
10	.6595	25	.9492	6	.7103	31	.2897	50	10	.7811	17	.9014	10	.8797	27	.1203	50
20	.6620	24	.9486	7	.7134	31	.2866	40	20	.7828	16	.9004	9	.8824	26	.1176	40
30	.6644	24	.9479	6	.7165	31	.2835	30	30	.7844	17	.8995	10	.8850	26	.1150	30
40	.6668	24	.9473	7	.7196	30	.2804	20	40	.7861	16	.8985	10	.8876	26	.1124	20
50	.6692	24	.9466	7	.7226	31	.2774	10	50	.7877	16	.8975	10	.8902	26	.1098	10
28 0	.6716	24	.9459	6	.7257	30	.2743	62 0	38 0	.7893	17	.8965	10	.8928	26	.1072	52 0
10	.6740	23	.9453	7	.7287	30	.2713	50	10	.7910	16	.8955	10	.8954	26	.1046	50
20	.6763	24	.9446	7	.7317	31	.2683	40	20	.7926	15	.8945	10	.8980	26	.1020	40
30	.6787	23	.9439	7	.7348	30	.2652	30	30	.7941	16	.8935	10	.9006	26	.0994	30
40	.6810	23	.9432	7	.7378	30	.2622	20	40	.7957	16	.8925	10	.9032	26	.0968	20
50	.6833	23	.9425	7	.7408	30	.2592	10	50	.7973	16	.8915	10	.9058	26	.0942	10
29 0	.6856	22	.9418	7	.7438	29	.2562	61 0	39 0	.7989	15	.8905	10	.9084	26	.0916	51 0
10	.6878	23	.9411	7	.7467	30	.2533	50	10	.8004	16	.8895	11	.9110	25	.0890	50
20	.6901	22	.9404	7	.7497	29	.2503	40	20	.8020	15	.8884	10	.9135	26	.0865	40
30	.6923	23	.9397	7	.7526	30	.2474	30	30	.8035	15	.8874	10	.9161	26	.0839	30
40	.6946	22	.9390	7	.7556	29	.2444	20	40	.8050	16	.8864	11	.9187	25	.0813	20
50	.6968	22	.9383	8	.7585	29	.2415	10	50	.8066	15	.8853	10	.9212	26	.0788	10
30 0	.6990	22	.9375	7	.7614	30	.2386	60 0	40 0	.8081	15	.8843	11	.9238	26	.0762	50 0
10	.7012	21	.9368	7	.7644	29	.2356	50	10	.8096	15	.8832	11	.9264	25	.0736	50
20	.7033	22	.9361	8	.7673	28	.2327	40	20	.8111	14	.8821	11	.9289	26	.0711	40
30	.7055	21	.9353	7	.7701	29	.2299	30	30	.8125	15	.8810	10	.9315	26	.0685	30
40	.7076	21	.9346	8	.7730	29	.2270	20	40	.8140	15	.8800	11	.9341	25	.0659	20
50	.7097	21	.9338	7	.7759	29	.2241	10	50	.8155	14	.8789	11	.9366	26	.0634	10
31 0	.7118	21	.9331	8	.7788	28	.2212	59 0	41 0	.8169	15	.8778	11	.9392	25	.0608	49 0
10	.7139	21	.9323	8	.7816	29	.2184	50	10	.8184	14	.8767	11	.9417	26	.0583	50
20	.7160	21	.9315	7	.7845	28	.2155	40	20	.8198	15	.8756	11	.9443	25	.0557	40
30	.7181	20	.9308	8	.7873	29	.2127	30	30	.8213	14	.8745	12	.9468	26	.0532	30
40	.7201	21	.9300	8	.7902	28	.2098	20	40	.8227	14	.8733	11	.9494	25	.0506	20
50	.7222	20	.9292	8	.7930	28	.2070	10	50	.8241	14	.8722	11	.9519	25	.0481	10
32 0	.7242	20	.9284	8	.7958	28	.2042	58 0	42 0	.8255	14	.8711	12	.9544	26	.0456	48 0
10	.7262	20	.9276	8	.7986	28	.2014	50	10	.8269	14	.8699	11	.9570	25	.0430	50
20	.7282	20	.9268	8	.8014	28	.1986	40	20	.8283	14	.8688	12	.9595	26	.0405	40
30	.7302	20	.9260	8	.8042	28	.1958	30	30	.8297	14	.8676	11	.9621	25	.0379	30
40	.7322	20	.9252	8	.8070	27	.1930	20	40	.8311	13	.8665	12	.9646	25	.0354	20
50	.7342	19	.9244	8	.8097	28	.1903	10	50	.8324	14	.8653	12	.9671	26	.0329	10
33 0	.7361	19	.9236	8	.8125	28	.1875	57 0	43 0	.8338	13	.8641	12	.9697	25	.0303	47 0
10	.7380	20	.9228	9	.8153	27	.1847	50	10	.8351	14	.8629	11	.9722	25	.0278	50
20	.7400	19	.9219	8	.8180	28	.1820	40	20	.8365	13	.8618	12	.9747	25	.0253	40
30	.7419	19	.9211	8	.8208	27	.1792	30	30	.8378	13	.8606	12	.9772	26	.0228	30
40	.7438	19	.9203	9	.8235	28	.1765	20	40	.8391	14	.8594	12	.9798	25	.0202	20
50	.7457	19	.9194	8	.8263	27	.1737	10	50	.8405	13	.8582	13	.9823	25	.0177	10
34 0	.7476	18	.9186	9	.8290	27	.1710	56 0	44 0	.8418	13	.8569	12	.9848	26	.0152	46 0
10	.7494	19	.9177	8	.8317	27	.1683	50	10	.8431	13	.8557	12	.9874	25	.0126	50
20	.7513	18	.9169	9	.8344	27	.1656	40	20	.8444	13	.8545	13	.9899	25	.0101	40
30	.7531	19	.9160	9	.8371	27	.1629	30	30	.8457	12	.8532	12	.9924	25	.0076	30
40	.7550	18	.9151	9	.8398	27	.1602	20	40	.8469	13	.8520	13	.9949	26	.0051	20
50	.7568	18	.9142	8	.8425	27	.1575	10	50	.8482	13	.8507	12	9.9975	25	.0025	10
35 0	9.7586	18	9.9134	9	9.8452	27	0.1548	55 0	45 0	9.8495		9.8495		0.0000		0.0000	45
Arc.	Cos.	Df.	Sin.	Df.	Cot.	Df.	Tan.	Arc.	Arc.	Cos.	Df.	Sin.	Df.	Cot.	Df.	Tan.	

TABLE IV.

Logarithmic Traverse Table. § 178.

Zero angle at South Point, and increasing to W. (90°), N. (180°), E. (270°).

Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.	Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.	Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.
0° 180°		**10.0000**	**180° 360°**	**1° 181°**	**8.2419**	**9.9999**	**179° 359°**	**2° 182°**	**8.5428**	**9.9997**	**178° 358°**
1′	6.4637	.0000	59′	1′	.2490	.9999	59′	1′	.5464	.9997	59′
2	.7648	.0000	58	2	.2561	.9999	58	2	.5500	.9997	58
3	6.9404	.0000	57	3	.2630	.9999	57	3	.5535	.9997	57
4	7.0658	.0000	56	4	.2699	.9999	56	4	.5571	.9997	56
— 5 —	.1627	.0000	— 55 —	— 5 —	.2766	.9999	— 55 —	— 5 —	.5605	.9997	— 55 —
6	.2419	.0000	54	6	.2832	.9999	54	6	.5640	.9997	54
7	.3088	.0000	53	7	.2898	.9999	53	7	.5674	.9997	53
8	.3668	.0000	52	8	.2962	.9999	52	8	.5708	.9997	52
9	.4180	.0000	51	9	.3025	.9999	51	9	.5742	.9997	51
10	**7.4637**	**10.0000**	**50**	**10**	**8.3088**	**9.9999**	**50**	**10**	**8.5776**	**9.9997**	**50**
11	.5051	.0000	49	11	.3150	.9999	49	11	.5809	.9997	49
12	.5429	.0000	48	12	.3210	.9999	48	12	.5842	.9997	48
13	.5777	.0000	47	13	.3270	.9999	47	13	.5875	.9997	47
14	.6099	.0000	46	14	.3229	.9999	46	14	.5907	.9997	46
— 15 —	.6398	.0000	— 45 —	— 15 —	.3388	.9999	— 45 —	— 15 —	.5939	.9997	— 45 —
16	.6678	.0000	44	16	.3445	.9999	44	16	.5972	.9997	44
17	.6942	.0000	43	17	.3502	.9999	43	17	.6003	.9997	43
18	.7190	.0000	[illegible]	18	.3558	.9999	42	18	.6035	.9996	42
19	.7425	.0000	[illegible]	19	.3613	.9999	41	19	.6066	.9996	41
20	**7.7648**	**10.0000**	**4**	**20**	**8.3668**	**9.9999**	**40**	**20**	**8.6097**	**9.9996**	**40**
21	.7859	.0000	39	21	.3722	.9999	39	21	.6128	.9996	39
22	.8061	.0000	38	22	.3775	.9999	38	22	.6159	.9996	38
23	.8255	.0000	37	23	.3828	.9999	37	23	.6189	.9996	37
24	.8439	.0000	36	24	.3880	.9999	36	24	.6220	.9996	36
— 25 —	.8617	.0000	— 35 —	— 25 —	.3931	.9999	— 35 —	— 25 —	.6250	.9996	— 35 —

26	.8787	.0000	34	26	.3982	.9999	34	26	.6279	.9996	34
27	.8951	.0000	33	27	.4032	.9999	33	27	.6309	.9996	33
28	.9109	.0000	32	28	.4082	.9999	32	28	.6339	.9996	32
29	.9261	.0000	31	29	.4131	.9999	31	29	.6368	.9996	31
30	**7.9408**	**10.0000**	**30**	**30**	**8.4179**	**9.9999**	**30**	**30**	**8.6397**	**9.9996**	**30**
31	.9551	.0000	29	31	.4227	.9998	29	31	.6426	.9996	29
32	.9687	.0000	28	32	.4275	.9998	28	32	.6454	.9996	28
33	.9822	.0000	27	33	.4322	.9998	27	33	.6483	.9996	27
34	7.9952	.0000	26	34	.4368	.9998	26	34	.6511	.9996	26
— 35 —	8.0078	.0000	— 25 —	— 35 —	.4414	.9998	— 25 —	— 35 —	.6539	.9966	— 25 —
36	.0200	.0000	24	36	.4459	.9998	24	36	.6567	.9996	24
37	.0319	.0000	23	37	.4504	.9998	23	37	.6595	.9995	23
38	.0435	.0000	22	38	.4549	.9998	22	38	.6622	.9995	22
39	.0548	.0000	21	39	.4593	.9998	21	39	.6650	.9995	21
40	**8.0658**	**10.0000**	**20**	**40**	**8.4637**	**9.9998**	**20**	**40**	**8.6677**	**9.9995**	**20**
41	.0765	.0000	19	41	.4680	.9998	19	41	.6704	.9995	19
42	.0870	.0000	18	42	.4723	.9998	18	42	.6731	.9995	18
43	.0972	.0000	17	43	.4765	.9998	17	43	.6758	.9995	17
44	.1072	.0000	16	44	.4807	.9998	16	44	.6784	.9995	16
— 45 —	.1169	.0000	— 15 —	— 45 —	.4848	.9998	— 15 —	— 45 —	.6810	.9995	— 15 —
46	.1265	.0000	14	46	.4890	.9998	14	46	.6837	.9995	14
47	.1358	.0000	13	47	.4930	.9998	13	47	.6863	.9995	13
48	.1450	.0000	12	48	.4971	.9998	12	48	.6889	.9995	12
49	.1539	.0000	11	49	.5011	.9998	11	49	.6914	.9995	11
50	**8.1627**	**10.0000**	**10**	**50**	**8.5050**	**9.9998**	**10**	**50**	**8.6940**	**9.9995**	**10**
51	.1713	.0000	9	51	.5090	.9998	9	51	.6965	.9995	9
52	.1797	.0000	8	52	.5129	.9998	8	52	.6991	.9995	8
53	.1880	9.9999	7	53	.5167	.9998	7	53	.7016	.9994	7
54	.1961	.9999	6	54	.5206	.9998	6	54	.7041	.9994	6
— 55 —	.2041	.9999	— 5 —	— 55 —	.5243	.9998	— 5 —	— 55 —	.7066	.9994	— 5 —
56	.2119	.9999	4	56	.5281	.9998	4	56	.7090	.9994	4
57	.2196	.9999	3	57	.5318	.9997	3	57	.7115	.9994	3
58	.2271	.9999	2	58	.5355	.9997	2	58	.7140	.9994	2
59	.2346	.9999	1	59	.5392	.9997	1	59	.7164	.9994	1
1° 181°	**8.2419**	**9.9999**	**179° 359°**	**2° 182°**	**8.5428**	**9.9997**	**178° 358°**	**3° 183°**	**8.7188**	**9.9994**	**177° 357°**

TABLE IV.—*Continued.*

Logarithmic Traverse Table.

Zero angle at South Point, and increasing to W. (90°), N. (180°), E. (270°).

Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Sin. Dif. for 1′.	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.	Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Sin. Dif. for 1′.	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.	Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Sin. Dif. for 1′.	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.
3° 183°	8.7188	23.5	9.9994	**177° 357°**	**17° 197°**	9.4659	4.1	9.9806	**163° 343°**	**31° 211°**	9.7118	2.1	9.9331	**149° 329°**
10′	.7423	22.2	.9993	50′	10′	.4700	4.1	.9802	50′	10′	.7139	2.1	.9323	50′
20	.7645	21.2	.9993	40	20	.4741	4.0	.9798	40	20	.7160	2.1	.9315	40
— 30 —	.7857	20.2	.9992	— 30 —	— 30 —	.4781	4.0	.9794	— 30 —	— 30 —	.7181	2.0	.9308	— 30 —
40	.8059	19.2	.9991	20	40	.4821	4.0	.9790	20	40	.7201	2.1	.9300	20
50	.8251	18.5	.9990	10	50	.4861	3.9	.9786	10	50	.7222	2.0	.9292	10
4° 184°	8.8436	17.7	9.9989	**176° 356°**	**18° 198°**	9.4900	3.9	9.9782	**162° 342°**	**32° 212°**	9.7242	2.0	9.9284	**148° 328°**
10	.8613	17.0	.9989	50	10	.4939	3.8	.9778	50	10	.7262	2.0	.9276	50
20	.8783	16.3	.9988	40	20	.4977	3.8	.9774	40	20	.7282	2.0	.9268	40
— 30 —	.8946	15.8	.9987	— 30 —	— 30 —	.5015	3.7	.9770	— 30 —	— 30 —	.7302	2.0	.9260	— 30 —
40	.9104	15.2	.9986	20	40	.5052	3.8	.9765	20	40	.7322	2.0	.9252	20
50	.9256	14.7	.9985	10	50	.5090	3.6	.9761	10	50	.7342	1.9	.9244	10
5° 185°	8.9403	14.2	9.9983	**175° 355°**	**19° 199°**	9.5126	3.7	9.9757	**161° 341°**	**33° 213°**	9.7361	1.9	9.9236	**147° 327°**
10	.9545	13.7	.9982	50	10	.5163	3.6	.9752	50	10	.7380	2.0	.9228	50
20	.9682	13.4	.9981	40	20	.5199	3.6	.9748	40	20	.7400	1.9	.9219	40
— 30 —	.9816	13.1	.9980	— 30 —	— 30 —	.5235	3.5	.9743	— 30 —	— 30 —	.7419	1.9	.9211	— 30 —
40	8.9945	12.5	.9979	20	40	.5270	3.6	.9739	20	40	.7438	1.9	.9203	20
50	9.0070	12.2	.9977	10	50	.5306	3.5	.9734	10	50	.7457	1.9	.9194	10
6° 186°	9.0192	11.9	9.9976	**174° 354°**	**20° 200°**	9.5341	3.4	9.9730	**160° 340°**	**34° 214°**	9.7476	1.8	9.9186	**146° 326°**
10	.0311	11.5	.9975	50	10	.5375	3.4	.9725	50	10	.7494	1.9	.9177	50
20	.0426	11.3	.9973	40	20	.5409	3.4	.9721	40	20	.7513	1.8	.9169	40
— 30 —	.0539	10.9	.9972	— 30 —	— 30 —	.5443	3.4	.9716	— 30 —	— 30 —	.7531	1.9	.9160	— 30 —
40	.0648	10.7	.9971	20	40	.5477	3.3	.9711	20	40	.7550	1.8	.9151	20
50	.0755	10.4	.9969	10	50	.5510	3.3	.9706	10	50	.7568	1.8	.9142	10
7° 187°	9.0859	10.2	9.9968	**173° 353°**	**21° 201°**	9.5543	3.3	9.9702	**159° 339°**	**35° 215°**	9.7586	1.8	9.9134	**145° 325°**
10	.0961	9.9	.9966	50	10	.5576	3.3	.9697	50	10	.7604	1.8	.9125	50
20	.1060	9.7	.9964	40	20	.5609	3.2	.9692	40	20	.7622	1.8	.9116	40
— 30 —	.1157	9.5	.9963	— 30 —	— 30 —	.5641	3.2	.9687	— 30 —	— 30 —	.7640	1.7	.9107	— 30 —
40	.1252	9.3	.9961	20	40	.5673	3.1	.9682	20	40	.7657	1.8	.9098	20
50	.1345	9.1	.9959	10	50	.5704	3.2	.9677	10	50	.7675	1.7	.9089	10
8° 188°	9.1436	8.9	9.9958	**172° 352°**	**22° 202°**	9.5736	3.1	9.9672	**158° 338°**	**36° 216°**	9.7692	1.8	9.9080	**144° 324°**
10	.1525	8.7	.9956	50	10	.5767	3.1	.9667	50	10	.7710	1.7	.9070	50
20	.1612	8.5	.9954	40	20	.5798	3.0	.9661	40	20	.7727	1.7	.9061	40
— 30 —	.1697	8.4	.9952	— 30 —	— 30 —	.5828	3.1	.9656	— 30 —	— 30 —	.7744	1.7	.9052	— 30 —
40	.1781	8.2	.9950	20	40	.5859	3.0	.9651	20	40	.7761	1.7	.9042	20
50	.1863	8.0	.9948	10	50	.5889	3.0	.9646	10	50	.7778	1.7	.9033	10
9° 189°	9.1943		9.9946	**171° 351°**	**23° 203°**	9.5919		9.9640	**157° 337°**	**37° 217°**	9.7795		9.9023	**143° 323°**

9° 189°	9.1943	7.9	9.9946	**171° 351°**	**23° 203°**	9.5919	2.9	9.9640	**157° 337°**	**37° 217°**	9.7795	1.6	9.9023	**143° 323°**
10	.2022	7.8	.9944	50	10	.5948	3.0	.9635	50	10	.7811	1.7	.9014	50
20	.2100	7.6	.9942	40	20	.5978	2.9	.9629	40	20	.7828	1.6	.9004	40
— 30 —	.2176	7.5	.9940	— 30 —	— 30 —	.6007	2.9	.9624	— 30 —	— 30 —	.7844	1.7	.8995	— 30 —
40	.2251	7.3	.9938	20	40	.6036	2.9	.9618	20	40	.7861	1.6	.8985	20
50	.2324	7.3	.9936	10	50	.6065	2.8	.9613	10	50	.7877	1.6	.8975	10
10° 190°	9.2397	7.1	9.9934	**170° 350°**	**24° 204°**	9.6093	2.8	9.9607	**156° 336°**	**38° 218°**	9.7893	1.7	9.8965	**142° 322°**
10	.2468	7.0	.9931	50	10	.6121	2.8	.9602	50	10	.7910	1.6	.8955	50
20	.2538	6.8	.9929	40	20	.6149	2.8	.9596	40	20	.7926	1.5	.8945	40
— 30 —	.2606	6.8	.9927	— 30 —	— 30 —	.6177	2.8	.9590	— 30 —	— 30 —	.7941	1.6	.8935	— 30 —
40	.2674	6.6	.9924	20	40	.6205	2.7	.9584	20	40	.7957	1.6	.8925	20
50	.2740	6.6	.9922	10	50	.6232	2.7	.9579	10	50	.7973	1.6	.8915	10
11° 191°	9.2806	6.4	9.9919	**169° 349°**	**25° 205°**	9.6259	2.7	9.9573	**155° 335°**	**39° 219°**	9.7989	1.5	9.8905	**141° 321°**
10	.2870	6.4	.9917	50	10	.6286	2.7	.9567	50	10	.8004	1.6	.8895	50
20	.2934	6.3	.9914	40	20	.6313	2.7	.9561	40	20	.8020	1.5	.8884	40
— 30 —	.2997	6.1	.9912	— 30 —	— 30 —	.6340	2.6	.9555	— 30 —	— 30 —	.8035	1.5	.8874	— 30 —
40	.3058	6.1	.9909	20	40	.6366	2.6	.9549	20	40	.8050	1.6	.8864	20
50	.3119	6.0	.9907	10	50	.6392	2.6	.9543	10	50	.8066	1.5	.8853	10
12° 192°	9.3179	5.9	9.9904	**168° 348°**	**26° 206°**	9.6418	2.6	9.9537	**154° 334°**	**40° 220°**	9.8081	1.5	9.8843	**140° 320°**
10	.3238	5.8	.9901	50	10	.6444	2.6	.9530	50	10	.8096	1.5	.8832	50
20	.3296	5.7	.9899	40	20	.6470	2.5	.9524	40	20	.8111	1.4	.8821	40
— 30 —	.3353	5.7	.9896	— 30 —	— 30 —	.6495	2.6	.9518	— 30 —	— 30 —	.8125	1.5	.8810	— 30 —
40	.3410	5.6	.9893	20	40	.6521	2.5	.9512	20	40	.8140	1.5	.8800	20
50	.3466	5.5	.9890	10	50	.6546	2.4	.9505	10	50	.8155	1.4	.8789	10
13° 193°	9.3521	5.4	9.9887	**167° 347°**	**27° 207°**	9.6570	2.5	9.9499	**153° 333°**	**41° 221°**	9.8169	1.5	9.8778	**139° 319°**
10	.3575	5.4	.9884	50	10	.6595	2.5	.9492	50	10	.8184	1.4	.8767	50
20	.3629	5.3	.9881	40	20	.6620	2.4	.9486	40	20	.8198	1.5	.8756	40
— 30 —	.3682	5.2	.9878	— 30 —	— 30 —	.6644	2.4	.9479	— 30 —	— 30 —	.8213	1.4	.8745	— 30 —
40	.3734	5.2	.9875	20	40	.6668	2.4	.9473	20	40	.8227	1.4	.8733	20
50	.3786	5.1	.9872	10	50	.6692	2.4	.9466	10	50	.8241	1.4	.8722	10
14° 194°	9.3837	5.0	9.9869	**166° 346°**	**28° 208°**	9.6716	2.4	9.9459	**152° 332°**	**42° 222°**	9.8255	1.4	9.8711	**138° 318°**
10	.3887	5.0	.9866	50	10	.6740	2.3	.9453	50	10	.8269	1.4	.8699	50
20	.3937	4.9	.9863	40	20	.6763	2.4	.9446	40	20	.8283	1.4	.8688	40
— 30 —	.3986	4.9	.9859	— 30 —	— 30 —	.6787	2.3	.9439	— 30 —	— 30 —	.8297	1.4	.8676	— 30 —
40	.4035	4.8	.9856	20	40	.6810	2.3	.9432	20	40	.8311	1.3	.8665	20
50	.4083	4.7	.9853	10	50	.6833	2.3	.9425	10	50	.8324	1.4	.8653	10
15° 195°	9.4130	4.7	9.9849	**165° 345°**	**29° 209°**	9.6856	2.2	9.9418	**151° 331°**	**43° 223°**	9.8338	1.3	9.8641	**137° 317°**
10	.4177	4.6	.9846	50	10	.6878	2.3	.9411	50	10	.8351	1.4	.8629	50
20	.4223	4.6	.9843	40	20	.6901	2.2	.9404	40	20	.8365	1.3	.8618	40
— 30 —	.4269	4.5	.9839	— 30 —	— 30 —	.6923	2.3	.9397	— 30 —	— 30 —	.8378	1.3	.8606	— 30 —
40	.4314	4.5	.9836	20	40	.6946	2.2	.9390	20	40	.8391	1.4	.8594	20
50	.4359	4.4	.9832	10	50	.6968	2.2	.9383	10	50	.8405	1.3	.8582	10
16° 196°	9.4403	4.4	9.9828	**164° 344°**	**30° 210°**	9.6990	2.2	9.9375	**150° 330°**	**44° 224°**	9.8418	1.3	9.8569	**136° 316°**
10	.4447	4.4	.9825	50	10	.7012	2.1	.9368	50	10	.8431	1.3	.8557	50
20	.4491	4.2	.9821	40	20	.7033	2.2	.9361	40	20	.8444	1.3	.8545	40
— 30 —	.4533	4.3	.9817	— 30 —	— 30 —	.7055	2.1	.9353	— 30 —	— 30 —	.8457	1.2	.8532	— 30 —
40	.4576	4.2	.9814	20	40	.7076	2.1	.9346	20	40	.8469	1.3	.8520	20
50	.4618	4.1	.9810	10	50	.7097	2.1	.9338	10	50	.8482	1.3	.8507	10
17° 197°	9.4659		9.9806	**163° 343°**	**31° 211°**	9.7118		9.9331	**149° 329°**	**45° 225°**	9.8495		9.8495	**135° 315°**

TABLE IV.

LOGARITHMIC TRAVERSE TABLE. § 178.

Zero angle at South Point, and increasing to W. (90°), N. (180°), E. (270°).

Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.	Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.	Arc 1st and 3d. Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th. Quadrants.
0° 180°		**10.0000**	**180° 360°**	**1° 181°**	**8.2419**	**9.9999**	**179° 359°**	**2° 182°**	**8.5428**	**9.9997**	**178° 358°**
1′	6.4637	.0000	59′	1′	.2490	.9999	59′	1′	.5464	.9997	59′
2	.7648	.0000	58	2	.2561	.9999	58	2	.5500	.9997	58
3	6.9404	.0000	57	3	.2630	.9999	57	3	.5535	.9997	57
4	7.0658	.0000	56	4	.2699	.9999	56	4	.5571	.9997	56
— 5 —	.1627	.0000	— 55 —	— 5 —	.2766	.9999	— 55 —	— 5 —	.5605	.9997	— 55 —
6	.2419	.0000	54	6	.2832	.9999	54	6	.5640	.9997	54
7	.3088	.0000	53	7	.2898	.9999	53	7	.5674	.9997	53
8	.3668	.0000	52	8	.2962	.9999	52	8	.5708	.9997	52
9	.4180	.0000	51	9	.3025	.9999	51	9	.5742	.9997	51
10	**7.4637**	**10.0000**	**50**	**10**	**8.3088**	**9.9999**	**50**	**10**	**8.5776**	**9.9997**	**50**
11	.5051	.0000	49	11	.3150	.9999	49	11	.5809	.9997	49
12	.5429	.0000	48	12	.3210	.9999	48	12	.5842	.9997	48
13	.5777	.0000	47	13	.3270	.9999	47	13	.5875	.9997	47
14	.6099	.0000	46	14	.3329	.9999	46	14	.5907	.9997	46
— 15 —	.6398	.0000	— 45 —	— 15 —	.3388	.9999	— 45 —	— 15 —	.5939	.9997	— 45 —
16	.6678	.0000	44	16	.3445	.9999	44	16	.5972	.9997	44
17	.6942	.0000	43	17	.3502	.9999	43	17	.6003	.9997	43
18	.7190	.0000		18	.3558	.9999	42	18	.6035	.9996	42
19	.7425	.0000		19	.3613	.9999	41	19	.6066	.9996	41
20	**7.7648**	**10.0000**	**4[illegible]**	**20**	**8.3668**	**9.9999**	**40**	**20**	**8.6097**	**9.9996**	**40**
21	.7859	.0000	39	21	.3722	.9999	39	21	.6128	.9996	39
22	.8061	.0000	38	22	.3775	.9999	38	22	.6159	.9996	38
23	.8255	.0000	37	23	.3828	.9999	37	23	.6189	.9996	37
24	.8439	.0000	36	24	.3880	.9999	36	24	.6220	.9996	36
— 25 —	.8617	.0000	— 35 —	— 25 —	.3931	.9999	— 35 —	— 25 —	.6250	.9996	— 35 —

51° 231°	9.8905	1.6	9.7989	129° 309°	65° 245°	9.9573	2.7	9.6259	115° 295°	79° 259°	9.9919	6.6	9.2806	101° 281°
10	.8915	1.6	.7973	50	10	.9579	2.7	.6232	50	10	.9922	6.6	.2740	50
20	.8925	1.6	.7957	40	20	.9584	2.8	.6205	40	20	.9924	6.8	.2674	40
— 30 —	.8935	1.5	.7941	— 30 —	— 30 —	.9590	2.8	.6177	— 30 —	— 30 —	.9927	6.8	.2606	— 30 —
40	.8945	1.6	.7926	20	40	.9596	2.8	.6149	20	40	.9929	7.0	.2538	20
50	.8955	1.7	.7910	10	50	.9602	2.8	.6121	10	50	.9931	7.1	.2468	10
52° 232°	9.8965	1.6	9.7893	128° 308°	66° 246°	9.9607	2.8	9.6093	114° 294°	80° 260°	9.9934	7.3	9.2397	100° 280°
10	.8975	1.6	.7877	50	10	.9613	2.9	.6064	50	10	.9936	7.3	.2324	50
20	.8985	1.7	.7861	40	20	.9618	2.9	.6036	40	20	.9938	7.5	.2251	40
— 30 —	.8995	1.6	.7844	— 30 —	— 30 —	.9624	2.9	.6007	— 30 —	— 30 —	.9940	7.6	.2176	— 30 —
40	.9004	1.7	.7828	20	40	.9629	3.0	.5978	20	40	.9942	7.8	.2100	20
50	.9014	1.6	.7811	10	50	.9635	2.9	.5948	10	50	.9944	7.9	.2022	10
53° 233°	9.9023	1.7	9.7795	127° 307°	67° 247°	9.9640	3.0	9.5919	113° 293°	81° 261°	9.9946	8.0	9.1943	99° 279°
10	.9033	1.7	.7778	50	10	.9646	3.0	.5889	50	10	.9948	8.2	.1863	50
20	.9042	1.7	.7761	40	20	.9651	3.1	.5859	40	20	.9950	8.4	.1781	40
— 30 —	.9052	1.7	.7744	— 30 —	— 30 —	.9656	3.0	.5828	— 30 —	— 30 —	.9952	8.5	.1697	— 30 —
40	.9061	1.7	.7727	20	40	.9661	3.1	.5798	20	40	.9954	8.7	.1612	20
50	.9070	1.8	.7710	10	50	.9667	3.1	.5767	10	50	.9956	8.9	.1525	10
54° 234°	9.9080	1.7	9.7692	126° 306°	68° 248°	9.9672	3.2	9.5736	112° 292°	82° 262°	9.9958	9.1	9.1436	98° 278°
10	.9089	1.8	.7675	50	10	.9677	3.1	.5704	50	10	.9959	9.3	.1345	50
20	.9098	1.7	.7657	40	20	.9682	3.2	.5673	40	20	.9961	9.5	.1252	40
— 30 —	.9107	1.8	.7640	— 30 —	— 30 —	.9687	3.2	.5641	— 30 —	— 30 —	.9963	9.7	.1157	— 30 —
40	.9116	1.8	.7622	20	40	.9692	3.3	.5609	20	40	.9964	9.9	.1060	20
50	.9125	1.8	.7604	10	50	.9697	3.3	.5576	10	50	.9966	10.2	.0961	10
55° 235°	9.9134	1.8	9.7586	125° 305°	69° 249°	9.9702	3.3	9.5543	111° 291°	83° 263°	9.9968	10.4	9.0859	97° 277°
10	.9142	1.8	.7568	50	10	.9706	3.3	.5510	50	10	.9969	10.7	.0755	50
20	.9151	1.9	.7550	40	20	.9711	3.4	.5477	40	20	.9971	10.9	.0648	40
— 30 —	.9160	1.8	.7531	— 30 —	— 30 —	.9716	3.4	.5443	— 30 —	— 30 —	.9972	11.3	.0539	— 30 —
40	.9169	1.9	.7513	20	40	.9721	3.4	.5409	20	40	.9973	11.5	.0426	20
50	.9177	1.8	.7494	10	50	.9725	3.4	.5375	10	50	.9975	11.9	.0311	10
56° 236°	9.9186	1.9	9.7476	124° 304°	70° 250°	9.9730	3.5	9.5341	110° 290°	84° 264°	9.9976	12.2	9.0192	96° 276°
10	.9194	1.9	.7457	50	10	.9734	3.6	.5306	50	10	.9977	12.5	9.0070	50
20	.9203	1.9	.7438	40	20	.9739	3.6	.5270	40	20	.9979	12.9	8.9945	40
— 30 —	.9211	1.9	.7419	— 30 —	— 30 —	.9743	3.6	.5235	— 30 —	— 30 —	.9980	13.4	.9816	— 30
40	.9219	2.0	.7400	20	40	.9748	3.6	.5199	20	40	.9981	13.7	.9682	20
50	.9228	1.9	.7380	10	50	.9752	3.5	.5163	10	50	.9982	14.2	.9545	10
57° 237°	9.9236	1.9	9.7361	123° 303°	71° 251°	9.9757	3.6	9.5126	109° 289°	85° 265°	9.9983	14.7	8.9403	95° 275°
10	.9244	2.0	.7342	50	10	.9761	3.8	.5090	50	10	.9985	15.2	.9256	50
20	.9252	2.0	.7322	40	20	.9765	3.7	.5052	40	20	.9986	15.8	.9104	40
— 30 —	.9260	2.0	.7302	— 30 —	— 30 —	.9770	3.8	.5015	— 30 —	— 30 —	.9987	16.3	.8946	— 30 —
40	.9268	2.0	.7282	20	40	.9774	3.8	.4977	20	40	.9988	17.0	.8783	20
50	.9276	2.0	.7262	10	50	.9778	3.9	.4939	10	50	.9989	17.7	.8613	10
58° 238°	9.9284	2.0	9.7242	122° 302°	72° 252°	9.9782	3.9	9.4900	108° 288°	86° 266°	9.9989	18.5	8.8436	94° 274°
10	.9292	2.1	.7222	50	10	.9786	4.0	.4861	50	10	.9990	19.2	.8251	50
20	.9300	2.0	.7201	40	20	.9790	4.0	.4821	40	20	.9991	20.2	.8059	40
— 30 —	.9308	2.1	.7181	— 30 —	— 30 —	.9794	4.0	.4781	— 30 —	— 30 —	.9992	21.2	.7857	— 30 —
40	.9315	2.1	.7160	20	40	.9798	4.1	.4741	20	40	.9993	22.2	.7645	20
50	.9323	2.1	.7139	10	50	.9802	4.1	.4700	10	50	.9993	23.5	.7423	10
59° 239°	9.9331		9.7118	121° 301°	73° 253°	9.9806		9.4659	107° 287°	87° 267°	9.9994		8.7188	93° 273°

TABLE IV.—*Continued.*

Logarithmic Traverse Table.

Zero angle at South Point, and increasing to W. (90°), N. (180°), E. (270°).

Arc 1st and 3d Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th Quadrants.	Arc 1st and 3d Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th Quadrants.	Arc 1st and 3d Quadrants.	Log. sin. (Dep.)	Log. cos. (Lat.)	Arc 2d and 4th Quadrants.
87° 267°	**9.9994**	**8.7188**	**93° 273°**	**88° 268°**	**9.9997**	**8.5428**	**92° 272°**	**89° 269°**	**9.9999**	**8.2419**	**91° 271°**
1′	.9994	.7164	59′	1′	.9997	.5392	59′	1′	.9999	.2346	59′
2	.9994	.7140	58	2	.9997	.5355	58	2	.9999	.2271	58
3	.9994	.7115	57	3	.9997	.5318	57	3	.9999	.2196	57
4	.9994	.7090	56	4	.9998	.5281	56	4	.9999	.2119	56
— 5 —	.9994	.7066	— 55 —	— 5 —	.9998	.5243	— 55 —	— 5 —	.9999	.2041	— 55 —
6	.9994	.7041	54	6	.9998	.5206	54	6	.9999	.1961	54
7	.9994	.7016	53	7	.9998	.5167	53	7	9.9999	.1880	53
8	.9995	.6991	52	8	.9998	.5129	52	8	10.0000	.1797	52
9	.9995	.6965	51	9	.9998	.5090	51	9	.0000	.1713	51
10	**9.9995**	**8.6940**	**50**	**10**	**9.9998**	**8.5050**	**50**	**10**	**10.0000**	**8.1627**	**50**
11	.9995	.6914	49	11	.9998	.5011	49	11	.0000	.1539	49
12	.9995	.6889	48	12	.9998	.4971	48	12	.0000	.1450	48
13	.9995	.6863	47	13	.9998	.4930	47	13	.0000	.1358	47
14	.9995	.6837	46	14	.9998	.4890	46	14	.0000	.1265	46
— 15 —	.9995	.6810	— 45 —	— 15 —	.9998	.4848	— 45 —	— 15 —	.0000	.1169	— 45 —
16	.9995	.6784	44	16	.9998	.4807	44	16	.0000	.1072	44
17	.9995	.6758	43	17	.9998	.4765	43	17	.0000	.0972	43
18	.9995	.6731	42	18	.9998	.4723	42	18	.0000	.0870	42
19	.9995	.6704	41	19	.9998	.4680	41	19	.0000	.0765	41
20	**9.9995**	**8.6677**	**40**	**20**	**9.9998**	**8.4637**	**40**	**20**	**10.0000**	**8.0658**	**40**
21	.9995	.6650	39	21	.9998	.4593	39	21	.0000	.0548	39
22	.9995	.6622	38	22	.9998	.4549	38	22	.0000	.0435	38
23	.9995	.6595	37	23	.9998	.4504	37	23	.0000	.0319	37
24	.9996	.6567	36	24	.9998	.4459	36	24	.0000	.0200	36
— 25 —	.9996	.6539	— 35 —	— 25 —	.9998	.4418	— 35 —	— 25 —	.0000	8.0078	— 35 —

26	.9996	.6511	34	26	.9998	.4368	34	26	.0000	7.9952	34
27	.9996	.6483	33	27	.9998	.4322	33	27	.0000	.9822	33
28	.9996	.6454	32	28	.9998	.4275	32	28	.0000	.9689	32
29	.9996	.6426	31	29	.9998	.4227	31	29	.0000	.9551	31
30	**9.9996**	**8.6397**	**30**	**30**	**9.9999**	**8.4179**	**30**	**30**	**10.0000**	**7.9408**	**30**
31	.9996	.6368	29	31	.9999	.4131	29	31	.0000	.9261	29
32	.9996	.6339	28	32	.9999	.4082	28	32	.0000	.9109	28
33	.9996	.6309	27	33	.9999	.4032	27	33	.0000	.8951	27
34	.9996	.6279	26	34	.9999	.3982	26	34	.0000	.8787	26
— 35 —	.9996	.6250	— 25 —	— 35 —	.9999	.3931	— 25 —	— 35 —	.0000	.8617	— 25 —
36	.9996	.6220	24	36	.9999	.3880	24	36	.0000	.8439	24
37	.9996	.6189	23	37	.9999	.3828	23	37	.0000	.8255	23
38	.9996	.6159	22	38	.9999	.3775	22	38	.0000	.8061	22
39	.9996	.6128	21	39	.9999	.3722	21	39	.0000	.7859	21
40	**9.9996**	**8.6097**	**20**	**40**	**9.9999**	**8.3668**	**20**	**40**	**10.0000**	**7.7648**	**20**
41	.9996	.6066	19	41	.9999	.3613	19	41	.0000	.7425	19
42	.9996	.6035	18	42	.9999	.3558	18	42	.0000	.7190	18
43	.9997	.6003	17	43	.9999	.3502	17	43	.0000	.6942	17
44	.9997	.5972	16	44	.9999	.3445	16	44	.0000	.6678	16
— 45 —	.9997	.5939	— 15 —	— 45 —	.9999	.3388	— 15 —	— 45 —	.0000	.6398	— 15 —
46	.9997	.5907	14	46	.9999	.3329	14	46	.0000	.6099	14
47	.9997	.5875	13	47	.9999	.3270	13	47	.0000	.5777	13
48	.9997	.5842	12	48	.9999	.3210	12	48	.0000	.5429	12
49	.9997	.5809	11	49	.9999	.3150	11	49	.0000	.5051	11
50	**9.9997**	**8.5776**	**10**	**50**	**9.9999**	**8.3088**	**10**	**50**	**10.0000**	**7.4637**	**10**
51	.9997	.5742	9	51	.9999	.3025	9	51	.0000	.4180	9
52	.9997	.5708	8	52	.9999	.2962	8	52	.0000	.3668	8
53	.9997	.5674	7	53	.9999	.2898	7	53	.0000	.3088	7
54	.9997	.5640	6	54	.9999	.2832	6	54	.0000	.2419	6
— 55 —	.9997	.5605	— 5 —	— 55 —	.9999	.2766	— 5 —	— 55 —	.0000	.1627	— 5 —
56	.9997	.5571	4	56	.9999	.2699	4	56	.0000	7.0658	4
57	.9997	.5535	3	57	.9999	.2630	3	57	.0000	6.9408	3
58	.9997	.5500	2	58	.9999	.2561	2	58	.0000	.7648	2
59′	.9997	.5464	1′	59′	.9999	.2490	1′	59′	.0000	6.4637	1′
88° 268°	**9.9997**	**8.5428**	**92° 272°**	**89° 269°**	**9.9999**	**8.2419**	**91° 271°**	**90° 270°**	**10.0000**		**90° 270°**

TABLE V.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS. § 204.

Minutes.	0° Hor. Dist.	0° Diff. Elev.	1° Hor. Dist.	1° Diff. Elev.	2° Hor. Dist.	2° Diff. Elev.	3° Hor. Dist.	3° Diff. Elev
0 . .	100.00	0.00	99.97	1.74	99.88	3.49	99.73	5.23
2 . .	"	0.06	"	1.80	99.87	3.55	99.72	5.28
4 . .	"	0.12	"	1.86	"	3.60	99.71	5.34
6 . .	"	0.17	99.96	1.92	"	3.66	"	5.40
8 . .	"	0.23	"	1.98	99.86	3.72	99.70	5.46
10 . .	"	0.29	"	2.04	"	3.78	99.69	5.52
12 . .	"	0.35	"	2.09	99.85	3.84	"	5.57
14 . .	"	0.41	99.95	2.15	"	3.90	99.68	5.63
16 . .	"	0.47	"	2.21	99.84	3.95	"	5.69
18 . .	"	0.52	"	2.27	"	4.01	99.67	5.75
20 . .	"	0.58	"	2.33	99.83	4.07	99.66	5.80
22 . .	"	0.64	99.94	2.38	"	4.13	"	5.86
24 . .	"	0.70	"	2.44	99.82	4.18	99.65	5.92
26 . .	99.99	0.76	"	2.50	"	4.24	99.64	5.98
28 . .	"	0.81	99.93	2.56	99.81	4.30	99.63	6.04
30 . .	"	0.87	"	2.62	"	4.36	"	6.09
32 . .	"	0.93	"	2.67	99.80	4.42	99.62	6.15
34 . .	"	0.99	"	2.73	"	4.48	"	6.21
36 . .	"	1.05	99.92	2.79	99.79	4.53	99.61	6.27
38 . .	"	1.11	"	2.85	"	4.59	99.60	6.33
40 . .	"	1.16	"	2.91	99.78	4.65	99.59	6.38
42 . .	"	1.22	99.91	2.97	"	4.71	"	6.44
44 . .	99.98	1.28	"	3.02	99.77	4.76	99.58	6.50
46 . .	"	1.34	99.90	3.08	"	4.82	99.57	6.56
48 . .	"	1.40	"	3.14	99.76	4.88	99.56	6.61
50 . .	"	1.45	"	3.20	"	4.94	"	6.67
52 . .	"	1.51	99.89	3.26	99.75	4.99	99.55	6.73
54 . .	"	1.57	"	3.31	99.74	5.05	99.54	6.78
56 . .	99.97	1.63	"	3.37	"	5.11	99.53	6.84
58 . .	"	1.69	99.88	3.43	99.73	5.17	99.52	6.90
60 . .	"	1.74	"	3.49	"	5.23	99.51	6.96
$c = 0.75$	0.75	0.01	0.75	0.02	0.75	0.03	0.75	0.05
$c = 1.00$	1.00	0.01	1.00	0.03	1.00	0.04	1.00	0.06
$c = 1.25$	1 25	0.02	1.25	0.03	1.25	0.05	1.25	0.08

* *This table was computed by* Mr. Arthur Winslow of the State Geological Survey of Penn*sia.* *See also Colby's* Slide Rule, p. 265.

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	4°		5°		6°		7°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	99.51	6.96	99.24	8.68	98.91	10.40	98.51	12.10
2 . .	"	7.02	99.23	8.74	98.90	10.45	98.50	12.15
4 . .	99.50	7.07	99.22	8.80	98.88	10.51	98.48	12.21
6 . .	99.49	7.13	99.21	8.85	98.87	10.57	98.47	12.26
8 . .	99.48	7.19	99.20	8.91	98.86	10.62	98.46	12.32
10 . .	99.47	7.25	99.19	8.97	98.85	10.68	98.44	12.38
12 . .	99.46	7.30	99.18	9.03	98.83	10.74	98.43	12.43
14 . .	"	7.36	99.17	9.08	98.82	10.79	98.41	12.49
16 . .	99.45	7.42	99.16	9.14	98.81	10.85	98.40	12.55
18 . .	99.44	7.48	99.15	9.20	98.80	10.91	98.39	12.60
20 . .	99.43	7.53	99.14	9.25	98.78	10.96	98.37	12.66
22 . .	99.42	7.59	99.13	9.31	98.77	11.02	98.36	12.72
24 . .	99.41	7.65	99.11	9.37	98.76	11.08	98.34	12.77
26 . .	99.40	7.71	99.10	9.43	98.74	11.13	98.33	12.83
28 . .	99.39	7.76	99.09	9.48	98.73	11.19	98.31	12.88
30 . .	99.38	7.82	99.08	9.54	98.72	11.25	98.29	12.94
32 . .	99.38	7.88	99.07	9.60	98.71	11.30	98.28	13.00
34 . .	99.37	7.94	99.06	9.65	98.69	11.36	98.27	13.05
36 . .	99.36	7.99	99.05	9.71	98.68	11.42	98.25	13.11
38 . .	99.35	8.05	99.04	9.77	98.67	11.47	98.24	13.17
40 . .	99.34	8.11	99.03	9.83	98.65	11.53	98.22	13.22
42 . .	99.33	8.17	99.01	9.88	98.64	11.59	98.20	13.28
44 . .	99.32	8.22	99.00	9.94	98.63	11.64	98.19	13.33
46 . .	99.31	8.28	98.99	10.00	98.61	11.70	98.17	13.39
48 . .	99.30	8.34	98.98	10.05	98.60	11.76	98.16	13.45
50 . .	99.29	8.40	98.97	10.11	98.58	11.81	98.14	13.50
52 . .	99.28	8.45	98.96	10.17	98.57	11.87	98.13	13.56
54 . .	99.27	8.51	98.94	10.22	98.56	11.93	98.11	13.61
56 . .	99.26	8.57	98.93	10.28	98.54	11.98	98.10	13.67
58 . .	99.25	8.63	98.92	10.34	98.53	12.04	98.08	13.73
60 . .	99.24	8.68	98.91	10.40	98.51	12.10	98.06	13.78
$c = 0.75$	0.75	0.06	0.75	0.07	0.75	0.08	0.74	0.10
$c = 1.00$	1.00	0.08	0.99	0.09	0.99	0.11	0.99	0.13
$c = 1.25$	1.25	0.10	1.24	0.11	1.24	0.14	1.24	0.16

TABLE V.—*Continued.*

Horizontal Distances and Elevations from Stadia Readings.

Minutes.	8°		9°		10°		11°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	98.06	13.78	97.55	15.45	96.98	17.10	96.36	18.73
2 . .	98.05	13.84	97.53	15.51	96.96	17.16	96.34	18.78
4 . .	98.03	13.89	97.52	15.56	96.94	17.21	96.32	18.84
6 . .	98.01	13.95	97.50	15.62	96.92	17.26	96.29	18.89
8 . .	98.00	14.01	97.48	15.67	96.90	17.32	96.27	18.95
10 . .	97.98	14.06	97.46	15.73	96.88	17.37	96.25	19.00
12 . .	97.97	14.12	97.44	15.78	96.86	17.43	96.23	19.05
14 . .	97.95	14.17	97.43	15.84	96.84	17.48	96.21	19.11
16 . .	97.93	14.23	97.41	15.89	96.82	17.54	96.18	19.16
18 . .	97.92	14.28	97.39	15.95	96.80	17.59	96.16	19.21
20 . .	97.90	14.34	97.37	16.00	96.78	17.65	96.14	19.27
22 . .	97.88	14.40	97.35	16.06	96.76	17.70	96.12	19.32
24 . .	97.87	14.45	97.33	16.11	96.74	17.76	96.09	19.38
26 . .	97.85	14.51	97.31	16.17	96.72	17.81	96.07	19.43
28 . .	97.83	14.56	97.29	16.22	96.70	17.86	96.05	19.48
30 . .	97.82	14.62	97.28	16.28	96.68	17.92	96.03	19.54
32 . .	97.80	14.67	97.26	16.33	96.66	17.97	96.00	19.59
34 . .	97.78	14.73	97.24	16.39	96.64	18.03	95.98	19.64
36 . .	97.76	14.79	97.22	16.44	96.62	18.08	95.96	19.70
38 . .	97.75	14.84	97.20	16.50	96.60	18.14	95.93	19.75
40 . .	97.73	14.90	97.18	16.55	96.57	18.19	95.91	19.80
42 . .	97.71	14.95	97.16	16.61	96.55	18.24	95.89	19.86
44 . .	97.69	15.01	97.14	16.66	96.53	18.30	95.86	19.91
46 . .	97.68	15.06	97.12	16.72	96.51	18.35	95.84	19.96
48 . .	97.66	15.12	97.10	16.77	96.49	18.41	95.82	20.02
50 . .	97.64	15.17	97.08	16.83	96.47	18.46	95.79	20.07
52 . .	97.62	15.23	97.06	16.88	96.45	18.51	95.77	20.12
54 . .	97.61	15.28	97.04	16.94	96.42	18.57	95.75	20.18
56 . .	97.59	15.34	97.02	16.99	96.40	18.62	95.72	20.23
58 . .	97.57	15.40	97.00	17.05	96.38	18.68	95.70	20.28
60 . .	97.55	15.45	96.98	17.10	96.36	18.73	95.68	20.34
$c = 0.75$	0.74	0.11	0.74	0.12	0.74	0.14	0.73	0.15
$c = 1.00$	0.99	0.15	0.99	0.16	0.98	0.18	0.98	0.20
$c = 1.25$	1.23	0.18	1.23	0.21	1.23	0.23	1.22	0.25

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	12°		13°		14°		15°	
	Hor. Dist	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	95.68	20.34	94.94	21.92	94.15	23.47	93.30	25.00
2 . .	95.65	20.39	94.91	21.97	94.12	23.52	93.27	25.05
4 . .	95.63	20.44	94.89	22.02	94.09	23.58	93.24	25.10
6 . .	95.61	20.50	94.86	22.08	94.07	23.63	93.21	25.15
8 . .	95.58	20.55	94.84	22.13	94.04	23.68	93.18	25.20
10 . .	95.56	20.60	94.81	22.18	94.01	23.73	93.16	25.25
12 . .	95.53	20.66	94.79	22.23	93.98	23.78	93.13	25.30
14 . .	95.51	20.71	94.76	22.28	93.95	23.83	93.10	25.35
16 . .	95.49	20.76	94.73	22.34	93.93	23.88	93.07	25.40
18 . .	95.46	20.81	94.71	22.39	93.90	23.93	93.04	25.45
20 . .	95.44	20.87	94.68	22.44	93.87	23.99	93.01	25.50
22 . .	95.41	20.92	94.66	22.49	93.84	24.04	92.98	25.55
24 . .	95.39	20.97	94.63	22.54	93.81	24.09	92.95	25.60
26 . .	95.36	21.03	94.60	22.60	93.79	24.14	92.92	25.65
28 . .	95.34	21.08	94.58	22.65	93.76	24.19	92.89	25.70
30 . .	95.32	21.13	94.55	22.70	93.73	24.24	92.86	25.75
32 . .	95.29	21.18	94.52	22.75	93.70	24.29	92.83	25.80
34 . .	95.27	21.24	94.50	22.80	93.67	24.34	92.80	25.85
36 . .	95.24	21.29	94.47	22.85	93.65	24.39	92.77	25.90
38 . .	95.22	21.34	94.44	22.91	93.62	24.44	92.74	25.95
40 . .	95.19	21.39	94.42	22.96	93.59	24.49	92.71	26.00
42 . .	95.17	21.45	94.39	23.01	93.56	24.55	92.68	26.05
44 . .	95.14	21.50	94.36	23.06	93.53	24.60	92.65	26.10
46 . .	95.12	21.55	94.34	23.11	93.50	24.65	92.62	26.15
48 . .	95.09	21.60	94.31	23.16	93.47	24.70	92.59	26.20
50 . .	95.07	21.66	94.28	23.22	93.45	24.75	92.56	26.25
52 . .	95.04	21.71	94.26	23.27	93.42	24.80	92.53	26.30
54 . .	95.02	21.76	94.23	23.32	93.39	24.85	92.49	26.35
56 . .	94.99	21.81	94.20	23.37	93.36	24.90	92.46	26.40
58 . .	94.97	21.87	94.17	23.42	93.33	24.95	92.43	26.45
60 . .	94.94	21.92	94.15	23.47	93.30	25.00	92.40	26.50
$c = 0.75$	0.73	0.16	0.73	0.17	0.73	0.19	0.72	0.20
$c = 1.00$	0.98	0.22	0.97	0.23	0.97	0.25	0.96	0.27
$c = 1.25$	1.22	0.27	1.21	0.29	1.21	0.31	1.20	0.34

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	16°		17°		18°		19°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	92.40	26.50	91.45	27.96	90.45	29.39	89.40	30.78
2 . .	92.37	26.55	91.42	28.01	90.42	29.44	89.36	30.83
4 . .	92.34	26.59	91.39	28.06	90.38	29.48	89.33	30.87
6 . .	92.31	26.64	91.35	28.10	90.35	29.53	89.29	30.92
8 . .	92.28	26.69	91.32	28.15	90.31	29.58	89.26	30.97
10 . .	92.25	26.74	91.29	28.20	90.28	29.62	89.22	31.01
12 . .	92.22	26.79	91.26	28.25	90.24	29.67	89.18	31.06
14 . .	92.19	26.84	91.22	28.30	90.21	29.72	89.15	31.10
16 . .	92.15	26.89	91.19	28.34	90.18	29.76	89.11	31.15
18 . .	92.12	26.94	91.16	28.39	90.14	29.81	89.08	31.19
20 . .	92.09	26.99	91.12	28.44	90.11	29.86	89.04	31.24
22 . .	92.06	27.04	91.09	28.49	90.07	29.90	89.00	31.28
24 . .	92.03	27.09	91.06	28.54	90.04	29.95	88.96	31.33
26 . .	92.00	27.13	91.02	28.58	90.00	30.00	88.93	31.38
28 . .	91.97	27.18	90.99	28.63	89.97	30.04	88.89	31.42
30 . .	91.93	27.23	90.96	28.68	89.93	30.09	88.86	31.47
32 . .	91.90	27.28	90.92	28.73	89.90	30.14	88.82	31.51
34 . .	91.87	27.33	90.89	28.77	89.86	30.19	88.78	31.56
36 . .	91.84	27.38	90.86	28.82	89.83	30.23	88.75	31.60
38 . .	91.81	27.43	90.82	28.87	89.79	30.28	88.71	31.65
40 . .	91.77	27.48	90.79	28.92	89.76	30.32	88.67	31.69
42 . .	91.74	27.52	90.76	28.96	89.72	30.37	88.64	31.74
44 . .	91.71	27.57	90.72	29.01	89.69	30.41	88.60	31.78
46 . .	91.68	27.62	90.69	29.06	89.65	30.46	88.56	31.83
48 . .	91.65	27.67	90.66	29.11	89.61	30.51	88.53	31.87
50 . .	91.61	27.72	90.62	29.15	89.58	30.55	88.49	31.92
52 . .	91.58	[illegible]	90.59	29.20	89.54	30.60	88.45	31.96
54 . .	91.55	27.81	90.55	29.25	89.51	30.65	88.41	32.01
56 . .	91.52	27.86	90.52	29.30	89.47	30.69	88.38	32.05
58 . .	91.48	27.91	90.48	29.34	89.44	30.74	88.34	32.09
60 . .	91.45	27.96	90.45	29.39	89.40	30.78	88.30	32.14
c = 0.75	0.72	0.21	0.72	0.23	0.71	0.24	0.71	0.25
c = 1.00	0.96	0.28	0.96	0.30	0.95	0.32	0.94	0.33
c = 1.25	1.20	0.35	1.20	0.38	1.19	0.40	1.18	0.42

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	20°		21°		22°		23°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	88.30	32.14	87.16	33.46	85.97	34.73	84.73	35.97
2 . .	88.26	32.18	87.12	33.50	85.93	34.77	84.69	36.01
4 . .	88.23	32.23	87.08	33.54	85.89	34.82	84.65	36.05
6 . .	88.19	32.27	87.04	33.59	85.85	34.86	84.61	36.09
8 . .	88.15	32.32	87.00	33.63	85.80	34.90	84.57	36.13
10 . .	88.11	32.36	86.96	33.67	85.76	34.94	84.52	36.17
12 . .	88.08	32.41	86.92	33.72	85.72	34.98	84.48	36.21
14 . .	88.04	32.45	86.88	33.76	85.68	35.02	84.44	36.25
16 . .	88.00	32.49	86.84	33.80	85.64	35.07	84.40	36.29
18 . .	87.96	32.54	86.80	33.84	85.60	35.11	84.35	36.33
20 . .	87.93	32.58	86.77	33.89	85.56	35.15	84.31	36.37
22 . .	87.89	32.63	86.73	33.93	85.52	35.19	84.27	36.41
24 . .	87.85	32.67	86.69	33.97	85.48	35.23	84.23	36.45
26 . .	87.81	32.72	86.65	34.01	85.44	35.27	84.18	36.49
28 . .	87.77	32.76	86.61	34.06	85.40	35.31	84.14	36.53
30 . .	87.74	32.80	86.57	34.10	85.36	35.36	84.10	36.57
32 . .	87.70	32.85	86.53	34.14	85.31	35.40	84.06	36.61
34 . .	87.66	32.89	86.49	34.18	85.27	35.44	84.01	36.65
36 . .	87.62	32.93	86.45	34.23	85.23	35.48	83.97	36.69
38 . .	87.58	32.98	86.41	34.27	85.19	35.52	83.93	36.73
40 . .	87.54	33.02	86.37	34.31	85.15	35.56	83.89	36.77
42 . .	87.51	33.07	86.33	34.35	85.11	35.60	83.84	36.80
44 . .	87.47	33.11	86.29	34.40	85.07	35.64	83.80	36.84
46 . .	87.43	33.15	86.25	34.44	85.02	35.68	83.76	36.88
48 . .	87.39	33.20	86.21	34.48	84.98	35.72	83.72	36.92
50 . .	87.35	33.24	86.17	34.52	84.94	35.76	83.67	36.96
52 . .	87.31	33.28	86.13	34.57	84.90	35.80	83.63	37.00
54 . .	87.27	33.33	86.09	34.61	84.86	35.85	83.59	37.04
56 . .	87.24	33.37	86.05	34.65	84.82	35.89	83.54	37.08
58 . .	87.20	33.41	86.01	34.69	84.77	35.93	83.50	37.12
60 . .	87.16	33.46	85.97	34.73	84.73	35.97	83.46	37.16
$c = 0.75$	0.70	0.26	0.70	0.27	0.69	0.29	0.69	0.30
$c = 1.00$	0.94	0.35	0.93	0.37	0.92	0.38	0.92	0.40
$c = 1.25$	1.17	0.44	1.16	0.46	1.15	0.48	1.15	0.50

[redacted]

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	8°		9°		10°		11°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	98.06	13.78	97.55	15.45	96.98	17.10	96.36	18.73
2 . .	98.05	13.84	97.53	15.51	96.96	17.16	96.34	18.78
4 . .	98.03	13.89	97.52	15.56	96.94	17.21	96.32	18.84
6 . .	98.01	13.95	97.50	15.62	96.92	17.26	96.29	18.89
8 . .	98.00	14.01	97.48	15.67	96.90	17.32	96.27	18.95
10 . .	97.98	14.06	97.46	15.73	96.88	17.37	96.25	19.00
12 . .	97.97	14.12	97.44	15.78	96.86	17.43	96.23	19.05
14 . .	97.95	14.17	97.43	15.84	96.84	17.48	96.21	19.11
16 . .	97.93	14.23	97.41	15.89	96.82	17.54	96.18	19.16
18 . .	97.92	14.28	97.39	15.95	96.80	17.59	96.16	19.21
20 . .	97.90	14.34	97.37	16.00	96.78	17.65	96.14	19.27
22 . .	97.88	14.40	97.35	16.06	96.76	17.70	96.12	19.32
24 . .	97.87	14.45	97.33	16.11	96.74	17.76	96.09	19.38
26 . .	97.85	14.51	97.31	16.17	96.72	17.81	96.07	19.43
28 . .	97.83	14.56	97.29	16.22	96.70	17.86	96.05	19.48
30 . .	97.82	14.62	97.28	16.28	96.68	17.92	96.03	19.54
32 . .	97.80	14.67	97.26	16.33	96.66	17.97	96.00	19.59
34 . .	97.78	14.73	97.24	16.39	96.64	18.03	95.98	19.64
36 . .	97.76	14.79	97.22	16.44	96.62	18.08	95.96	19.70
38 . .	97.75	14.84	97.20	16.50	96.60	18.14	95.93	19.75
40 . .	97.73	14.90	97.18	16.55	96.57	18.19	95.91	19.80
42 . .	97.71	14.95	97.16	16.61	96.55	18.24	95.89	19.86
44 . .	97.69	15.01	97.14	16.66	96.53	18.30	95.86	19.91
46 . .	97.68	15.06	97.12	16.72	96.51	18.35	95.84	19.96
48 . .	97.66	15.12	97.10	16.77	96.49	18.41	95.82	20.02
50 . .	97.64	15.17	97.08	16.83	96.47	18.46	95.79	20.07
52 . .	97.62	15.23	97.06	16.88	96.45	18.51	95.77	20.12
54 . .	97.61	15.28	97.04	16.94	96.42	18.57	95.75	20.18
56 . .	97.59	15.34	97.02	16.99	96.40	18.62	95.72	20.23
58 . .	97.57	15.40	97.00	17.05	96.38	18.68	95.70	20.28
60 . .	97.55	15.45	96.98	17.10	96.36	18.73	95.68	20.34
$c = 0.75$	0.74	0.11	0.74	0.12	0.74	0.14	0.73	0.15
$c = 1.00$	0.99	0.15	0.99	0.16	0.98	0.18	0.98	0.20
$c = 1.25$	1.23	0.18	1.23	0.21	1.23	0.23	1.22	0.25

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	12°		13°		14°		15°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	95.68	20.34	94.94	21.92	94.15	23.47	93.30	25.00
2 . .	95.65	20.39	94.91	21.97	94.12	23.52	93.27	25.05
4 . .	95.63	20.44	94.89	22.02	94.09	23.58	93.24	25.10
6 . .	95.61	20.50	94.86	22.08	94.07	23.63	93.21	25.15
8 . .	95.58	20.55	94.84	22.13	94.04	23.68	93.18	25.20
10 . .	95.56	20.60	94.81	22.18	94.01	23.73	93.16	25.25
12 . .	95.53	20.66	94.79	22.23	93.98	23.78	93.13	25.30
14 . .	95.51	20.71	94.76	22.28	93.95	23.83	93.10	25.35
16 . .	95.49	20.76	94.73	22.34	93.93	23.88	93.07	25.40
18 . .	95.46	20.81	94.71	22.39	93.90	23.93	93.04	25.45
20 . .	95.44	20.87	94.68	22.44	93.87	23.99	93.01	25.50
22 . .	95.41	20.92	94.66	22.49	93.84	24.04	92.98	25.55
24 . .	95.39	20.97	94.63	22.54	93.81	24.09	92.95	25.60
26 . .	95.36	21.03	94.60	22.60	93.79	24.14	92.92	25.65
28 . .	95.34	21.08	94.58	22.65	93.76	24.19	92.89	25.70
30 . .	95.32	21.13	94.55	22.70	93.73	24.24	92.86	25.75
32 . .	95.29	21.18	94.52	22.75	93.70	24.29	92.83	25.80
34 . .	95.27	21.24	94.50	22.80	93.67	24.34	92.80	25.85
36 . .	95.24	21.29	94.47	22.85	93.65	24.39	92.77	25.90
38 . .	95.22	21.34	94.44	22.91	93.62	24.44	92.74	25.95
40 . .	95.19	21.39	94.42	22.96	93.59	24.49	92.71	26.00
42 . .	95.17	21.45	94.39	23.01	93.56	24.55	92.68	26.05
44 . .	95.14	21.50	94.36	23.06	93.53	24.60	92.65	26.10
46 . .	95.12	21.55	94.34	23.11	93.50	24.65	92.62	26.15
48 . .	95.09	21.60	94.31	23.16	93.47	24.70	92.59	26.20
50 . .	95.07	21.66	94.28	23.22	93.45	24.75	92.56	26.25
52 . .	95.04	21.71	94.26	23.27	93.42	24.80	92.53	26.30
54 . .	95.02	21.76	94.23	23.32	93.39	24.85	92.49	26.35
56 . .	94.99	21.81	94.20	23.37	93.36	24.90	92.46	26.40
58 . .	94.97	21.87	94.17	23.42	93.33	24.95	92.43	26.45
60 . .	94.94	21.92	94.15	23.47	93.30	25.00	92.40	26.50
$c = 0.75$	0.73	0.16	0.73	0.17	0.73	0.19	0.72	0.20
$c = 1.00$	0.98	0.22	0.97	0.23	0.97	0.25	0.96	0.27
$c = 1.25$	1.22	0.27	1.21	0.29	1.21	0.31	1.20	0.34

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	16°		17°		18°		19°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	92.40	26.50	91.45	27.96	90.45	29.39	89.40	30.78
2 . .	92.37	26.55	91.42	28.01	90.42	29.44	89.36	30.83
4 . .	92.34	26.59	91.39	28.06	90.38	29.48	89.33	30.87
6 . .	92.31	26.64	91.35	28.10	90.35	29.53	89.29	30.92
8 . .	92.28	26.69	91.32	28.15	90.31	29.58	89.26	30.97
10 . .	92.25	26.74	91.29	28.20	90.28	29.62	89.22	31.01
12 . .	92.22	26.79	91.26	28.25	90.24	29.67	89.18	31.06
14 . .	92.19	26.84	91.22	28.30	90.21	29.72	89.15	31.10
16 . .	92.15	26.89	91.19	28.34	90.18	29.76	89.11	31.15
18 . .	92.12	26.94	91.16	28.39	90.14	29.81	89.08	31.19
20 . .	92.09	26.99	91.12	28.44	90.11	29.86	89.04	31.24
22 . .	92.06	27.04	91.09	28.49	90.07	29.90	89.00	31.28
24 . .	92.03	27.09	91.06	28.54	90.04	29.95	88.96	31.33
26 . .	92.00	27.13	91.02	28.58	90.00	30.00	88.93	31.38
28 . .	91.97	27.18	90.99	28.63	89.97	30.04	88.89	31.42
30 . .	91.93	27.23	90.96	28.68	89.93	30.09	88.86	31.47
32 . .	91.90	27.28	90.92	28.73	89.90	30.14	88.82	31.51
34 . .	91.87	27.33	90.89	28.77	89.86	30.19	88.78	31.56
36 . .	91.84	27.38	90.86	28.82	89.83	30.23	88.75	31.60
38 . .	91.81	27.43	90.82	28.87	89.79	30.28	88.71	31.65
40 . .	91.77	27.48	90.79	28.92	89.76	30.32	88.67	31.69
42 . .	91.74	27.52	90.76	28.96	89.72	30.37	88.64	31.74
44 . .	91.71	27.57	90.72	29.01	89.69	30.41	88.60	31.78
46 . .	91.68	27.62	90.69	29.06	89.65	30.46	88.56	31.83
48 . .	91.65	27.67	90.66	29.11	89.61	30.51	88.53	31.87
50 . .	91.61	27.72	90.62	29.15	89.58	30.55	88.49	31.92
52 . .	91.58	27.77	90.59	29.20	89.54	30.60	88.45	31.96
54 . .	91.55	27.81	90.55	29.25	89.51	30.65	88.41	32.01
56 . .	91.52	27.86	90.52	29.30	89.47	30.69	88.38	32.05
58 . .	91.48	27.91	90.48	29.34	89.44	30.74	88.34	32.09
60 . .	91.45	27.96	90.45	29.39	89.40	30.78	88.30	32.14
$c = 0.75$	0.72	0.21	0.72	0.23	0.71	0.24	0.71	0.25
$c = 1.00$	0.86	0.28	0.95	0.30	0.95	0.32	0.94	0.33
$c = 1.25$	1.20	0.35	1.19	0.38	1.19	0.40	1.18	0.42

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	20°		21°		22°		23°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	88.30	32.14	87.16	33.46	85.97	34.73	84.73	35.97
2 . .	88.26	32.18	87.12	33.50	85.93	34.77	84.69	36.01
4 . .	88.23	32.23	87.08	33.54	85.89	34.82	84.65	36.05
6 . .	88.19	32.27	87.04	33.59	85.85	34.86	84.61	36.09
8 . .	88.15	32.32	87.00	33.63	85.80	34.90	84.57	36.13
10 . .	88.11	32.36	86.96	33.67	85.76	34.94	84.52	36.17
12 . .	88.08	32.41	86.92	33.72	85.72	34.98	84.48	36.21
14 . .	88.04	32.45	86.88	33.76	85.68	35.02	84.44	36.25
16 . .	88.00	32.49	86.84	33.80	85.64	35.07	84.40	36.29
18 . .	87.96	32.54	86.80	33.84	85.60	35.11	84.35	36.33
20 . .	87.93	32.58	86.77	33.89	85.56	35.15	84.31	36.37
22 . .	87.89	32.63	86.73	33.93	85.52	35.19	84.27	36.41
24 . .	87.85	32.67	86.69	33.97	85.48	35.23	84.23	36.45
26 . .	87.81	32.72	86.65	34.01	85.44	35.27	84.18	36.49
28 . .	87.77	32.76	86.61	34.06	85.40	35.31	84.14	36.53
30 . .	87.74	32.80	86.57	34.10	85.36	35.36	84.10	36.57
32 . .	87.70	32.85	86.53	34.14	85.31	35.40	84.06	36.61
34 . .	87.66	32.89	86.49	34.18	85.27	35.44	84.01	36.65
36 . .	87.62	32.93	86.45	34.23	85.23	35.48	83.97	36.69
38 . .	87.58	32.98	86.41	34.27	85.19	35.52	83.93	36.73
40 . .	87.54	33.02	86.37	34.31	85.15	35.56	83.89	36.77
42 . .	87.51	33.07	86.33	34.35	85.11	35.60	83.84	36.80
44 . .	87.47	33.11	86.29	34.40	85.07	35.64	83.80	36.84
46 . .	87.43	33.15	86.25	34.44	85.02	35.68	83.76	36.88
48 . .	87.39	33.20	86.21	34.48	84.98	35.72	83.72	36.92
50 . .	87.35	33.24	86.17	34.52	84.94	35.76	83.67	36.96
52 . .	87.31	33.28	86.13	34.57	84.90	35.80	83.63	37.00
54 . .	87.27	33.33	86.09	34.61	84.86	35.85	83.59	37.04
56 . .	87.24	33.37	86.05	34.65	84.82	35.89	83.54	37.08
58 . .	87.20	33.41	86.01	34.69	84.77	35.93	83.50	37.12
60 . .	87.16	33.46	85.97	34.73	84.73	35.97	83.46	37.16
$c = 0.75$	0.70	0.26	0.70	0.27	0.69	0.29	0.69	0.30
$c = 1.00$	0.94	0.35	0.93	0.37	0.92	0.38	0.92	0.40
$c = 1.25$	1.17	0.44	1.16	0.46	1.15	0.48	1.15	0.50

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	24°		25°		26°		27°	
	Hor. Dist	Diff. Elev.	Hor. Dist.	Diff. Elev	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	83.46	37.16	82.14	38.30	80.78	39.40	79.39	40.45
2 . .	83.41	37.20	82.09	38.34	80.74	39.44	79.34	40.49
4 . .	83.37	37.23	82.05	38.38	80.69	39.47	79.30	40.52
6 . .	83.33	37.27	82.01	38.41	80.65	39.51	79.25	40.55
8 . .	83.28	37.31	81.96	38.45	80.60	39.54	79.20	40.59
10 . .	83.24	37.35	81.92	38.49	80.55	39.58	79.15	40.62
12 . .	83.20	37.39	81.87	38.53	80.51	39.61	79.11	40.66
14 . .	83.15	37.43	81.83	38.56	80.46	39.65	79.06	40.69
16 . .	83.11	37.47	81.78	38.60	80.41	39.69	79.01	40.72
18 . .	83.07	37.51	81.74	38.64	80.37	39.72	78.96	40.76
20 . .	83.02	37.54	81.69	38.67	80.32	39.76	78.92	40.79
22 . .	82.98	37.58	81.65	38.71	80.28	39.79	78.87	40.82
24 . .	82.93	37.62	81.60	38.75	80.23	39.83	78.82	40.86
26 . .	82.89	37.66	81.56	38.78	80.18	39.86	78.77	40.89
28 . .	82.85	37.70	81.51	38.62	80.14	39.90	78.73	40.92
30 . .	82.80	37.74	81.47	38.86	80.09	39.93	78.68	40.96
32 . .	82.76	37.77	81.42	38.89	80.04	39.97	78.63	40.99
34 . .	82.72	37.81	81.38	38.93	80.00	40.00	78.58	41.02
36 . .	82.67	37.85	81.33	38.97	79.95	40.04	78.54	41.06
38 . .	82.63	37.89	81.28	39.00	79.90	40.07	78.49	41.09
40 . .	82.58	37.93	81.24	39.04	79.86	40.11	78.44	41.12
42 . .	82.54	37.96	81.19	39.08	79.81	40.14	78.39	41.16
44 . .	82.49	38.00	81.15	39.11	79.76	40.18	78.34	41.19
46 . .	82.45	38.04	81.10	39.15	79.72	40.21	78.30	41.22
48 . .	82.41	38.08	81.06	39.18	79.67	40.24	78.25	41.26
50 . .	82.36	38.11	81.01	39.22	79.62	40.28	78.20	41.29
52 . .	82.32	38.15	80.97	39.26	79.58	40.31	78.15	41.32
54 . .	82.27	38.19	80.92	39.29	79.53	40.35	78.10	41.35
56 . .	82.23	38.23	80.87	39.33	79.48	40.38	78.06	41.39
58 . .	82.18	38.26	80.83	39.36	79.44	40.42	78.01	41.42
60 . .	82.14	38.30	80.78	39.40	79.39	40.45	77.96	41.45
$c = 0.75$	0.68	0.31	0.68	0.32	0.67	0.33	0.66	0.35
$c = 1.00$	0.91	0.41	0.90	0.43	0.89	0.45	0.89	0.46
$c = 1.25$	1.14	0.52	1.13	0.54	1.12	0.56	1.11	0.58

TABLE V.—*Continued.*

HORIZONTAL DISTANCES AND ELEVATIONS FROM STADIA READINGS.

Minutes.	28°		29°		30°	
	Hor. Dist.	Diff. Elev.	Hor. Dist	Diff. Elev.	Hor. Dist.	Diff. Elev.
0 . .	77.96	41.45	76.50	42.40	75.00	43.30
2 . .	77.91	41.48	76.45	42.43	74.95	43.33
4 . .	77.86	41.52	76.40	42.46	74.90	43.36
6 . .	77.81	41.55	76.35	42.49	74.85	43.39
8 . .	77.77	41.58	76.30	42.53	74.80	43.42
10 . .	77.72	41.61	76.25	42.56	74.75	43.45
12 . .	77.67	41.65	76.20	42.59	74.70	43.47
14 . .	77.62	41.68	76.15	42.62	74.65	43.50
16 . .	77.57	41.71	76.10	42.65	74.60	43.53
18 . .	77.52	41.74	76.05	42.68	74.55	43.56
20 . .	77.48	41.77	76.00	42.71	74.49	43.59
22 . .	77.42	41.81	75.95	42.74	74.44	43.62
24 . .	77.38	41.84	75.90	42.77	74.39	43.65
26 . .	77.33	41.87	75.85	42.80	74.34	43.67
28 . .	77.28	41.90	75.80	42.83	74.29	43.70
30 . .	77.23	41.93	75.75	42.86	74.24	43.73
32 . .	77.18	41.97	75.70	42.89	74.19	43.76
34 . .	77.13	42.00	75.65	42.92	74.14	43.79
36 . .	77.09	42.03	75.60	42.95	74.09	43.82
38 . .	77.04	42.06	75.55	42.98	74.04	43.84
40 . .	76.99	42.09	75.50	43.01	73.99	43.87
42 . .	76.94	42.12	75.45	43.04	73.93	43.90
44 . .	76.89	42.15	75.40	43.07	73.88	43.93
46 . .	76.84	42.19	75.35	43.10	73.83	43.95
48 . .	76.79	42.22	75.30	43.13	73.78	43.98
50 . .	76.74	42.25	75.25	43.16	73.73	44.01
52 . .	76.69	42.28	75.20	43.18	73.68	44.04
54 . .	76.64	42.31	75.15	43.21	73.63	44.07
56 . .	76.59	42.34	75.10	43.24	73.58	44.09
58 . .	76.55	42.37	75.05	43.27	73.52	44.12
60 . .	76.50	42.40	75.00	43.30	73.47	44.15
$c = 0.75$	0.66	0.36	0.65	0.37	0.65	0.38
$c = 1.00$	0.88	0.48	0.87	0.49	0.86	0.51
$c = 1.25$	1.10	0.60	1.09	0.62	1.08	0.64

TABLE VI.

Natural Sines and Cosines.

′	0° Sine	0° Cosin	1° Sine	1° Cosin	2° Sine	2° Cosin	3° Sine	3° Cosin	4° Sine	4° Cosin	′
0	.00000	One.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	One.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	One.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	One.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	One.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	One.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	One.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	One.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	One.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	One.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	One.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	89°		88°		87°		86°		85°		

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	5°		6°		7°		8°		9°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	84°		83°		82°		81°		80°		

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	10°		11°		12°		13°		14°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	59
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	58
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	57
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	54
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	53
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	52
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	49
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	48
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	47
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	44
17	.17852	.98394	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	43
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	42
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	39
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	38
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	37
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	34
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.96837	33
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	32
29	.18195	.98331	.19908	.97998	.21616	.97636	.23316	.97244	.25010	.96822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	29
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	28
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	27
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	24
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	23
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	22
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	19
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	18
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	17
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	14
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	13
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	12
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	9
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	8
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	7
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	4
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	3
58	.19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	2
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	
′	79°		78°		77°		76°		75°		′

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	15° Sine	15° Cosin	16° Sine	16° Cosin	17° Sine	17° Cosin	18° Sine	18° Cosin	19° Sine	19° Cosin	′
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.96347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.95231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.94580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
′	74° Cosin	74° Sine	73° Cosin	73° Sine	72° Cosin	72° Sine	71° Cosin	71° Sine	70° Cosin	70° Sine	′

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	20°		21°		22°		23°		24°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.34229	.93959	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.34257	.93949	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.34284	.93939	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.34311	.93929	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.34366	.93909	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.34393	.93899	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.34421	.93889	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.34448	.93879	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.34503	.93859	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.34530	.93849	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.34557	.93839	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.34584	.93829	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.34639	.93809	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.34666	.93799	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.34694	.93789	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.34721	.93779	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.34775	.93759	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.34803	.93748	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.34830	.93738	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.34857	.93728	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.34884	.93718	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.34912	.93708	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.34939	.93698	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.34966	.93688	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.34993	.93677	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.35048	.93657	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.35075	.93647	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.35102	.93637	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.35130	.93626	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.35184	.93606	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.35211	.93596	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.35239	.93585	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.35266	.93575	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.35320	.93555	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.35347	.93544	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.35375	.93534	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.35402	.93524	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.35456	.93503	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.35484	.93493	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.35511	.93483	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.35538	.93472	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.35592	.93452	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.35619	.93441	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.35647	.93431	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.35674	.93420	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.35728	.93400	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.35755	.93389	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.35782	.93379	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.35810	.93368	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	69°		68°		67°		66°		65°		

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	25° Sine	25° Cosin	26° Sine	26° Cosin	27° Sine	27° Cosin	28° Sine	28° Cosin	29° Sine	29° Cosin	′
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.49141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	64°		63°		62°		61°		60°		

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	30° Sine	30° Cosin	31° Sine	31° Cosin	32° Sine	32° Cosin	33° Sine	33° Cosin	34° Sine	34° Cosin	′
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	59°		58°		57°		56°		55°		

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	35° Sine	35° Cosin	36° Sine	36° Cosin	37° Sine	37° Cosin	38° Sine	38° Cosin	39° Sine	39° Cosin	′
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	49
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	.60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60136	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
′	54° Cosin	54° Sine	53° Cosin	53° Sine	52° Cosin	52° Sine	51° Cosin	51° Sine	50° Cosin	50° Sine	′

TABLE VI.—*Continued.*

NATURAL SINES AND COSINES.

′	40° Sine	40° Cosin	41° Sine	41° Cosin	42° Sine	42° Cosin	43° Sine	43° Cosin	44° Sine	44° Cosin	′
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74352	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	49°		48°		47°		46°		45°		

TABLE VII.

Natural Tangents and Cotangents.

′	0° Tang	0° Cotang	1° Tang	1° Cotang	2° Tang	2° Cotang	3° Tang	3° Cotang	′
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	60
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	59
2	.00058	1718.87	.01804	55.4415	.03550	28.1664	.05299	18.8711	58
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	33
28	.00815	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	.01338	74.7292	.03084	32.4213	.04833	20.6932	.06584	15.1893	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	89°		88°		87°		86°		

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	4°		5°		6°		7°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.06993	14.3007	.08749	11.4301	.10510	9.51436	.12278	8.14435	60
1	.07022	14.2411	.08778	11.3919	.10540	9.48781	.12308	8.12481	59
2	.07051	14.1821	.08807	11.3540	.10569	9.46141	.12338	8.10536	58
3	.07080	14.1235	.08837	11.3163	.10599	9.43515	.12367	8.08600	57
4	.07110	14.0655	.08866	11.2789	.10628	9.40904	.12397	8.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	.07168	13.9507	.08925	11.2048	.10687	9.35724	.12456	8.02848	54
7	.07197	13.8940	.08954	11.1681	.10716	9.33155	.12485	8.00948	53
8	.07227	13.8378	.08983	11.1316	.10746	9.30599	.12515	7.99058	52
9	.07256	13.7821	.09013	11.0954	.10775	9.28058	.12544	7.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	.07314	13.6719	.09071	11.0237	.10834	9.23016	.12603	7.93438	49
12	.07344	13.6174	.09101	10.9882	.10863	9.20516	.12633	7.91582	48
13	.07373	13.5634	.09130	10.9529	.10893	9.18028	.12662	7.89734	47
14	.07402	13.5098	.09159	10.9178	.10922	9.15554	.12692	7.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	.07461	13.4039	.09218	10.8483	.10981	9.10646	.12751	7.84242	44
17	.07490	13.3515	.09247	10.8139	.11011	9.08211	.12781	7.82428	43
18	.07519	13.2996	.09277	10.7797	.11040	9.05789	.12810	7.80622	42
19	.07548	13.2480	.09306	10.7457	.11070	9.03379	.12840	7.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	.07607	13.1461	.09365	10.6783	.11128	8.98598	.12899	7.75254	39
22	.07636	13.0958	.09394	10.6450	.11158	8.96227	.12929	7.73480	38
23	.07665	13.0458	.09423	10.6118	.11187	8.93867	.12958	7.71715	37
24	.07695	12.9962	.09453	10.5789	.11217	8.91520	.12988	7.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	.07753	12.8981	.09511	10.5136	.11276	8.86862	.13047	7.66466	34
27	.07782	12.8496	.09541	10.4813	.11305	8.84551	.13076	7.64732	33
28	.07812	12.8014	.09570	10.4491	.11335	8.82252	.13106	7.63005	32
29	.07841	12.7536	.09600	10.4172	.11364	8.79964	.13136	7.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	.07899	12.6591	.09658	10.3538	.11423	8.75425	.13195	7.57872	29
32	.07929	12.6124	.09688	10.3224	.11452	8.73172	.13224	7.56176	28
33	.07958	12.5660	.09717	10.2913	.11482	8.70931	.13254	7.54487	27
34	.07987	12.5199	.09746	10.2602	.11511	8.68701	.13284	7.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	.08046	12.4288	.09805	10.1988	.11570	8.64275	.13343	7.49465	24
37	.08075	12.3838	.09834	10.1683	.11600	8.62078	.13372	7.47806	23
38	.08104	12.3390	.09864	10.1381	.11629	8.59893	.13402	7.46154	22
39	.08134	12.2946	.09893	10.1080	.11659	8.57718	.13432	7.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	.08192	12.2067	.09952	10.0483	.11718	8.53402	.13491	7.41240	19
42	.08221	12.1632	.09981	10.0187	.11747	8.51259	.13521	7.39616	18
43	.08251	12.1201	.10011	9.98931	.11777	8.49128	.13550	7.37999	17
44	.08280	12.0772	.10040	9.96007	.11806	8.47007	.13580	7.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	.08339	11.9923	.10099	9.90211	.11865	8.42795	.13639	7.33190	14
47	.08368	11.9504	.10128	9.87338	.11895	8.40705	.13669	7.31600	13
48	.08397	11.9087	.10158	9.84482	.11924	8.38625	.13698	7.30018	12
49	.08427	11.8673	.10187	9.81641	.11954	8.36555	.13728	7.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	.08485	11.7853	.10246	8.76009	.12013	8.32446	.13787	7.25310	9
52	.08514	11.7448	.10275	9.73217	.12042	8.30406	.13817	7.23754	8
53	.08544	11.7045	.10305	9.70441	.12072	8.28376	.13846	7.22204	7
54	.08573	11.6645	.10334	9.67680	.12101	8.26355	.13876	7.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8.24345	.13906	7.19125	5
56	.08632	11.5853	.10393	9.62205	.12160	8.22344	.13935	7.17594	4
57	.08661	11.5461	.10422	9.59490	.12190	8.20352	.13965	7.16071	3
58	.08690	11.5072	.10452	9.56791	.12219	8.18370	.13995	7.14553	2
59	.08720	11.4685	.10481	9.54106	.12249	8.16398	.14024	7.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	85°		84°		83°		82°		

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	8°		9°		10°		11°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	60
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	59
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	58
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	57
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	54
7	.14262	7.01174	.16047	6.23160	.17843	5.60452	.19649	5.08921	53
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	52
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	49
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	48
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	47
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	44
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	43
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	42
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	39
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	38
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	37
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	36
25	.14796	6.75838	.16585	6.02962	.18384	5.43966	.20194	4.95201	35
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	34
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	33
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	32
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	29
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	28
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	27
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.88605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	25
36	.15124	6.61219	.16914	5.91236	.18714	5.34345	.20527	4.87162	24
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	23
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	22
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	.15272	6.54777	.17063	5.86051	.18865	5.30080	.20679	4.83590	19
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	18
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	17
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	14
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	13
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	12
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	9
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	8
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	7
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	4
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	3
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	2
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	81°		80°		79°		78°		

TABLE VII.—*Continued.*

Natural Tangents and Cotangents.

′	12° Tang	12° Cotang	13° Tang	13° Cotang	14° Tang	14° Cotang	15° Tang	15° Cotang	′
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	60
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	59
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	58
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	57
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	56
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	55
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	54
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	53
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	52
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	49
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	48
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	47
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	44
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	43
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	42
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	39
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	38
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	37
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	34
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	33
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	32
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3.60588	30
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	29
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	28
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	27
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	26
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27921	3.58160	24
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57758	23
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	22
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	19
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	18
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	17
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	14
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	13
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	12
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	9
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	8
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	7
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	.22964	4.35459	.24809	4.03076	.26670	3.74950	.28549	3.50279	4
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	3
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	2
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	77°		76°		75°		74°		

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	16° Tang	16° Cotang	17° Tang	17° Cotang	18° Tang	18° Cotang	19° Tang	19° Cotang	′
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35020	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35118	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	73° Cotang	73° Tang	72° Cotang	72° Tang	71° Cotang	71° Tang	70° Cotang	70° Tang	′

TABLE VII.—*Continued*

NATURAL TANGENTS AND COTANGENTS

′	20°		21°		22°		23°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	[illegible]	[illegible]	[illegible]	[illegible]	.40403	2.47509	.42447	2.35585	60
1	[illegible]	[illegible]	[illegible]	[illegible]	.40436	2.47302	.42482	2.35395	59
2	[illegible]	[illegible]	[illegible]	[illegible]	.40470	2.47095	.42516	2.35205	58
3	[illegible]	[illegible]	[illegible]	[illegible]	.40504	2.46888	.42551	2.35015	57
4	[illegible]	[illegible]	[illegible]	[illegible]	.40538	2.46682	.42585	2.34825	56
5	[illegible]	[illegible]	[illegible]	[illegible]	.40572	2.46476	.42619	2.34636	55
6	[illegible]	[illegible]	[illegible]	[illegible]	.40606	2.46270	.42654	2.34447	54
7	[illegible]	[illegible]	[illegible]	[illegible]	.40640	2.46065	.42688	2.34258	53
8	[illegible]	[illegible]	[illegible]	[illegible]	.40674	2.45860	.42722	2.34069	52
9	[illegible]	[illegible]	[illegible]	[illegible]	.40707	2.45655	.42757	2.33881	51
10	[illegible]	[illegible]	[illegible]	[illegible]	.40741	2.45451	.42791	2.33693	50
11	[illegible]	[illegible]	[illegible]	[illegible]	.40775	2.45246	.42826	2.33505	49
12	[illegible]	[illegible]	[illegible]	[illegible]	.40809	2.45043	.42860	2.33317	48
13	[illegible]	[illegible]	[illegible]	[illegible]	.40843	2.44839	.42894	2.33130	47
14	[illegible]	[illegible]	[illegible]	[illegible]	.40877	2.44636	.42929	2.32943	46
15	[illegible]	[illegible]	[illegible]	[illegible]	.40911	2.44433	.42963	2.32756	45
16	[illegible]	[illegible]	[illegible]	[illegible]	.40945	2.44230	.42998	2.32570	44
17	[illegible]	[illegible]	[illegible]	[illegible]	.40979	2.44027	.43032	2.32383	43
18	[illegible]	[illegible]	[illegible]	[illegible]	.41013	2.43825	.43067	2.32197	42
19	[illegible]	[illegible]	[illegible]	[illegible]	.41047	2.43623	.43101	2.32012	41
20	[illegible]	[illegible]	[illegible]	[illegible]	.41081	2.43422	.43136	2.31826	40
21	[illegible]	[illegible]	[illegible]	[illegible]	.41115	2.43220	.43170	2.31641	39
22	[illegible]	[illegible]	[illegible]	[illegible]	.41149	2.43019	.43205	2.31456	38
23	[illegible]	[illegible]	[illegible]	[illegible]	.41183	2.42819	.43239	2.31271	37
24	[illegible]	[illegible]	[illegible]	[illegible]	.41217	2.42618	.43274	2.31086	36
25	[illegible]	[illegible]	[illegible]	[illegible]	.41251	2.42418	.43308	2.30902	35
26	[illegible]	[illegible]	[illegible]	[illegible]	.41285	2.42218	.43343	2.30718	34
27	[illegible]	[illegible]	[illegible]	[illegible]	.41319	2.42019	.43378	2.30534	33
28	[illegible]	[illegible]	[illegible]	[illegible]	.41353	2.41819	.43412	2.30351	32
29	[illegible]	[illegible]	[illegible]	[illegible]	.41387	2.41620	.43447	2.30167	31
30	[illegible]	[illegible]	[illegible]	[illegible]	.41421	2.41421	.43481	2.29984	30
31	[illegible]	[illegible]	[illegible]	[illegible]	.41455	2.41223	.43516	2.29801	29
32	[illegible]	[illegible]	[illegible]	[illegible]	.41490	2.41025	.43550	2.29619	28
33	[illegible]	[illegible]	[illegible]	[illegible]	.41524	2.40827	.43585	2.29437	27
34	[illegible]	[illegible]	[illegible]	[illegible]	.41558	2.40629	.43620	2.29254	26
35	[illegible]	[illegible]	[illegible]	[illegible]	.41592	2.40432	.43654	2.29073	25
36	[illegible]	[illegible]	[illegible]	[illegible]	.41626	2.40235	.43689	2.28891	24
37	[illegible]	[illegible]	[illegible]	[illegible]	.41660	2.40038	.43724	2.28710	23
38	[illegible]	[illegible]	[illegible]	[illegible]	.41694	2.39841	.43758	2.28528	22
39	[illegible]	[illegible]	[illegible]	[illegible]	.41728	2.39645	.43793	2.28348	21
40	[illegible]	[illegible]	[illegible]	[illegible]	.41763	2.39449	.43828	2.28167	20
41	[illegible]	[illegible]	[illegible]	[illegible]	.41797	2.39253	.43862	2.27987	19
42	[illegible]	[illegible]	[illegible]	[illegible]	.41831	2.39058	.43897	2.27806	18
43	[illegible]	[illegible]	[illegible]	[illegible]	.41865	2.38863	.43932	2.27626	17
44	[illegible]	[illegible]	[illegible]	[illegible]	.41899	2.38668	.43966	2.27447	16
45	[illegible]	[illegible]	[illegible]	[illegible]	.41933	2.38473	.44001	2.27267	15
46	[illegible]	[illegible]	[illegible]	[illegible]	.41968	2.38279	.44036	2.27088	14
47	[illegible]	[illegible]	[illegible]	[illegible]	.42002	2.38084	.44071	2.26909	13
48	[illegible]	[illegible]	[illegible]	[illegible]	.42036	2.37891	.44105	2.26730	12
49	[illegible]	[illegible]	[illegible]	[illegible]	.42070	2.37697	.44140	2.26552	11
50	[illegible]	[illegible]	[illegible]	[illegible]	.42105	2.37504	.44175	2.26374	10
51	[illegible]	[illegible]	[illegible]	[illegible]	.42139	2.37311	.44210	2.26196	9
52	[illegible]	[illegible]	[illegible]	[illegible]	.42173	2.37118	.44244	2.26018	8
53	[illegible]	[illegible]	[illegible]	[illegible]	.42207	2.36925	.44279	2.25840	7
54	[illegible]	[illegible]	[illegible]	[illegible]	.42242	2.36733	.44314	2.25663	6
55	[illegible]	[illegible]	[illegible]	[illegible]	.42276	2.36541	.44349	2.25486	5
56	[illegible]	[illegible]	[illegible]	[illegible]	.42310	2.36349	.44384	2.25309	4
57	[illegible]	[illegible]	[illegible]	[illegible]	.42345	2.36158	.44418	2.25132	3
58	[illegible]	[illegible]	[illegible]	[illegible]	.42379	2.35967	.44453	2.24956	2
59	[illegible]	[illegible]	[illegible]	[illegible]	.42413	2.35776	.44488	2.24780	1
60	[illegible]	[illegible]	[illegible]	[illegible]	.42447	2.35585	.44523	2.24604	0
	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	69°		68°		67°		66°		

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	24°		25°		26°		27°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45538	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97681	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52985	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	
′	65°		64°		63°		62°		′

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	28° Tang	28° Cotang	29° Tang	29° Cotang	30° Tang	30° Cotang	31° Tang	31° Cotang	′
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1.66209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65554	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56501	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58905	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76629	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1.61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	.61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	61°		60°		59°		58°		

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	32° Tang	32° Cotang	33° Tang	33° Cotang	34° Tang	34° Cotang	35° Tang	35° Cotang	′
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65024	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62933	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50797	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45139	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72167	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69977	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
′	Cotang 57°	Tang 57°	Cotang 56°	Tang 56°	Cotang 55°	Tang 55°	Cotang 54°	Tang 54°	′

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	36° Tang	36° Cotang	37° Tang	37° Cotang	38° Tang	38° Cotang	39° Tang	39° Cotang	′
0	[illegible]	[illegible]	[illegible]	[illegible]	.78129	1.27994	.80978	1.23490	60
1	[illegible]	[illegible]	[illegible]	[illegible]	.78175	1.27917	.81027	1.23416	59
2	[illegible]	[illegible]	[illegible]	[illegible]	.78222	1.27841	.81075	1.23343	58
3	[illegible]	[illegible]	[illegible]	[illegible]	.78269	1.27764	.81123	1.23270	57
4	[illegible]	[illegible]	[illegible]	[illegible]	.78316	1.27688	.81171	1.23196	56
5	[illegible]	[illegible]	[illegible]	[illegible]	.78363	1.27611	.81220	1.23123	55
6	[illegible]	[illegible]	[illegible]	[illegible]	.78410	1.27535	.81268	1.23050	54
7	[illegible]	[illegible]	[illegible]	[illegible]	.78457	1.27458	.81316	1.22977	53
8	[illegible]	[illegible]	[illegible]	[illegible]	.78504	1.27382	.81364	1.22904	52
9	[illegible]	[illegible]	[illegible]	[illegible]	.78551	1.27306	.81413	1.22831	51
10	[illegible]	[illegible]	[illegible]	[illegible]	.78598	1.27230	.81461	1.22758	50
11	[illegible]	[illegible]	[illegible]	[illegible]	.78645	1.27153	.81510	1.22685	49
12	[illegible]	[illegible]	[illegible]	1.31745	.78692	1.27077	.81558	1.22612	48
13	[illegible]	[illegible]	[illegible]	[illegible]	.78739	1.27001	.81606	1.22539	47
14	[illegible]	[illegible]	[illegible]	[illegible]	.78786	1.26925	.81655	1.22467	46
15	[illegible]	[illegible]	[illegible]	[illegible]	.78834	1.26849	.81703	1.22394	45
16	[illegible]	[illegible]	[illegible]	1.31427	.78881	1.26774	.81752	1.22321	44
17	[illegible]	[illegible]	.76131	[illegible]	.78928	1.26698	.81800	1.22249	43
18	[illegible]	[illegible]	[illegible]	[illegible]	.78975	1.26622	.81849	1.22176	42
19	[illegible]	[illegible]	[illegible]	[illegible]	.79022	1.26546	.81898	1.22104	41
20	[illegible]	[illegible]	[illegible]	1.31110	.79070	1.26471	.81946	1.22031	40
21	[illegible]	[illegible]	[illegible]	1.31031	.79117	1.26395	.81995	1.21959	39
22	[illegible]	[illegible]	[illegible]	[illegible]	.79164	1.26319	.82044	1.21886	38
23	[illegible]	[illegible]	.76410	[illegible]	.79212	1.26244	.82092	1.21814	37
24	[illegible]	[illegible]	[illegible]	[illegible]	.79259	1.26169	.82141	1.21742	36
25	[illegible]	[illegible]	[illegible]	[illegible]	.79306	1.26093	.82190	1.21670	35
26	[illegible]	[illegible]	[illegible]	[illegible]	.79354	1.26018	.82238	1.21598	34
27	[illegible]	[illegible]	[illegible]	[illegible]	.79401	1.25943	.82287	1.21526	33
28	[illegible]	[illegible]	[illegible]	[illegible]	.79449	1.25867	.82336	1.21454	32
29	[illegible]	[illegible]	[illegible]	[illegible]	.79496	1.25792	.82385	1.21382	31
30	[illegible]	[illegible]	[illegible]	[illegible]	.79544	1.25717	.82434	1.21310	30
31	[illegible]	[illegible]	[illegible]	[illegible]	.79591	1.25642	.82483	1.21238	29
32	[illegible]	[illegible]	[illegible]	[illegible]	.79639	1.25567	.82531	1.21166	28
33	[illegible]	[illegible]	[illegible]	[illegible]	.79686	1.25492	.82580	1.21094	27
34	[illegible]	[illegible]	[illegible]	[illegible]	.79734	1.25417	.82629	1.21023	26
35	[illegible]	[illegible]	[illegible]	[illegible]	.79781	1.25343	.82678	1.20951	25
36	[illegible]	[illegible]	[illegible]	[illegible]	.79829	1.25268	.82727	1.20879	24
37	[illegible]	[illegible]	[illegible]	[illegible]	.79877	1.25193	.82776	1.20808	23
38	[illegible]	[illegible]	[illegible]	[illegible]	.79924	1.25118	.82825	1.20736	22
39	[illegible]	[illegible]	[illegible]	[illegible]	.79972	1.25044	.82874	1.20665	21
40	[illegible]	[illegible]	[illegible]	[illegible]	.80020	1.24969	.82923	1.20593	20
41	[illegible]	[illegible]	[illegible]	[illegible]	.80067	1.24895	.82972	1.20522	19
42	[illegible]	[illegible]	[illegible]	[illegible]	.80115	1.24820	.83022	1.20451	18
43	[illegible]	[illegible]	[illegible]	[illegible]	.80163	1.24746	.83071	1.20379	17
44	[illegible]	[illegible]	[illegible]	[illegible]	.80211	1.24672	.83120	1.20308	16
45	[illegible]	[illegible]	[illegible]	[illegible]	.80258	1.24597	.83169	1.20237	15
46	[illegible]	[illegible]	[illegible]	[illegible]	.80306	1.24523	.83218	1.20166	14
47	[illegible]	[illegible]	[illegible]	[illegible]	.80354	1.24449	.83268	1.20095	13
48	[illegible]	[illegible]	[illegible]	[illegible]	.80402	1.24375	.83317	1.20024	12
49	[illegible]	[illegible]	[illegible]	[illegible]	.80450	1.24301	.83366	1.19953	11
50	[illegible]	[illegible]	[illegible]	[illegible]	.80498	1.24227	.83415	1.19882	10
51	[illegible]	[illegible]	[illegible]	[illegible]	.80546	1.24153	.83465	1.19811	9
52	[illegible]	[illegible]	[illegible]	[illegible]	.80594	1.24079	.83514	1.19740	8
53	[illegible]	[illegible]	[illegible]	[illegible]	.80642	1.24005	.83564	1.19669	7
54	[illegible]	[illegible]	[illegible]	[illegible]	.80690	1.23931	.83613	1.19599	6
55	[illegible]	[illegible]	[illegible]	[illegible]	.80738	1.23858	.83662	1.19528	5
56	[illegible]	[illegible]	[illegible]	[illegible]	.80786	1.23784	.83712	1.19457	4
57	[illegible]	[illegible]	[illegible]	[illegible]	.80834	1.23710	.83761	1.19387	3
58	[illegible]	[illegible]	[illegible]	[illegible]	.80882	1.23637	.83811	1.19316	2
59	[illegible]	[illegible]	[illegible]	[illegible]	.80930	1.23563	.83860	1.19246	1
60	[illegible]	[illegible]	[illegible]	[illegible]	.80978	1.23490	.83910	1.19175	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	53°		52°		51°		50°		

TABLE VII.—*Continued.*

NATURAL TANGENTS AND COTANGENTS.

′	40° Tang	40° Cotang	41° Tang	41° Cotang	42° Tang	42° Cotang	43° Tang	43° Cotang	′
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	33
28	.85308	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.64158	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	49°		48°		47°		46°		

TABLE VII.—*Continued*

NATURAL TANGENTS AND COTANGENTS

′	44° Tang	44° Cotang	′	′	44° Tang	44° Cotang	′	′	44° Tang	44° Cotang	′
0	[illegible]	[illegible]	60	20	[illegible]	[illegible]	40	40	[illegible]	[illegible]	20
1	[illegible]	[illegible]	59	21	[illegible]	[illegible]	39	41	[illegible]	[illegible]	19
2	[illegible]	[illegible]	58	22	[illegible]	[illegible]	38	42	[illegible]	[illegible]	18
3	[illegible]	[illegible]	57	23	[illegible]	[illegible]	37	43	[illegible]	[illegible]	17
4	[illegible]	[illegible]	56	24	[illegible]	[illegible]	36	44	[illegible]	[illegible]	16
5	[illegible]	[illegible]	55	25	[illegible]	[illegible]	35	45	[illegible]	[illegible]	15
6	[illegible]	[illegible]	54	26	[illegible]	[illegible]	34	46	[illegible]	[illegible]	14
7	[illegible]	[illegible]	53	27	[illegible]	[illegible]	33	47	[illegible]	[illegible]	13
8	[illegible]	[illegible]	52	28	[illegible]	[illegible]	32	48	[illegible]	[illegible]	12
9	[illegible]	[illegible]	51	29	[illegible]	[illegible]	31	49	[illegible]	[illegible]	11
10	[illegible]	[illegible]	50	30	[illegible]	[illegible]	30	50	[illegible]	[illegible]	10
11	[illegible]	[illegible]	49	31	[illegible]	[illegible]	29	51	[illegible]	[illegible]	9
12	[illegible]	[illegible]	48	32	[illegible]	[illegible]	28	52	[illegible]	[illegible]	8
13	[illegible]	[illegible]	47	33	[illegible]	[illegible]	27	53	[illegible]	[illegible]	7
14	[illegible]	[illegible]	46	34	[illegible]	[illegible]	26	54	[illegible]	[illegible]	6
15	[illegible]	[illegible]	45	35	[illegible]	[illegible]	25	55	[illegible]	[illegible]	5
16	[illegible]	[illegible]	44	36	[illegible]	[illegible]	24	56	[illegible]	[illegible]	4
17	[illegible]	[illegible]	43	37	[illegible]	[illegible]	23	57	[illegible]	[illegible]	3
18	[illegible]	[illegible]	42	38	[illegible]	[illegible]	22	58	[illegible]	1.00116	2
19	[illegible]	[illegible]	41	39	[illegible]	[illegible]	21	59	[illegible]	[illegible]	1
20	[illegible]	[illegible]	40	40	[illegible]	1.01170	20	60	1.0000	1.0000	0
′	45° Cotang	45° Tang	′	′	45° Cotang	45° Tang	′	′	45° Cotang	45° Tang	′

TABLE VIII.

CO-ORDINATES OF POINTS OF INTERSECTION OF PARALLELS AND MERIDIANS IN POLYCONIC PROJECTION. § 417.

Latitude.	Length of 1° Latitude, in Statute Miles.	Length of Side of Tangent Cone, in Statute Miles.	Meridian Distances for 1° Longitude.				Divergence of Parallels for 1° Longitude.			
			In Yards.	In Metres.	In Miles.	Factor.	In Yards.	In Metres.	In Miles.	Factor.
30°	68.875	6869	105507	96476	59.95	$n \cos (0.288n^\circ)$	460.4	421.0	0.2617	n^2
32°	68.897	6348	103327	94481	58.71	$n \cos (0.304n^\circ)$	477.8	436.8	0.2715	n^2
34°	68.918	5881	101022	92373	57.40	$n \cos (0.320n^\circ)$	493.0	450.7	0.2800	n^2
36°	68.941	5461	98593	90152	56.02	$n \cos (0.337n^\circ)$	505.7	462.4	0.2873	n^2
38°	68.964	5079	96044	87822	54.57	$n \cos (0.353n^\circ)$	516.0	471.8	0.2932	n^2
40°	68.987	4729	93377	85383	53.06	$n \cos (0.369n^\circ)$	523.8	479.0	0.2976	n^2
42°	69.011	4408	90596	82840	51.48	$n \cos (0.386n^\circ)$	529.0	483.8	0.3006	n^2
44°	69.036	4110	87704	80197	49.83	$n \cos (0.402n^\circ)$	531.7	486.2	0.3022	n^2
46°	69.060	3833	84704	77452	48.13	$n \cos (0.418n^\circ)$	531.7	486.2	0.3022	n^2
48°	69.084	3575	81601	74615	46.37	$n \cos (0.435n^\circ)$	529.2	484.0	0.3007	n^2
50°	69.108	3332	78398	71686	44.54	$n \cos (0.451n^\circ)$	524.1	479.2	0.2978	n^2

n = number degrees of longitude between the given meridian and the prime meridian of the map.

TABLE IX.

GIVING VALUES OF C IN KUTTER'S FORMULA WHEN $s = 0.001$. § 259.

r in feet.	Values of n. .009	.010	.011	.012	.013	.015	.017	.020	.0225	.025	.030	.035	r in feet.
.1	108.3	93.8	82.2	72.7	65.0	53.2	44.6	35.5	30.0	25.0	20.1	16.3	.1
.2	129.5	113.1	100.0	80.1	80.2	66.3	56.2	45.2	38.6	33.6	26.3	21.5	.2
.3	141.8	123.3	111.0	98.8	90.2	75.0	63.4	51.8	44.6	38.4	30.3	25.1	.3
.4	150.3	132.5	118.0	106.0	96.2	80.4	68.8	56.2	48.4	42.4	33.7	27.8	.4
.5	156.8	138.6	123.8	111.2	101.2	85.1	72.8	60.0	51.8	45.4	36.2	30.0	.5
.6	161.9	143.3	128.3	115.7	105.3	88.8	76.4	62.9	54.5	48.0	38.5	32.0	.6
.7	166.1	147.4	131.9	119.3	108.7	92.0	79.3	65.4	56.9	50.2	40.3	33.6	.7
.8	169.7	150.8	135.1	122.3	111.6	94.6	81.9	67.7	59.0	52.2	42.0	35.1	.8
.9	172.8	153.7	137.8	125.1	114.2	97.0	84.2	69.7	60.8	53.8	43.4	36.3	.9
1.0	175.4	156.2	140.5	127.4	116.5	99.1	86.0	71.5	62.5	55.4	44.9	37.7	1.0
1.2	180.0	160.4	144.6	131.5	120.4	102.7	89.4	74.5	65.3	58.1	47.1	39.7	1.2
1.4	183.6	164.0	147.9	134.7	123.7	105.7	92.2	77.0	67.7	60.2	49.2	41.5	1.4
1.6	186.7	167.0	150.8	137.4	126.2	108.2	94.5	79.3	69.9	62.3	51.0	43.2	1.6
1.8	189.2	169.5	153.2	139.7	128.7	110.3	96.6	81.1	71.6	64.0	52.6	44.6	1.8
2.0	191.4	171.6	155.4	141.8	130.5	112.3	98.4	82.9	73.4	65.5	54.0	45.9	2.0
2.2	193.3	173.5	157.3	143.7	132.3	114.0	100.0	84.3	74.7	66.9	55.2	47.0	2.2
2.4	195.0	175.2	159.0	145.3	133.9	115.4	101.4	85.6	76.0	68.1	56.3	48.0	2.4
2.6	196.7	176.8	160.5	146.8	135.3	116.8	102.8	87.0	77.1	69.2	57.4	49.0	2.6
2.8	198.0	178.2	161.8	148.1	136.7	118.0	104.0	88.2	78.2	70.3	58.4	49.9	2.8
3.0	199.3	179.4	163.2	149.3	137.9	119.2	105.1	89.3	79.2	71.3	59.2	50.6	3.0
3.4	201.7	181.7	165.3	151.4	140.0	121.3	107.1	91.1	81.0	73.0	60.8	52.0	3.4
3.8	203.7	183.6	167.2	153.3	141.8	123.0	108.8	92.7	82.5	74.5	62.3	53.5	3.8
4.2	205.4	185.3	168.8	155.0	143.4	124.6	110.3	94.3	83.9	75.8	63.5	54.7	4.2
4.6	207.0	186.8	170.3	156.4	144.8	125.9	111.6	95.4	85.2	77.0	64.7	55.7	4.6
5.0	208.3	188.1	171.6	157.7	146.0	127.2	112.9	96.6	86.3	78.1	65.7	56.7	5.0

TABLE X. § 259.

GIVING DIAMETERS IN FEET OF CIRCULAR BRICK CONDUITS FOR VARIOUS INCLINATIONS AND RATES OF DISCHARGE.

Conduit full to point of maximum discharge. (By Kutter's formula.) § 259.

Cubic Feet per Second.	FALL PER 100 FEET.													Cubic Feet per Second.
	.10	.15	.20	.30	.50	.75	1.0	1.5	2.0	3.0	4.0	6.0	10.0	
1	.9	.9	.8	.8	.7	.7	.6	.6	.6	.5	.5	.5	.4	1
2	1.2	1.1	1.1	1.0	.9	.8	.8	.7	.7	.7	.6	.6	.5	2
3	1.4	1.3	1.3	1.2	1.1	1.0	.9	.9	.8	.8	.7	.7	.6	3
4	1.6	1.5	1.4	1.3	1.2	1.1	1.0	1.0	.9	.8	.8	.7	.7	4
5	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	1.0	.9	.9	.8	.7	5
10	2.2	2.0	1.9	1.8	1.6	1.5	1.4	1.3	1.3	1.2	1.1	1.0	.9	10
15	2.6	2.4	2.3	2.1	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	15
20	2.9	2.6	2.5	2.3	2.1	2.0	1.9	1.7	1.6	1.5	1.4	1.3	1.2	20
25	3.1	2.9	2.7	2.5	2.3	2.1	2.0	1.9	1.8	1.6	1.6	1.5	1.3	25
30	3.3	3.1	2.9	2.7	2.5	2.3	2.2	2.0	1.9	1.8	1.7	1.5	1.4	30
35	3.5	3.3	3.1	2.9	2.6	2.4	2.3	2.1	2.0	1.9	1.8	1.6	1.5	35
40	3.7	3.4	3.2	3.0	2.7	2.5	2.4	2.2	2.1	2.0	1.9	1.7	1.6	40
45	3.9	3.6	3.4	3.1	2.9	2.6	2.5	2.3	2.2	2.0	1.9	1.8	1.6	45
50	4.0	3.7	3.5	3.3	3.0	2.7	2.6	2.4	2.3	2.1	2.0	1.9	1.7	50
60	4.3	4.0	3.7	3.5	3.2	2.9	2.8	2.6	2.5	2.3	2.1	2.0	1.8	60
70	4.5	4.2	4.0	3.7	3.4	3.1	2.9	2.7	2.6	2.4	2.3	2.1	1.9	70
80	4.8	4.4	4.2	4.0	3.5	3.3	3.1	2.9	2.7	2.5	2.4	2.2	2.0	80
90	5.0	4.6	4.4	4.1	3.7	3.4	3.2	3.0	2.9	2.7	2.5	2.3	2.1	90
100	5.2	4.8	4.6	4.2	3.8	3.6	3.4	3.1	3.0	2.7	2.6	2.4	2.2	100
125	5.7	5.2	5.0	4.6	4.2	3.9	3.6	3.4	3.2	3.0	2.8	2.6	2.4	125
150	6.1	5.6	5.3	4.9	4.5	4.1	3.9	3.6	3.4	3.2	3.0	2.8	2.6	150
175	6.4	5.9	5.6	5.2	4.6	4.4	4.2	3.8	3.6	3.4	3.2	3.0	2.7	175
200	6.8	6.3	5.9	5.5	5.0	4.6	4.4	4.0	3.8	3.6	3.4	3.1	2.8	200
250	7.3	6.8	6.4	5.9	5.4	5.0	4.7	4.4	4.2	3.9	3.7	3.4	3.1	250
300	7.9	7.3	6.9	6.4	5.8	5.4	5.1	4.7	4.5	4.1	3.9	3.6	3.3	300
350	8.3	7.7	7.3	6.8	6.1	5.7	5.4	5.0	4.7	4.4	4.2	3.9	3.5	350
400	8.8	8.1	7.7	7.1	6.5	6.0	5.7	5.2	5.0	4.6	4.4	4.0	3.7	400
500	9.5	8.8	8.4	7.8	7.0	6.5	6.2	5.7	5.4	5.0	4.7	4.4	4.0	500
600	10.2	9.5	9.0	8.3	7.5	7.0	6.6	6.1	5.8	5.4	5.1	4.7	4.3	600
700	10.8	10.0	9.5	8.8	8.0	7.4	7.0	6.5	6.1	5.7	5.4	5.0	4.5	700
800	11.4	10.6	10.0	9.3	8.4	7.8	7.4	6.8	6.5	6.0	5.7	5.3	4.7	800
900	11.9	11.1	10.5	9.7	8.8	8.1	7.7	7.1	6.7	6.2	5.9	5.5	5.0	900
1,000	12.4	11.5	10.9	10.1	9.2	8.5	8.0	7.4	7.0	6.5	6.2	5.7	5.2	1,000

TABLE XI.

Volumes by the Prismoidal Formula. § 320.

Widths	Heights. 1	2	3	4	5	6	7	8	9	10	Corrections for tenths in height.	
1	0	1	1	1	2	2	2	2	3	3	.1	0
2	1	1	2	2	3	4	4	5	6	6	.2	0
3	1	2	3	4	5	6	6	7	8	9	.3	0
4	1	2	4	5	6	7	9	10	11	12	.4	1
5	2	3	5	6	—8	—9	—11	—12	—14	—15	.5	1
6	2	4	6	7	9	11	13	15	17	19	.6	1
7	2	4	6	9	11	13	15	17	19	22	.7	1
8	2	5	7	10	12	15	17	20	22	25	.8	1
9	3	6	8	11	14	17	19	22	25	28	.9	1
10	3	6	9	12	15	19	22	25	28	31		
11	3	7	10	14	17	20	24	27	31	34	.1	0
12	4	7	11	15	19	22	26	30	33	37	.2	1
13	4	8	12	16	20	24	28	32	36	40	.3	1
14	4	9	13	17	22	26	30	35	39	43	.4	2
15	5	9	14	19	—23	—28	—32	—37	—42	—46	.5	2
16	5	10	15	20	25	30	35	40	44	49	.6	3
17	5	10	16	21	26	31	37	42	47	52	.7	3
18	6	11	17	22	28	33	39	44	50	56	.8	4
19	6	12	18	23	29	35	41	47	53	59	.9	4
20	6	12	19	25	31	37	43	49	56	62		
21	6	13	19	26	32	39	45	52	58	65	.1	1
22	7	14	20	27	34	41	48	54	61	68	.2	2
23	7	14	21	28	35	43	50	57	64	71	.3	2
24	7	15	22	30	37	44	52	59	67	74	.4	3
25	8	15	23	31	39	—46	—54	62	—69	—77	.5	4
26	8	16	24	32	40	48	56	64	72	80	.6	5
27	8	17	25	33	42	50	58	67	75	83	.7	5
28	9	17	26	35	43	52	60	69	78	86	.8	6
29	9	18	27	36	45	54	63	72	81	90	.9	7
30	9	19	28	37	46	56	65	74	83	93		
31	10	19	29	38	48	57	67	77	86	96	.1	1
32	10	20	30	40	49	59	69	79	89	99	.2	2
33	10	20	31	41	51	61	71	81	92	102	.3	3
34	10	21	31	42	52	63	73	84	94	105	.4	4
35	11	22	32	43	54	—65	76	—86	—97	—108	.5	5
36	11	22	33	44	56	67	78	89	100	111	.6	6
37	11	23	34	46	57	69	80	91	103	114	.7	8
38	12	23	35	47	59	70	82	94	106	117	.8	9
39	12	24	36	48	60	72	84	96	108	120	.9	10
40	12	25	37	49	62	74	86	99	111	123		
41	13	25	38	51	63	76	89	101	114	127	.1	1
42	13	26	39	52	65	78	91	104	117	130	.2	3
43	13	27	40	53	66	80	93	106	119	133	.3	4
44	14	27	41	54	68	81	95	109	122	136	.4	6
45	14	28	42	56	69	83	97	111	—125	—139	.5	7
46	14	28	43	57	71	85	99	114	128	142	.6	8
47	15	29	44	58	73	87	102	116	131	145	.7	10
48	15	30	44	59	74	89	104	119	133	148	.8	11
49	15	30	45	60	76	91	106	121	136	151	.9	13
50	15	31	46	62	77	93	108	123	139	154		
	1	2	3	4	5	6	7	8	9	10		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	0	0	0	1	1	1	1	1	1			

TABLE XI.—*Continued.*

VOLUMES BY THE PRISMOIDAL FORMULA.

Widths.	Heights. 1	2	3	4	5	6	7	8	9	10	Corrections for tenths in height.	
51	16	31	47	63	79	94	110	126	142	157	.1	2
52	16	32	48	64	80	96	112	128	144	160	.2	3
53	16	33	49	65	82	98	115	131	147	163	.3	5
54	17	33	50	67	83	100	117	133	150	167	.4	7
55	—17	—34	—51	—68	—85	—102	—119	—136	—153	—170	.5	8
56	17	35	52	69	86	104	121	138	156	173	.6	10
57	18	35	53	70	88	106	123	141	158	176	.7	12
58	18	36	54	72	90	107	125	143	161	179	.8	14
59	18	36	55	73	91	109	127	146	164	182	.9	15
60	19	37	56	74	98	111	130	148	167	185		
61	19	38	56	75	94	113	132	151	169	188	.1	2
62	19	38	57	77	96	115	134	153	172	191	.2	4
63	19	39	58	78	97	117	136	156	175	194	.3	6
64	20	40	59	79	99	119	138	158	178	197	.4	8
65	—20	—40	—60	—80	—100	—120	—140	—160	—181	—201	.5	10
66	20	41	61	81	102	122	143	163	183	204	.6	12
67	21	41	62	83	103	124	145	165	186	207	.7	14
68	21	42	63	84	105	126	147	168	189	210	.8	16
69	21	43	64	85	106	128	149	170	192	213	.9	18
70	22	43	65	86	108	130	151	173	194	216		
71	22	44	66	88	100	131	153	175	197	219	.1	2
72	22	44	67	89	111	133	156	178	200	222	.2	5
73	23	45	68	90	113	135	158	180	203	225	.3	7
74	23	46	69	91	114	137	160	183	206	228	.4	9
75	—23	—46	—69	—93	—116	—139	—162	—185	—208	—231	.5	12
76	23	47	70	94	117	141	164	188	211	235	.6	14
77	24	48	71	95	119	143	166	190	214	238	.7	16
78	24	48	72	96	120	144	169	193	217	241	.8	19
79	24	49	73	98	122	146	171	195	219	244	.9	21
80	25	49	74	99	123	148	173	198	222	247		
81	25	50	75	100	125	150	175	200	225	250	.1	3
82	25	51	76	101	127	152	177	202	228	253	.2	5
83	26	51	77	102	128	154	179	205	231	256	.3	8
84	26	52	78	104	130	156	181	207	233	259	.4	10
85	—26	—52	—79	—105	—131	—157	—184	—210	—236	—262	.5	13
86	27	53	80	106	133	159	186	212	239	265	.6	16
87	27	54	81	107	134	161	188	215	242	269	.7	18
88	27	54	81	109	136	163	190	217	244	272	.8	21
89	27	55	82	110	137	165	192	220	247	275	.9	24
90	28	56	83	111	139	167	194	222	250	278		
91	28	56	84	112	140	169	197	225	253	281	.1	3
92	28	57	85	114	142	170	199	227	256	284	.2	6
93	29	57	86	115	144	172	201	230	258	287	.3	9
94	29	58	87	116	145	174	203	232	261	290	.4	12
95	—29	—59	—88	—117	—147	—176	—205	—235	—264	—293	.5	15
96	30	59	89	119	148	178	207	237	267	296	.6	18
97	30	60	90	120	150	180	210	240	269	299	.7	21
98	30	60	91	121	151	181	212	242	272	302	.8	23
99	31	61	92	122	153	183	214	244	275	306	.9	26
100	31	62	93	123	154	185	216	247	278	309		
	1	2	3	4	5	6	7	8	9	10		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	0	0	0	1	1	1	1	1	1			

TABLE XI.—*Continued.*

Volumes by the Prismoidal Formula.

Widths.	Heights. 11	12	13	14	15	16	17	18	19	20	Corrections for tenths in height.	
1	3	4	4	4	5	5	5	6	6	6	.1	0
2	7	7	8	9	9	10	10	11	12	12	.2	0
3	10	11	12	13	14	15	16	17	18	19	.3	0
4	14	15	16	17	19	20	21	22	23	25	.4	1
5	—17	—19	—20	—22	—23	—25	—26	—28	—29	—31	.5	1
6	20	22	24	26	28	30	31	33	35	37	.6	1
7	24	26	28	30	32	35	37	39	41	43	.7	1
8	27	30	32	35	37	40	42	44	47	49	.8	1
9	31	33	36	39	42	44	47	50	53	56	.9	1
10	34	37	40	43	46	49	52	56	59	62		
11	37	41	44	48	51	54	58	61	65	68	.1	0
12	41	44	48	52	56	59	63	67	70	74	.2	1
13	44	48	52	56	60	64	68	72	76	80	.3	1
14	48	52	56	60	65	69	73	78	82	86	.4	2
15	—51	—56	—60	—65	—69	—74	—79	—83	—88	—93	.5	2
16	54	59	64	69	74	79	84	89	94	99	.6	3
17	58	63	68	73	79	84	89	94	100	105	.7	3
18	61	67	72	78	83	89	94	100	106	111	.8	4
19	65	70	76	82	88	94	100	106	111	117	.9	4
20	68	74	80	86	93	99	105	111	117	123		
21	71	78	84	91	97	104	110	117	123	130	.1	1
22	75	81	88	95	102	109	115	122	129	136	.2	2
23	78	85	92	99	106	114	121	128	135	142	.3	2
24	81	89	96	104	111	119	126	133	141	148	.4	3
25	—85	—93	—100	—108	—116	—123	—131	—139	—147	—154	.5	4
26	88	96	104	112	120	128	136	144	152	160	.6	5
27	92	100	108	117	125	133	142	150	158	167	.7	5
28	95	104	112	121	130	138	147	156	164	173	.8	6
29	98	107	116	125	134	143	152	161	170	179	.9	7
30	102	111	120	130	139	148	157	167	176	185		
31	105	115	124	134	144	153	163	172	182	191	.1	1
32	109	119	128	138	148	158	168	178	188	198	.2	2
33	112	122	132	143	153	163	173	183	194	204	.3	3
34	115	126	136	147	157	168	178	189	199	210	.4	4
35	—119	—130	—140	—151	—162	—173	—184	—194	—205	—216	.5	5
36	122	133	144	156	167	178	189	200	211	222	.6	6
37	126	137	148	160	171	183	194	206	217	228	.7	8
38	129	141	152	164	176	188	199	211	223	235	.8	9
39	132	144	156	169	181	193	205	217	229	241	.9	10
40	136	148	160	173	185	198	210	222	235	247		
41	139	152	165	177	190	202	215	228	240	253	.1	1
42	143	156	169	181	194	207	220	233	246	259	.2	3
43	146	159	173	186	199	212	226	239	252	265	.3	4
44	149	163	177	190	204	217	231	244	258	272	.4	6
45	—153	—167	—181	—194	—208	—222	—236	—250	—264	—278	.5	7
46	156	170	185	199	213	227	241	256	270	284	.6	8
47	160	174	189	203	218	232	247	261	276	290	.7	10
48	163	178	193	207	222	237	252	267	281	296	.8	11
49	166	181	197	212	227	242	257	272	287	302	.9	13
50	170	185	201	216	231	247	262	278	293	309		
	11	12	13	14	15	16	17	18	19	20		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	0	1	1	2	2	3	3	4	4			

TABLE XI.—*Continued.*

VOLUMES BY THE PRISMOIDAL FORMULA.

Widths.	Heights.										Corrections for tenths in height.	
	11	12	13	14	15	16	17	18	19	20		
51	173	189	205	220	236	252	268	283	299	315	.1	2
52	177	193	209	225	241	257	273	289	305	321	.2	3
53	180	196	213	229	245	262	278	294	311	327	.3	5
54	183	200	217	233	250	267	283	300	317	333	.4	7
55	187	204	221	238	255	272	289	306	323	340	.5	8
56	190	207	225	242	259	277	294	311	328	346	.6	10
57	194	211	229	246	264	281	299	317	334	352	.7	12
58	197	215	233	251	269	286	304	322	340	358	.8	14
59	200	219	237	255	273	291	310	328	346	364	.9	15
60	204	222	241	259	278	296	315	333	352	370		
61	207	226	245	264	282	301	320	339	358	377	.1	2
62	210	230	249	268	287	306	325	344	364	383	.2	4
63	214	233	253	272	292	311	331	350	369	389	.3	6
64	217	237	257	277	296	316	336	356	375	395	.4	8
65	221	241	261	281	301	321	341	361	381	401	.5	10
66	224	244	265	285	306	326	346	367	387	407	.6	12
67	227	248	269	290	310	331	352	372	393	414	.7	14
68	231	252	273	294	315	336	357	378	399	420	.8	16
69	234	256	277	298	319	341	362	383	405	426	.9	18
70	238	259	281	302	324	346	367	389	410	432		
71	241	263	285	307	329	351	373	394	416	438	.1	2
72	244	267	289	311	333	356	378	400	422	444	.2	5
73	248	270	293	315	338	360	383	406	428	451	.3	7
74	251	274	297	320	343	365	388	411	434	457	.4	9
75	255	278	301	324	347	370	394	417	440	463	.5	12
76	258	281	305	328	352	375	399	422	446	469	.6	14
77	261	285	309	333	356	380	404	428	452	475	.7	16
78	265	289	313	337	361	385	409	433	457	481	.8	19
79	268	293	317	341	366	390	415	439	463	488	.9	21
80	272	296	321	346	370	395	420	444	469	494		
81	275	300	325	350	375	400	425	450	475	500	.1	3
82	278	304	329	354	380	405	430	456	481	506	.2	5
83	282	307	333	359	384	410	435	461	487	512	.3	8
84	285	311	337	363	389	415	441	467	493	519	.4	10
85	289	315	341	367	394	420	446	472	498	525	.5	13
86	292	319	345	372	398	425	451	478	504	531	.6	16
87	295	322	349	376	403	430	456	483	510	537	.7	18
88	299	326	353	380	407	435	462	489	516	543	.8	21
89	303	330	357	385	412	440	467	494	522	549	.9	24
90	306	333	361	389	417	444	472	500	528	556		
91	309	337	365	393	421	449	477	506	534	562	.1	3
92	312	341	369	398	426	454	483	511	540	568	.2	6
93	316	344	373	402	431	459	488	517	545	574	.3	9
94	319	348	377	406	435	464	493	522	551	580	.4	12
95	323	352	381	410	440	469	498	528	557	586	.5	15
96	326	356	385	415	444	474	504	533	563	593	.6	18
97	329	359	389	419	449	479	509	539	569	599	.7	21
98	333	363	393	423	454	484	514	544	575	605	.8	23
99	336	367	397	428	458	489	519	550	581	611	.9	26
100	340	370	401	432	463	494	525	556	586	617		
	11	12	13	14	15	16	17	18	19	20		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	0	1	1	2	2	3	3	4	4			

TABLE XI.—*Continued.*

VOLUMES BY THE PRISMOIDAL FORMULA.

Widths.	Heights. 21	22	23	24	25	26	27	28	29	30	Corrections for tenths in height.	
1	6	7	7	7	8	8	8	9	9	9	.1	0
2	13	14	14	15	15	16	17	17	18	19	.2	0
3	19	20	21	22	23	24	25	26	27	28	.3	0
4	26	27	28	30	31	32	33	35	36	37	.4	1
5	32	34	—35	—37	—39	—40	—42	—43	—45	—46	.5	1
6	39	41	43	44	46	48	50	52	54	56	.6	1
7	45	48	50	52	54	56	58	60	63	65	.7	1
8	52	54	57	59	62	64	67	69	72	74	.8	1
9	58	61	64	67	69	72	75	78	81	83	.9	1
10	65	68	71	74	77	80	83	86	90	93		
11	71	75	78	81	85	88	92	95	98	102	.1	0
12	78	81	85	89	93	96	100	104	107	111	.2	1
13	84	88	92	96	100	114	108	112	116	120	.3	1
14	91	95	99	104	108	112	117	121	125	130	.4	2
15	—97	—102	—106	—111	—116	—120	—125	—130	—134	—139	.5	2
16	104	109	114	119	123	128	133	138	143	148	.6	3
17	110	115	121	126	131	136	142	147	152	157	.7	3
18	117	122	128	133	139	144	150	156	161	167	.8	4
19	123	129	135	141	147	152	158	164	170	176	.9	4
20	130	136	142	148	154	160	167	173	179	185		
21	136	143	149	156	162	169	175	181	188	194	.1	1
22	143	149	156	163	170	177	183	190	197	204	.2	2
23	149	156	163	170	177	185	192	199	206	213	.3	2
24	156	163	170	178	185	193	200	207	215	222	.4	3
25	—162	—170	—177	—185	—193	—201	—208	—216	—224	—231	.5	4
26	169	177	185	193	201	209	217	225	233	241	.6	5
27	175	183	192	200	208	217	225	233	242	250	.7	5
28	181	190	199	207	216	225	233	242	251	259	.8	6
29	188	197	206	215	224	233	242	251	260	269	.9	7
30	194	204	213	222	231	241	250	259	269	278		
31	201	210	220	230	239	249	258	268	277	287	.1	1
32	207	217	227	237	247	257	267	277	286	296	.2	2
33	214	224	234	244	255	265	275	285	295	306	.3	3
34	220	231	241	252	262	273	283	294	304	315	.4	4
35	227	238	248	259	—270	—281	—292	—302	—313	—324	.5	5
36	233	244	256	267	278	289	300	311	322	333	.6	6
37	240	251	263	274	285	297	308	320	331	343	.7	8
38	246	258	270	281	293	305	317	328	340	352	.8	9
39	253	265	277	289	301	313	325	337	349	361	.9	10
40	259	272	284	296	309	321	333	346	358	370		
41	266	278	291	304	316	329	342	354	367	380	.1	1
42	272	285	298	311	324	337	350	363	376	389	.2	3
43	279	292	305	319	332	345	358	372	385	398	.3	4
44	285	299	312	326	340	353	367	380	394	407	.4	6
45	292	306	319	333	347	361	—375	—389	—403	—417	.5	7
46	298	312	327	341	355	369	383	398	412	426	.6	8
47	305	319	334	348	363	377	392	406	421	435	.7	10
48	311	326	341	356	370	385	400	415	430	444	.8	11
49	318	333	348	363	378	393	408	423	439	454	.9	13
50	324	340	355	370	386	401	417	432	448	463		
	21	22	23	24	25	26	27	28	29	30		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	1	2	2	3	4	5	5	6	7			

TABLE XI.—*Continued.*

VOLUMES BY THE PRISMOIDAL FORMULA.

Widths.	Heights.										Corrections for tenths in height.	
	21	22	23	24	25	26	27	28	29	30		
51	331	346	362	378	394	409	425	441	456	472	.1	2
52	337	353	369	385	401	417	433	449	465	481	.2	3
53	344	360	376	393	409	425	442	458	474	491	.3	5
54	350	367	383	400	417	433	450	467	483	500	.4	7
55	—356	—373	—390	—407	—424	—441	—458	—475	—492	—509	.5	8
56	363	380	398	415	432	449	467	484	501	519	.6	10
57	369	387	405	422	440	457	475	493	510	528	.7	12
58	376	394	412	430	448	465	483	501	519	537	.8	14
59	382	401	419	437	455	473	492	510	528	546	.9	15
60	389	407	426	444	463	481	500	519	537	556		
61	395	414	433	452	471	490	508	527	546	565	.1	2
62	402	421	440	459	478	498	517	536	555	574	.2	4
63	408	428	447	467	486	506	525	544	564	583	.3	6
64	415	435	454	474	494	514	533	553	573	593	.4	8
65	—421	—441	—461	—481	—502	—522	—542	—563	—582	—602	.5	10
66	428	448	469	489	509	530	550	570	591	611	.6	12
67	434	455	476	496	517	538	558	579	600	620	.7	14
68	441	462	483	504	525	546	567	588	609	630	.8	16
69	447	469	490	511	532	554	575	596	618	639	.9	18
70	454	475	497	519	540	562	583	605	627	648		
71	460	482	504	526	548	570	592	614	635	657	.1	2
72	467	489	511	533	556	578	600	622	644	667	.2	5
73	473	496	518	541	563	586	608	631	653	676	.3	7
74	480	502	525	548	571	594	617	640	662	685	.4	9
75	—486	—509	—532	—556	—579	—601	—625	—648	—671	—694	.5	12
76	493	516	540	563	586	610	633	657	680	704	.6	14
77	499	523	547	570	594	618	642	665	689	713	.7	16
78	506	530	554	578	602	626	650	674	698	722	.8	19
79	512	536	561	585	610	634	658	683	707	731	.9	21
80	519	543	568	593	617	642	667	691	716	741		
81	525	550	575	600	625	650	675	700	725	750	.1	3
82	531	557	582	607	633	658	683	709	734	759	.2	5
83	538	564	589	615	640	666	692	717	743	769	.3	8
84	544	570	596	622	648	674	700	726	752	778	.4	10
85	—551	—577	—603	—630	—656	—682	—708	—735	—761	—787	.5	13
86	557	584	610	637	664	690	717	743	770	796	.6	16
87	564	591	618	644	671	698	725	752	779	806	.7	18
88	570	598	625	652	679	706	733	760	788	815	.8	21
89	577	604	632	659	687	714	742	769	797	824	.9	24
90	583	611	639	667	694	722	750	777	806	833		
91	590	618	646	674	702	730	758	786	815	843	.1	3
92	596	625	653	681	710	738	767	795	823	852	.2	6
93	603	631	660	689	718	746	775	804	832	861	.3	9
94	609	638	667	696	725	754	783	812	841	870	.4	12
95	—616	—645	—674	—704	—733	—762	—792	—821	—850	—880	.5	15
96	622	652	681	711	741	770	800	830	859	889	.6	18
97	629	659	689	719	748	778	808	838	868	898	.7	21
98	635	665	696	726	756	786	817	847	877	907	.8	23
99	642	672	703	733	764	794	825	856	886	917	.9	26
100	648	679	710	741	772	802	833	864	895	926		
	21	22	23	24	25	26	27	28	29	30		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	1	2	2	3	4	5	5	6	7			

TABLE XI.—*Continued.*

VOLUMES BY THE PRISMOIDAL FORMULA.

Widths.	Heights. 31	32	33	34	35	36	37	38	39	40	Corrections for tenths in height.	
1	10	10	10	10	11	11	11	12	12	12	.1	0
2	19	20	20	21	22	22	23	23	24	25	.2	0
3	29	30	31	31	32	33	34	35	36	37	.3	0
4	38	40	41	42	43	44	46	47	48	49	.4	1
5	48	49	51	52	54	56	57	59	60	62	.5	1
6	57	59	61	63	65	67	68	70	72	74	.6	1
7	67	69	71	73	76	78	80	82	84	86	.7	1
8	77	79	81	84	86	89	91	94	96	97	.8	1
9	86	89	92	94	97	100	103	106	108	111	.9	1
10	96	99	102	105	108	111	114	117	120	123		
11	105	109	112	115	119	122	126	129	132	136	.1	0
12	115	119	122	126	130	133	137	141	144	148	.2	1
13	124	128	132	136	140	144	148	152	156	160	.3	1
14	134	138	143	147	151	156	160	164	169	173	.4	2
15	144	148	153	157	162	167	171	176	181	185	.5	2
16	153	158	163	168	173	178	183	188	193	198	.6	3
17	163	168	173	178	183	189	194	199	205	210	.7	3
18	172	178	183	189	194	200	206	211	217	222	.8	4
19	182	188	194	199	205	211	217	223	229	235	.9	4
20	191	198	204	210	216	222	228	235	241	247		
21	201	207	214	220	227	233	240	246	253	259	.1	1
22	210	217	224	231	238	244	251	258	265	272	.2	2
23	220	227	234	241	248	256	263	270	277	284	.3	2
24	230	237	244	252	259	267	274	281	289	296	.4	3
25	239	247	255	262	270	278	285	293	301	309	.5	4
26	249	257	265	273	281	289	297	305	313	321	.6	5
27	258	267	275	283	292	300	308	317	325	333	.7	5
28	268	277	285	294	302	311	320	328	337	346	.8	6
29	277	286	295	304	313	322	331	340	349	358	.9	7
30	287	296	306	315	324	333	343	352	361	370		
31	297	306	316	325	335	344	354	364	373	383	.1	1
32	306	316	326	336	346	356	365	375	385	395	.2	2
33	316	326	336	346	356	367	377	387	397	407	.3	3
34	325	336	346	357	367	378	388	399	409	420	.4	4
35	335	346	356	367	378	389	400	410	421	432	.5	5
36	344	356	367	378	389	400	411	422	433	444	.6	6
37	354	365	377	388	400	411	423	434	445	457	.7	8
38	364	375	387	399	410	422	434	446	457	469	.8	9
39	373	385	397	409	421	433	445	457	469	481	.9	10
40	383	395	407	420	432	444	457	469	481	494		
41	392	405	418	430	443	456	468	481	494	506	.1	1
42	402	415	428	441	454	467	480	493	506	519	.2	3
43	411	425	438	451	465	478	491	504	518	531	.3	4
44	421	435	448	462	475	489	502	516	530	543	.4	6
45	431	444	458	472	486	500	514	528	542	556	.5	7
46	440	454	469	483	497	511	525	540	554	568	.6	8
47	450	464	479	493	508	522	537	551	566	580	.7	10
48	459	474	489	504	519	533	548	563	578	593	.8	11
49	469	484	499	514	529	544	560	575	590	605	.9	13
50	478	494	500	525	540	556	571	586	602	617		
	31	32	33	34	35	36	37	38	39	40		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	1	2	3	4	5	6	8	9	10			

TABLE XI.—*Continued.*

Volumes by the Prismoidal Formula.

Widths.	Heights. 31	32	33	34	35	36	37	38	39	40	Corrections for tenths in height.	
51	488	504	519	535	551	567	582	598	614	630	.1	2
52	498	514	530	546	562	578	594	610	626	642	.2	3
53	507	523	540	556	573	589	605	622	638	654	.3	5
54	517	533	550	567	583	600	617	633	650	667	.4	7
55	—526	—543	—560	—577	—594	—611	—628	—645	—662	—679	.5	8
56	536	553	570	588	605	622	640	657	674	691	.6	10
57	545	563	581	598	616	633	651	669	686	704	.7	12
58	555	573	591	609	627	644	662	680	698	716	.8	14
59	565	583	601	619	637	656	674	692	710	728	.9	15
60	574	593	611	630	648	667	685	704	722	741		
61	584	602	621	640	659	678	697	715	734	753	.1	2
62	593	612	631	651	670	689	708	727	746	765	.2	4
63	603	622	642	661	681	700	719	739	758	778	.3	6
64	612	632	652	672	691	711	731	751	770	790	.4	8
65	—622	—642	—662	—682	—702	—722	—742	—762	—782	—802	.5	10
66	631	652	672	693	713	733	754	774	794	815	.6	12
67	641	662	682	703	724	744	765	786	806	827	.7	14
68	651	672	693	714	735	756	777	798	819	840	.8	16
69	660	681	703	724	745	767	788	809	831	852	.9	18
70	670	691	713	735	756	778	799	821	843	864		
71	679	701	723	745	767	789	811	833	855	877	.1	2
72	689	711	733	756	778	800	822	844	867	889	.2	5
73	698	721	744	766	789	811	834	856	879	901	.3	7
74	708	731	754	777	799	822	845	868	891	914	.4	9
75	—718	—741	—764	—787	—810	—833	—856	—880	—903	—926	.5	12
76	727	751	774	798	821	844	868	891	915	938	.6	14
77	737	760	784	808	832	856	879	903	927	951	.7	16
78	746	770	794	819	843	867	891	915	939	963	.8	19
79	756	780	805	829	853	878	902	927	951	975	.9	21
80	765	790	815	840	864	889	914	938	963	988		
81	775	800	825	850	875	900	925	950	975	1000	.1	3
82	785	810	835	860	886	911	936	962	987	1012	.2	5
83	794	820	845	871	897	922	948	973	999	1025	.3	8
84	804	830	856	881	907	933	959	985	1011	1037	.4	10
85	—813	—840	—866	—892	—918	—944	—971	—997	—1023	—1049	.5	13
86	823	849	876	902	929	956	982	1009	1035	1062	.6	16
87	832	859	886	913	940	967	994	1020	1047	1074	.7	18
88	842	869	896	923	951	978	1005	1032	1059	1086	.8	21
89	852	879	906	934	961	989	1016	1044	1071	1098	.9	24
90	861	889	917	944	972	1000	1028	1056	1083	1111		
91	871	899	927	955	983	1011	1039	1067	1095	1123	.1	3
92	880	909	937	965	994	1022	1051	1079	1107	1136	.2	6
93	890	919	947	976	1005	1033	1062	1091	1119	1148	.3	9
94	899	928	957	986	1015	1044	1073	1102	1131	1160	.4	12
95	—909	—938	—968	—997	—1026	—1056	—1085	—1114	—1144	—1173	.5	15
96	919	948	978	1007	1037	1067	1096	1126	1156	1185	.6	18
97	928	958	988	1018	1048	1078	1108	1138	1168	1198	.7	21
98	938	968	998	1028	1059	1089	1119	1149	1180	1210	.8	23
99	947	978	1008	1039	1069	1100	1131	1161	1192	1222	.9	26
100	957	988	1019	1049	1080	1111	1142	1173	1204	1235		
	31	32	33	34	35	36	37	38	39	40		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	1	2	3	4	5	6	8	9	10			

TABLE XI.—*Continued.*

Volumes by the Prismoidal Formula.

Widths.	Heights. 41	42	43	44	45	46	47	48	49	50	Corrections for tenths in height.	
1	13	13	13	14	14	14	15	15	15	15	.1	0
2	25	26	27	27	28	28	29	30	30	31	.2	0
3	38	39	40	41	42	43	44	44	45	46	.3	0
4	51	52	53	54	56	57	58	59	60	62	.4	1
5	63	65	66	68	69	71	73	74	76	77	.5	1
6	76	78	80	81	83	85	87	89	91	93	.6	1
7	89	91	93	95	97	99	102	104	106	108	.7	1
8	101	104	106	109	111	114	116	119	121	123	.8	1
9	114	117	119	122	125	128	131	133	136	139	.9	1
10	127	130	133	136	139	142	145	148	151	154		
11	139	143	146	149	153	156	160	163	166	170	.1	0
12	152	156	159	163	167	170	174	178	181	185	.2	1
13	165	169	173	177	181	185	189	193	197	201	.3	1
14	177	181	186	190	194	199	203	207	212	216	.4	2
15	190	194	199	204	208	213	218	222	227	231	.5	2
16	203	207	212	217	222	227	232	237	242	247	.6	3
17	215	220	226	231	236	241	247	252	257	262	.7	3
18	228	233	239	244	250	256	261	267	272	278	.8	4
19	240	246	252	258	264	270	276	281	287	293	.9	4
20	253	259	265	272	278	284	290	296	302	309		
21	266	272	279	285	292	298	305	311	318	324	.1	1
22	278	285	292	299	306	312	319	326	333	340	.2	2
23	291	298	305	312	319	327	334	341	348	355	.3	2
24	304	311	319	326	333	341	348	356	363	370	.4	3
25	316	324	332	340	347	355	363	370	378	386	.5	4
26	329	337	345	353	361	369	377	385	393	401	.6	5
27	342	350	358	367	375	383	392	400	408	417	.7	5
28	354	363	372	380	389	398	406	415	423	432	.8	6
29	367	376	385	394	403	412	421	430	439	448	.9	7
30	380	389	398	407	417	426	435	444	454	463		
31	392	402	411	421	431	440	450	459	469	478	.1	1
32	405	415	425	435	444	454	464	474	484	494	.2	2
33	418	428	438	448	458	469	479	489	499	509	.3	3
34	430	441	451	462	472	483	493	504	514	525	.4	4
35	443	454	465	475	486	497	508	519	529	540	.5	5
36	456	467	478	489	500	511	522	533	544	556	.6	6
37	468	480	491	502	514	525	537	548	560	671	.7	8
38	481	493	504	516	528	540	551	563	575	586	.8	9
39	494	506	518	530	542	554	566	578	590	602	.9	10
40	506	519	531	543	556	568	580	593	605	617		
41	519	531	544	557	569	582	595	607	620	633	.1	1
42	531	544	557	570	583	596	609	622	635	648	.2	3
43	544	557	571	584	597	610	624	637	650	664	.3	4
44	557	570	584	598	611	625	638	652	665	679	.4	6
45	569	583	597	611	625	639	653	667	681	694	.5	7
46	582	596	610	625	639	653	667	681	696	710	.6	8
47	595	609	624	638	653	667	682	696	711	725	.7	10
48	607	622	637	652	667	681	696	711	726	741	.8	11
49	620	635	650	665	681	696	710	726	741	756	.9	13
50	633	648	664	679	694	710	725	741	756	772		
	41	42	43	44	45	46	47	48	49	50		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	1	3	4	6	7	8	10	11	13			

TABLE XI.—*Continued.*

VOLUMES BY THE PRISMOIDAL FORMULA.

Widths.	Heights.										Corrections for tenths in height.	
	41	42	43	44	45	46	47	48	49	50		
51	645	661	677	693	708	724	740	756	771	787	.1	2
52	658	674	690	706	722	738	754	770	786	802	.2	3
53	671	687	703	720	736	752	768	785	802	818	.3	5
54	683	700	717	733	750	767	783	800	817	833	.4	7
55	—696	—713	—730	—747	—764	—781	—798	—815	—832	—849	.5	8
56	709	726	743	760	778	795	812	830	847	864	.6	10
57	721	739	756	774	792	809	827	844	862	880	.7	12
58	734	752	770	788	806	823	841	859	877	895	.8	14
59	747	765	783	801	819	833	856	874	892	910	.9	15
60	759	778	796	815	833	852	870	889	907	926		
61	772	791	810	828	847	866	885	991	923	941	.1	2
62	785	804	823	842	861	880	899	919	938	957	.2	4
63	797	817	836	856	875	894	914	933	953	972	.3	6
64	810	830	849	869	889	909	928	948	968	988	.4	8
65	—823	—843	—863	—883	—903	—923	—943	—963	—983	—1003	.5	10
66	835	856	876	896	917	937	957	978	998	1019	.6	12
67	848	869	889	910	931	951	972	993	1013	1034	.7	14
68	860	881	902	923	944	965	986	1007	1028	1049	.8	16
69	873	894	916	937	958	980	1001	1022	1044	1065	.9	18
70	886	907	929	951	972	994	1015	1037	1059	1080		
71	898	920	942	964	986	1008	1030	1052	1074	1096	.1	2
72	911	933	956	978	1000	1022	1044	1067	1089	1111	.2	3
73	924	946	969	991	1014	1036	1059	1081	1104	1127	.3	7
74	936	959	982	1005	1028	1051	1073	1096	1119	1142	.4	9
75	—949	—972	—995	—1019	—1042	—1065	—1088	—1111	—1134	—1157	.5	12
76	962	985	1009	1032	1056	1079	1102	1126	1149	1173	.6	14
77	974	998	1022	1046	1069	1093	1117	1141	1165	1188	.7	16
78	987	1011	1035	1059	1083	1107	1131	1156	1180	1204	.8	19
79	1000	1024	1048	1073	1097	1122	1146	1170	1195	1219	.9	21
80	1012	1037	1062	1086	1111	1136	1160	1185	1210	1235		
81	1025	1050	1075	1100	1125	1150	1175	1200	1225	1250	.1	3
82	1038	1063	1088	1114	1139	1164	1190	1215	1240	1265	.2	5
83	1050	1076	1102	1127	1153	1178	1204	1230	1255	1281	.3	8
84	1063	1089	1115	1141	1167	1193	1219	1244	1270	1296	.4	10
85	—1076	—1102	—1128	—1154	—1181	—1207	—1233	—1259	—1285	—1312	.5	13
86	1088	1115	1141	1168	1194	1221	1248	1274	1301	1327	.6	16
87	1101	1128	1155	1181	1208	1235	1262	1289	1316	1343	.7	18
88	1114	1141	1168	1195	1222	1249	1277	1304	1331	1358	.8	21
89	1126	1154	1181	1209	1236	1264	1291	1319	1346	1373	.9	24
90	1139	1167	1194	1222	1250	1278	1306	1333	1361	1389		
91	1152	1180	1208	1236	1264	1292	1320	1348	1376	1404	.1	3
92	1164	1193	1221	1249	1278	1306	1335	1363	1391	1420	.2	6
93	1177	1206	1234	1263	1292	1320	1349	1378	1406	1435	.3	9
94	1190	1219	1248	1277	1306	1335	1364	1393	1422	1451	.4	12
95	—1202	—1231	—1261	—1290	—1319	—1349	—1378	—1407	—1437	—1466	.5	15
96	1215	1244	1274	1304	1333	1363	1393	1422	1452	1481	.6	18
97	1227	1257	1287	1317	1347	1377	1407	1437	1467	1497	.7	21
98	1240	1270	1301	1331	1361	1391	1422	1452	1482	1512	.8	23
99	1253	1283	1314	1344	1375	1406	1436	1467	1497	1528	.9	26
100	1265	1296	1327	1358	1389	1420	1451	1481	1512	1543		
	41	42	43	44	45	46	47	48	49	50		
	.1	.2	.3	.4	.5	.6	.7	.8	.9	Corrections for tenths in width.		
	1	3	4	6	7	8	10	11	13			

TABLE III.—AZIMUTHS OF POLARIS

THE STAR AND THE AZIMUTH ARE W. OF N. when the hour angle is less
THE ANGLE NEXT to the star's hour angle or 12h. 58min.
TO FIND THE TRUE MERIDIAN the azimuth [illegible] add off to the east when the

Hour angle	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Azimuths for latitude— 30	32	34	36	38	40	42	44	46	48	50
h.	m	m	m	m	m	° ′	′	′	′	′	′	′	′	′	′	′
0	4	4	4	4	4	0 20	20	20	20	20	20	20	20	20	20	20
	8	8	8	8	8	2	2	2	2	2	4	4	4	4	4	4
	12	12	12	12	12	5	5	5	5	5	5	6	6	6	6	6
	16	16	16	16	16	6	6	7	7	7	7	7	8	8	8	8
	20	20	20	20	21	8	8	8	8	9	9	9	9	10	10	11
	24	24	24	24	25	9	10	10	10	10	11	11	11	12	12	13
	28	28	28	29	29	11	11	11	12	12	12	13	13	14	14	15
	33	32	32	33	33	12	12	12	13	14	14	15	15	16	16	17
	36	36	37	37	37	14	14	15	15	15	16	16	17	18	18	19
	40	40	41	41	41	15	16	16	17	17	18	18	19	20	20	21
	44	44	45	45	45	17	17	18	18	19	19	20	21	21	23	23
	48	49	49	49	50	19	19	19	20	21	22	22	23	23	24	25
	53	53	53	53	54	20	21	21	22	22	23	24	24	25	26	27
0	56	57	57	58	58	22	22	23	23	24	25	25	26	27	28	29
1	0	1	1	2	2	23	24	24	25	26	26	27	28	29	30	32
	5	6	6	7	7	25	26	26	27	28	29	29	30	31	33	34
	10	11	11	12	13	27	27	28	29	30	31	32	33	34	35	37
	15	16	17	17	18	29	29	30	31	32	33	34	35	36	38	39
	20	21	22	22	23	31	31	32	33	34	35	36	37	38	40	42
	26	26	27	28	28	32	33	34	35	36	37	38	39	41	42	44
	31	31	32	33	34	34	35	36	37	38	39	40	42	43	45	47
	36	36	37	38	39	36	37	38	39	40	41	42	44	45	47	49
	40	41	42	43	44	38	39	40	41	42	43	44	46	47	49	51
	45	46	47	48	49	39	40	41	42	44	45	46	48	50	52	54
	50	52	53	54	55	41	42	43	44	46	47	48	50	52	54	56
1	55	57	58	59	0	43	44	45	46	47	49	50	52	54	56	0 59
2	1	2	3	4	5	45	46	47	48	49	51	52	54	56	0 58	1 1
	6	7	8	9	10	46	47	49	50	51	53	54	56	0 58	1 1	3
	11	12	13	14	16	48	49	50	51	53	54	56	0 58	1 0	3	5
	16	17	18	20	21	50	51	52	53	55	56	0 58	1 0	2	5	8
	21	22	24	25	26	51	52	54	55	57	0 58	1 0	2	4	7	10
	26	27	28	30	32	53	54	55	57	0 58	1 0	2	4	6	9	12
	31	32	34	35	37	54	55	57	0 58	1 0	2	4	6	8	11	14
	36	38	39	41	42	56	57	0 58	1 0	2	3	6	8	10	13	16
	41	43	44	46	48	57	0 59	1 0	2	3	5	7	9	12	15	18
	46	48	49	51	53	0 59	1 0	2	3	5	7	9	11	14	17	20
	51	54	55	57	58	1 0	2	3	5	7	8	11	13	16	19	22
2	56	58	0	2	3	2	3	5	6	8	10	12	15	17	20	24
3	1	3	5	7	9	3	4	6	8	10	12	14	16	19	22	26
	7	10	12	14	16	5	6	8	10	12	13	16	18	21	24	28
	13	16	18	20	23	6	8	9	11	13	15	18	20	23	27	30
	19	22	24	27	29	8	9	11	13	15	17	19	22	25	28	32
	25	28	31	33	36	9	11	13	14	16	19	21	24	27	30	34
	32	35	37	40	43	11	13	14	16	18	20	23	25	29	32	36
	38	41	44	48	51	12	14	16	17	20	22	25	27	31	34	38
	44	47	50	55	58	14	15	17	19	21	24	26	29	32	36	40
3	50	54	0	4	7	15	17	19	21	23	25	28	31	34	38	42
4	0	4	10	13	17	17	19	21	23	25	27	30	33	36	40	44
	13	17	21	25	30	19	21	23	24	27	29	32	35	39	43	47
	21	24	33	38	43	20	22	24	26	29	31	34	37	41	45	49
	34	40	45	50	57	22	24	26	28	30	33	36	39	42	47	51
4	50	57	1	12	21	24	26	28	30	33	35	38	41	45	49	54
5	7	17	20	30	.	26	27	30	32	34	37	40	43	47	51	56
	30	.	..		.	27	28	31	33	36	39	42	45	49	53	58
6	..	...	....	. .	....	1 29	1 30	1 32	1 33	1 37	1 40	1 43	1 47	1 50	1 55	1 59

Date. 1888.	
Jan.	1
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Feb.	1
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Mar.	1
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Apr.	1
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May	1
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June	1
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July	1
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Aug.	1
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Sept.	1
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Oct.	1
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Nov.	1
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Dec.	1
	15

Tabular

Days.
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FOR ALL HOUR ANGLES. § 381A.

than $11^h 58^m$ and E. of N. when the hour angle is *greater* than $11^h 58^m$.
minus the star's hour angle), for the years given.
hour angle is *less* than $11^h 58^m$, and to the *west* when it is *greater* than $11^h 58^m$.

Time of upper Culmination after mean noon.	Hours.	1892.	1894.	1896.	1898.	1900.	Azimuths for latitude— 30°	32°	34°	36°	38°	40°	42°	44°	46°	48°	50°
h. m.	h.	m.	m.	m.	m.	m.	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
6 32.3	11	54	54	54	54	54	0 1	0 1	0 1	0 2	0 2	0 2	0 2	0 2	0 2	0 2	0 2
		50	50	50	50	50	3	3	3	3	3	3	3	4	4	4	4
5 37.0		46	46	46	45.	45.	5	5	5	5	5	5	5	5	6	6	6
4 29.9		42	42	41.	41.	41.	6	6	6	6	7	7	7	7	8	8	8
		38	37.	37.	37.	37.	8	8	8	8	8	8	9	9	9	10	10
3 34.6		34	33.	33.	34.	33	9	9	9	10	10	10	11	11	11	12	12
2 39.4		30	29.	29.	29	29	11	11	11	11	12	12	12	13	13	14	14
		26	25.	25.	25	25	12	12	13	13	13	14	14	14	15	15	16
1 44.3		22	21.	21	21	21	14	14	14	15	15	15	16	16	17	17	18
0 37.3		18	17.	17	17	16.	15	15	16	16	17	17	18	18	19	19	20
23 38.4		14	13.	13	12.	12.	17	17	17	18	18	19	19	20	21	21	22
		10	9.	9	8.	8	18	18	19	19	20	20	21	22	22	23	25
22 35.5		6	5.	5	4.	4	20	20	20	21	22	22	23	23	24	25	26
21 40.6	11	2	1.	1	0.	0	21	21	22	23	23	24	24	25	26	27	28
20 34.0	10	57.	57	56.	56.	55.	23	23	24	24	25	25	26	27	28	29	30
		52.	52	51.	51	50.	24	25	25	26	27	27	28	29	30	31	32
19 39.1		47.	47	46.	46	45	26	27	27	28	29	29	30	31	32	34	35
18 36.5		42.	42.	41.	40.	40	28	29	29	30	31	31	32	33	35	36	37
		37.	37	36	35.	35	30	30	31	22	33	34	35	36	37	38	40
17 41.6		32.	32	31	30.	29.	32	32	33	34	35	36	37	38	39	41	42
16 35.1		27.	26.	26	25	24.	33	34	35	36	37	38	29	40	41	43	44
15 40.2		22.	21.	21	20	19	35	36	37	37	39	40	41	42	43	45	47
		17.	16.	15.	15	14	37	38	39	39	40	41	43	44	46	47	49
14 33.6		12.	11.	10.	9.	8.	39	39	40	41	42	43	45	46	48	49	51
13 38.7		7.	6.	5.	4.	3.	40	41	42	43	44	45	47	48	50	52	54
	10	2.	1.	0	59	58	42	48	44	45	46	47	49	50	52	54	56
12 35.9	9	57.	56.	55.	54	53	44	45	46	47	48	49	51	52	54	56	0 58
11 40.8		52.	51	50.	49	47.	45	46	47	48	50	51	53	54	56	0 58	1 0
10 34.0		47.	46	44.	43.	42	47	48	49	50	51	53	54	56	0 58	1 0	2
		42	40.	39.	38.	37	49	50	51	52	53	55	56	0 58	1 0	2	5
9 38.9		37	35.	34.	33	31.	50	51	52	53	55	56	0 58	1 0	2	4	7
		32	30.	29	28	26.	51	53	54	55	57	0 58	1 0	2	4	6	9
8 35.8		27	25.	24	22.	21	53	54	56	57	0 58	1 0	2	4	6	8	11
7 40.6		22	20.	19	17.	15.	55	56	57	0 58	1 0	2	4	5	8	10	13
		17	15	13.	12	10	56	57	0 59	1 0	2	3	5	7	10	12	15
		12	10	8.	6.	5	58	0 59	1 0	2	3	5	7	9	11	14	17
difference.		7	5	3	1.	59.	0 59	1 0	2	3	5	7	9	11	13	16	19
	9	2	59.	58	56	54	1 0	2	3	5	6	8	10	12	15	18	21
m.	8	56.	54.	52.	51	49	2	3	5	6	8	10	12	14	17	20	23
8.9		50.	48.	46.	44.	42.	3	5	6	8	10	12	14	16	19	22	25
7.9		44.	42.	40	38	35.	5	6	8	9	11	13	16	18	21	24	27
11.8		38.	36	33.	31.	29	7	8	10	11	13	15	17	20	22	26	29
15.7		32.	29.	27	25	23.	8	9	11	13	15	17	19	22	24	28	31
19.6		26	23	21	17.	15.	10	11	13	14	16	18	21	23	26	29	33
23.6		19	16	13.	10.	7.	11	13	14	16	18	20	22	25	28	31	35
27.5		12	8.	5.	2.	59.	13	14	16	18	20	22	25	27	30	33	37
31.4	8	5	1	58	55	51.	14	16	18	19	21	24	26	29	32	35	39
35.4	7	55	51.	48	44.	40.	16	18	19	21	24	26	28	31	34	38	42
39.3		45	40.	37	33	28.	18	19	21	23	26	28	31	33	37	40	44
43.2		34.	29.	25	20.	15.	20	21	23	25	27	30	33	35	39	43	47
47.2		24	18.	13.	7.	1.	21	23	25	27	29	32	35	37	41	45	49
51.1	7	8	1	54	45.	35.	23	25	27	29	32	34	37	40	43	47	52
55.0	6	51.	41	30	11	....	25	27	29	31	34	36	39	42	46	50	54
58.9	6	26	....	....	....	....	27	29	31	33	36	38	41	44	48	52	57
62.9	5	....	....	....	....	....	1 29	1 30	1 32	1 35	1 37	1 40	1 43	1 47	1 50	1 55	1 59

ROMAN CAPITALS.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Roman Small.

a b c d e f g h i j k l m n o p q r s t u v w x y z

ITALIC CAPITALS.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Italic Small.

a b c d e f g h i j k l m n o p q r s t u v w x y z

Stump Writing.

a b c d e f g h i j k l m n o p q r s t u v w x y z

I II III IV V VI VII VIII IX X L (50) C (100) D (500) M (1000) **0 1 2 3 4 5 6 7 8 9**

0 1 2 3 4 5 6 7 8 9

CAPITALS, FORTRESSES of 1st Class, CHAIN of MOUNTAINS,

Oceans and Frontiers.

CITIES, BOUNDARIES of a STATE, FORTRESSES of 2d Rank,

Seas, Great Lakes, Mountain Branches.

FORTRESSES of 3d Rank, *STREAMS, NAVIGABLE RIVERS*, PLATEAUS,

Glaciers and Forests of large extensions.

Towns, Forts, ***Navigable Canals,*** Parts of Mountains, Summits, Forks, Abysses, ***Glaciers***

of less extensions, Boundaries of Counties.

Large Villages, Fortified Passes, Rivers navigable by Rafts, Lakes, Ponds, Forests.

Smaller Villages; Hamlets; Rivers not navigable; Creeks, Brooks, etc., not fordable.

Isolated Churches, Convents, Castles, Factories, Farmhouses, Mansions, and other smaller objects.

Note. — *All names and words connected with Water to be slanting Capitals and Italics.*

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